Exam Correction ALG3
2024/2025

Exo1. 1). $P(\lambda)=\det(A-\lambda I_3)=\begin{vmatrix}1-\lambda & -1 & 0\\ -1 & 2-\lambda & -1\\ 0 & -1 & 1-\lambda\end{vmatrix}$

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$P(\lambda)=\lambda^3-4\lambda^2+3\lambda=\lambda(1-\lambda)(\lambda-3)$.

2) $P(\lambda)=0 \Rightarrow \begin{cases}\lambda_1=0 \;;\; m_1=1\\ \lambda_2=1 \;;\; m_2=1\\ \lambda_3=3 \;;\; m_3=1\end{cases}$ 0,5

* For $\lambda=\lambda_1=1$: $E(\lambda_1)=E(0)=\langle V_1\rangle$, where $V_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$

$\dim E(\lambda_1)=1=m_1$ 0,5

* For $\lambda=\lambda_2=1$: $E(\lambda_2)=E(1)=\langle V_2\rangle$, where $V_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}$

$\dim E(\lambda_2)=1=m_2$ 0,5

* For $\lambda=\lambda_3=3$: $E(\lambda_3)=E(3)=\langle V_3\rangle$, where $V_3=\begin{pmatrix}1\\-2\\1\end{pmatrix}$

The new basis $B'=(V_1,V_2,V_3)$ and: 0,5

$$D=\begin{pmatrix}0&0&0\\0&1&0\\0&0&3\end{pmatrix}=P^{-1}AP \;;\; \text{where } P=(V_1V_2V_3)=\begin{pmatrix}1&-1&1\\1&0&-2\\1&1&1\end{pmatrix}$$

Compute P^{-1}: we have $X'=PX \Leftrightarrow X=P^{-1}X'$.

$$\begin{cases}x'=x-y+z\\ y'=x+2z\\ z'=x+y+z\end{cases} \Rightarrow \begin{cases}x=\frac{1}{3}x'+\frac{1}{3}y'+\frac{1}{3}z'\\ y=-\frac{1}{2}x'+\frac{1}{2}z'\\ z=\frac{1}{6}x'-\frac{1}{3}y'+\frac{1}{6}z'\end{cases}$$ 1

$$P^{-1}=\begin{pmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\ -\frac{1}{2}&0&\frac{1}{2}\\ \frac{1}{6}&-\frac{1}{3}&\frac{1}{6}\end{pmatrix}$$

3) we have $D = P^{-1}AP \Rightarrow A = PDP^{-1} \Rightarrow \boxed{A^n = PD^nP^{-1}}$

Then $A^n = \begin{pmatrix}1 & -1 & 1\\ 1 & 0 & -2\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3^n\end{pmatrix}\begin{pmatrix}1/3 & 1/3 & 1/3\\ -1/2 & 0 & 1/2\\ 1/6 & -1/3 & 1/6\end{pmatrix}$.

$$A^n = \begin{pmatrix}\frac{1}{6}\cdot 3^n + \frac{1}{2} & -\frac{1}{3}\cdot 3^n & \frac{1}{6}\cdot 3^n - \frac{1}{2}\\ -\frac{1}{3}\cdot 3^n & \frac{2}{3}\cdot 3^n & -\frac{1}{3}\cdot 3^n\\ \frac{1}{6}\cdot 3^n - \frac{1}{2} & -\frac{1}{3}\cdot 3^n & \frac{1}{6}\cdot 3^n + \frac{1}{2}\end{pmatrix}$$

4) we have $X_{n+1} = AX_n \Rightarrow X_n = A^n X_0$, $X_0 = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$.

$$\Rightarrow \begin{pmatrix}x_n\\ y_n\\ z_n\end{pmatrix} = A^n\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \Rightarrow \begin{cases} x_n = \left(\frac{3^n}{6}+\frac{1}{2}\right)\alpha - \frac{3^n}{3}\beta + \left(\frac{3^n}{6}-\frac{1}{2}\right)\gamma\\ y_n = \left(-\frac{3^n}{3}\right)\alpha + \left(\frac{2\cdot 3^n}{3}\right)\beta - \frac{3^n}{3}\gamma.\\ z_n = \left(\frac{3^n}{6}-\frac{1}{2}\right)\alpha - \frac{3^n}{3}\beta + \left(\frac{3^n}{6}+\frac{1}{2}\right)\gamma,\end{cases}$$

<u>EXO2</u>:

1) $P(\lambda) = \det(B - \lambda I_3) = \begin{vmatrix}9-\lambda & 7 & 3\\ -9 & -7-\lambda & -4\\ 4 & 4 & 4-\lambda\end{vmatrix} = \lambda^3 - 6\lambda^2 + 12\lambda - 8$

$P(\lambda) = (K-\lambda)^3$

$N = B - 2I_3 = \begin{pmatrix}7 & 7 & 3\\ -9 & -9 & -4\\ 4 & 4 & 2\end{pmatrix} \neq O_3$

2) $\begin{cases} Q(\lambda) = \lambda - 2 \longrightarrow N = B - 2I_3\\ \text{or}\\ Q(\lambda) = (\lambda-2)^2 \longrightarrow N^2 = (B-2I_3)^2 = \begin{pmatrix}-2 & -2 & -1\\ 2 & 2 & 1\\ 0 & 0 & 0\end{pmatrix} \neq O_3\\ \text{or}\\ Q(\lambda) = (\lambda-2)^3 \longrightarrow N^3 = (B-2I_3)^3 = O_3\end{cases}$

Then $\boxed{Q(\lambda) = \cancel{B} (\lambda-2)^3}$

We deduce that B is not diagonalizable

3) $P(\lambda)=0 \Rightarrow \lambda_1 = 2$; $m_1 = 3$.

For $\underline{\lambda=\lambda_1=2}$: $(B-\lambda_1 I_3)X=0 \Rightarrow (B-2I_3)X=0$

$$\Rightarrow \begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & -4 \\ 4 & 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} 7x+7y+3z=0 \\ -9x-9y-4z=0 \\ 4x+4y+2z=0 \end{cases} \Rightarrow \begin{cases} y=-x \\ z=0 \end{cases}$$

$$X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x \\ -x \\ 0 \end{pmatrix}=x\underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{V_1}$$; $E(\lambda_1)=E(2)=\langle U_1 \rangle$

and $\boxed{\dim E(\lambda_1)=1 \neq m_1=3}$

The Jordan basis $B'=(V_1,V_2,V_3)$ where:

$$\begin{cases} V_3 \notin \operatorname{Ker}(B-\lambda_1 I_3)^2 = \operatorname{Ker}(B-2I_3)^2 = \operatorname{Ker} N^2 \\ V_2 = (B-2I_3)V_3 = N V_3 \\ V_1 = (B-2I_3)V_2 \in E(\lambda_1). \end{cases}$$

0,5

Since $\operatorname{Ker} N^2$: $N^2 X = 0 \Rightarrow \begin{pmatrix} -2 & -2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} -2x-2y-z=0 \\ 2x+2y+z=0 \end{cases} \Rightarrow \boxed{z=-2x-2y}$$

$$X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x \\ y \\ -2x-2y \end{pmatrix}=x\underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}_{U_2}+y\underbrace{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}_{U_3}.$$

0,8

We put $V_3 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and

$$V_2 = \begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & -4 \\ 4 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 7 \\ -9 \\ 4 \end{pmatrix}$$

0,5

$$V_1 = \begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & -4 \\ 4 & 4 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ -9 \\ 4 \end{pmatrix}=\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \in E(\lambda_1).$$

The Jordan form of B :

$$\boxed{J_3(2) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = P^{-1} B P}$$

0,5

where $P = (V_1 V_2 V_3) = \begin{pmatrix} -2 & 7 & 1 \\ 2 & -9 & 0 \\ 0 & 4 & 0 \end{pmatrix}$.

and $P^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{9}{8} \\ 0 & 0 & \frac{1}{4} \\ 1 & 1 & \frac{1}{2} \end{pmatrix}$

4). By 2) $N^3 = O_3$, N is nilpotent of the index 3.

$N = B - 2 I_3 \Rightarrow \boxed{B = N + 2 I_3}$.

we have $e^{tB} = e^{t(N+2I_3)}$

$= e^{tN + 2tI_3}$

$= e^{tN} \cdot e^{2tI_3}$

1

since $e^{2tI_3} = I_3 e^{2t}$, then $e^{tB} = e^{2t} I_3 e^{tN}$

$$\boxed{e^{tB} = e^{2t} \cdot e^{tN}}$$

We have $X' = BX \Rightarrow X = e^{tB} \cdot X_0$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $X_0 = \begin{pmatrix} 9 \\ 4 \\ 9 \end{pmatrix}$

Since $e^{tB} = e^{2t} \cdot e^{tN} = e^{2t}\left(I_3 + tN + \frac{(tN)^2}{2}\right)$

1

$= e^{2t}\left(I_3 + tN + \frac{t^2}{2} N^2\right)$

$= e^{2t}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & -4 \\ 4 & 4 & 2 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -2 & -2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right]$

Finally
$$\textcircled{1}\quad\begin{cases} x = \left[(1+7t-t^2)c_1 + (7t-t^2)c_2 + (3t-\frac{t^2}{2})c_3\right] e^{2t} \\ y = \left[(-9t+t^2)c_1 + (1-9t+t^2)c_2 + (-4t+\frac{t^2}{2})c_3\right] e^{2t} \\ z = \left[4t c_1 + 4t c_2 + (1+2t)c_3\right] e^{4t} \end{cases}$$

<u>Exo 3.</u> We have $J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

1)
$$\left[J_2(\lambda)\right]^1 = \begin{bmatrix} \lambda^1 & 1\lambda^{1-1} \\ 0 & \lambda^1 \end{bmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = J_2(\lambda).$$

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- $P(1)$ is true
- $P(n)$ true $\Rightarrow P(n+1)$ true

Since $P(n+1)$: $\left[J_2(\lambda)\right]^{n+1} = \left[J_2(\lambda)\right]^n \times \left[J_2(\lambda)\right] =$
$$= \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix}$$

2) We have
$$e^{J_2(\lambda)} = \sum_{n\geq 0} \frac{\left[J_2(\lambda)\right]^n}{n!} = \sum_{n\geq 0} \frac{\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}}{n!} =$$
$$= \sum_{n\geq 0} \begin{pmatrix} \frac{\lambda^n}{n!} & \frac{\lambda^{n-1}}{(n-1)!} \\ 0 & \frac{\lambda^n}{n!} \end{pmatrix} = \begin{pmatrix} \sum_{n\geq 0} \frac{\lambda^n}{n!} & \sum_{n\geq 1} \frac{\lambda^{n-1}}{(n-1)!} \\ 0 & \sum_{n\geq 0} \frac{\lambda^n}{n!} \end{pmatrix}$$

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$$= \begin{pmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{pmatrix}.$$