Graphs Theory
Cycles and Cocycles(3)

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Speciality:

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L3

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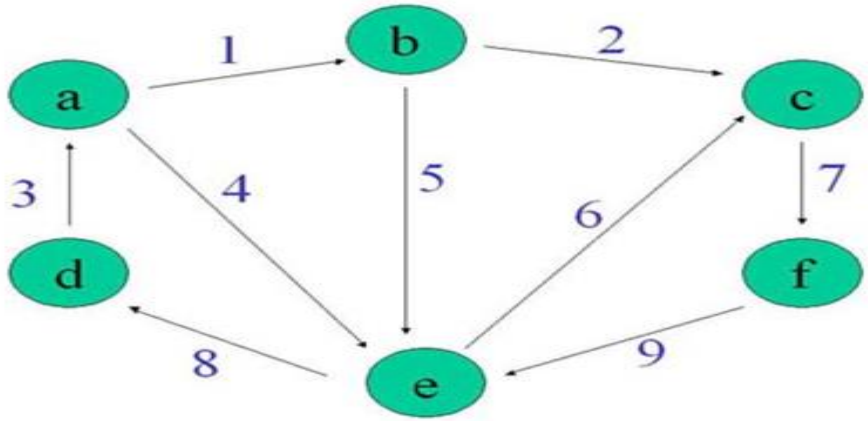
Cycle

- A cycle is a closed chain (the start end-point and the terminal end point are identicals)
- The length of a cycle is the number of edges that contains.
- An elementary cycle is a cycle where no vertex appears more than one time (except the initial and the terminal).

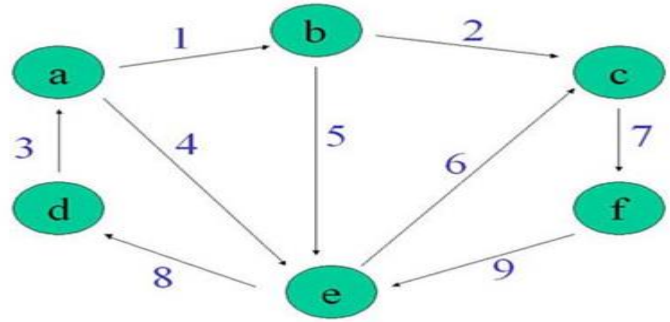
Example

the cycle c1=(a,e,d,a) is an elementary cycle its length =3.
C2=(a,b,c,e,a) is an elementary cycle its length =4



Cycle

❑Vector representation



Let G $G=(V,E)$ **be a graph** $m=|E|$
A cycle may be represented by a vector

$$Vi=\begin{cases} 0, \text{if the edge(i)} \notin \text{cycle} \\ +1, \text{if edge(i)} \in \text{cycle and travelled in right direction} \\ -1, \text{if the edge(i)} \in \text{cycle and travelled in inverse direction} \end{cases}$$

Example: c1=(a,b,e,a) is represented by the vector (+1,0,0,-1,+1,0,0,0,0)
C2 = (1,2,7,9,8,3) , Is represented by the vector (+1,+1,+1,0,0,0,+1,+1,+1)

Cycle

- We can therefore compare a cycle to a vector and define the sum of 2 or more cycles

Example :

C3 = (3, 4, 9, 7, 6. 8) is the sum of two cycles (3,4,8) and (9,7,6).

- **Property 1**:Any cycle is the sum of elementary cycles without a common edge.
- **Property 2** :A cycle is elementary if and only if it is minimal
- **The base of cycles** : each set of cycle linear independent :

$$\alpha_1 \overrightarrow{v_1} + \alpha_2 \overrightarrow{v_2} + \ldots + \alpha_k \overrightarrow{v_k} = \vec{0}$$

$$(\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0)$$

Cycle

- **the cyclomatic number:**

The number of vectors in base of cycles is called cyclomatic number
Denoted by $$\mu(G) = |E| - |V| + p$$

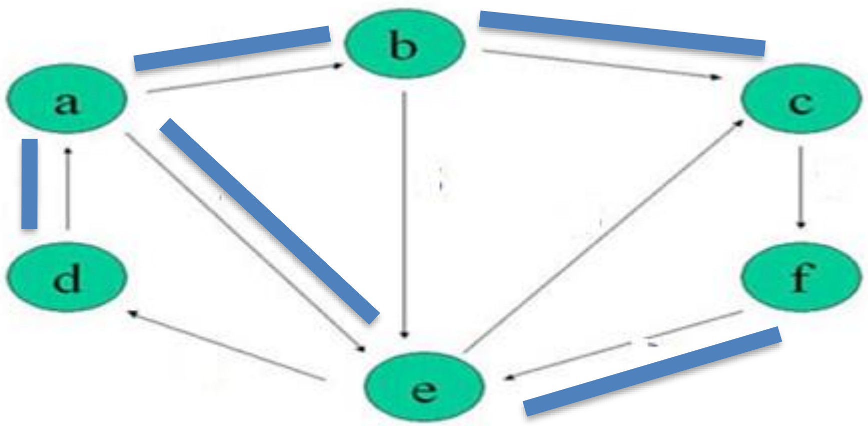
P: is the number of the connected components in the graph.

- **Base of cycles**

For obtaining the base of cycle in a connected graph G(V,E) ;
1-construct the tree (partial graph G1(V,E1) $E1 \subset E$ **)**
2-put in the base each cycle created by adding an edge
$ei \in E - E1$ **In the tree.**

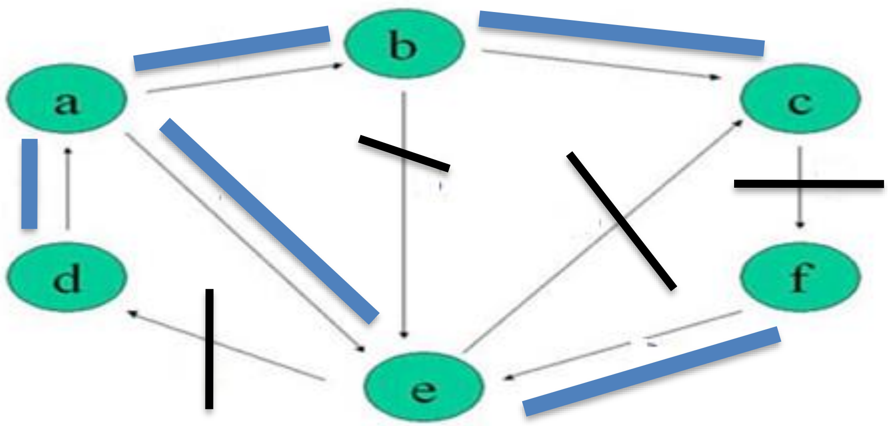
Cycle

Example :

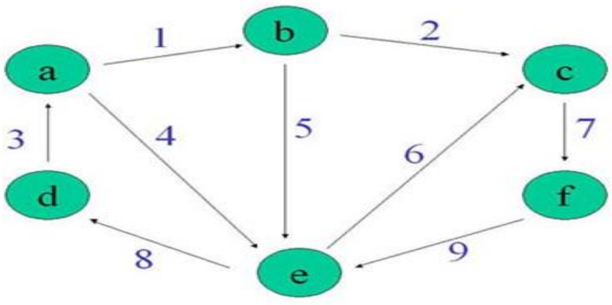


Cycle

Example :



U1=(0,0,+1,+1,0,0,0,+1,0)
U2=(+1,0,0,-1,+1,0,0,0,0)
U3=(+1,+1,0,-1,0,-1,0,0,0)
U4=(+1,+1,0,-1,0,0,+1,0,+1)



Co-cycle

❑Definition

A partition $\{A, V-A\}$ is a partition of V, the set $E(A, V-A)$ Of all the edges of G crossing this partition is called a cut (Cocycle). Denoted by

$$\omega(A) = \omega^{+}(A) \cup \omega^{-}(A)$$

$\omega^{+}(A)$: arcs incident from A,(those which leave A , will be noted positively).

$\omega^{-}(A)$: arcs incident towards A, (those which point towards A will be noted negatively).

Co-cycle

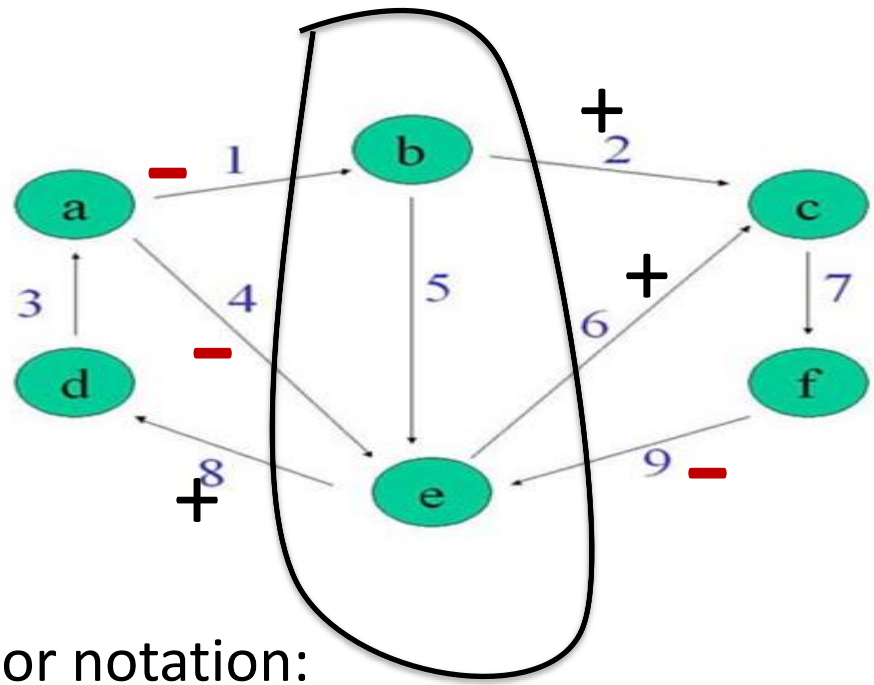
- Each cocycle is represented by a vecteur

$$\omega = (\omega_1, \omega_2, \ldots, \omega_m)$$

$$\omega_i = \begin{cases} +1, \text{if the edge incide from A} \\ -1, \text{if the idge incide to A} \\ 0, \text{else} \end{cases}$$

Co-cycle

Example: $A = \{b, e\}$ $\omega(A)$



Vector notation:
Vector (-1,+1,0,-1,0,+1,0,+1,-1)

Co-cycle

❑Definition

A cocycle will be said to be elementary when it is composed of edges connecting two subsets of connected components.

Example : this cocyle does not elementary

(-1,+1,0,-1,0,+1,0,+1,-1). Because the set {a,c,d,f}.

Not connected .But the cocycle where its vector is :

(0,1,0,1,1,0,0,-1,0) is elementary.

❑Property

Every cocycle is the sum of elementary cocycles without a common arc.

❑property

A cocycle is elementary if and only if it is minimal.

Co-cycle

- Base of Cocycle

A cocycle base is a set of cocycle lineary indipendent.

- Cocyclomatic number

Let $G=(V,E)$ be a graph with p connected components.

The cocyclomatic number is the number of coccycle in the cpcycle base. Denoted by :

$$\lambda(G)=|V|-p$$

Co-cycle

❑ Creation of a base of Cocycle

Let $G = (V, E)$ be a graph

1-construct a tree. (partial graph $E1 \subset E$)

2-remove an edge from the tree.

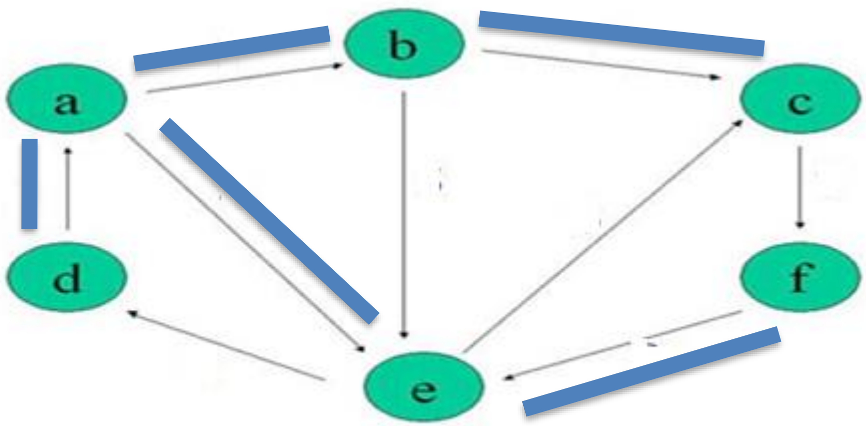
3-add the correspondent cocycle in the base

4-if all the edges are treated then stop

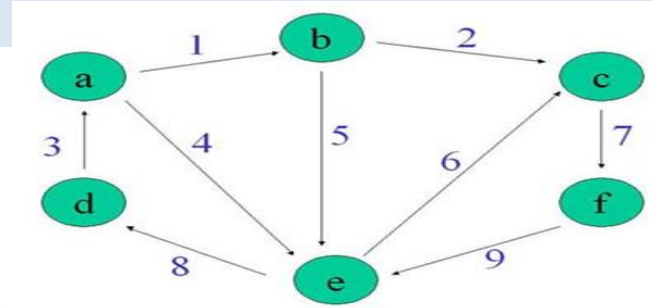
5-else go to step 2.

CoCycle

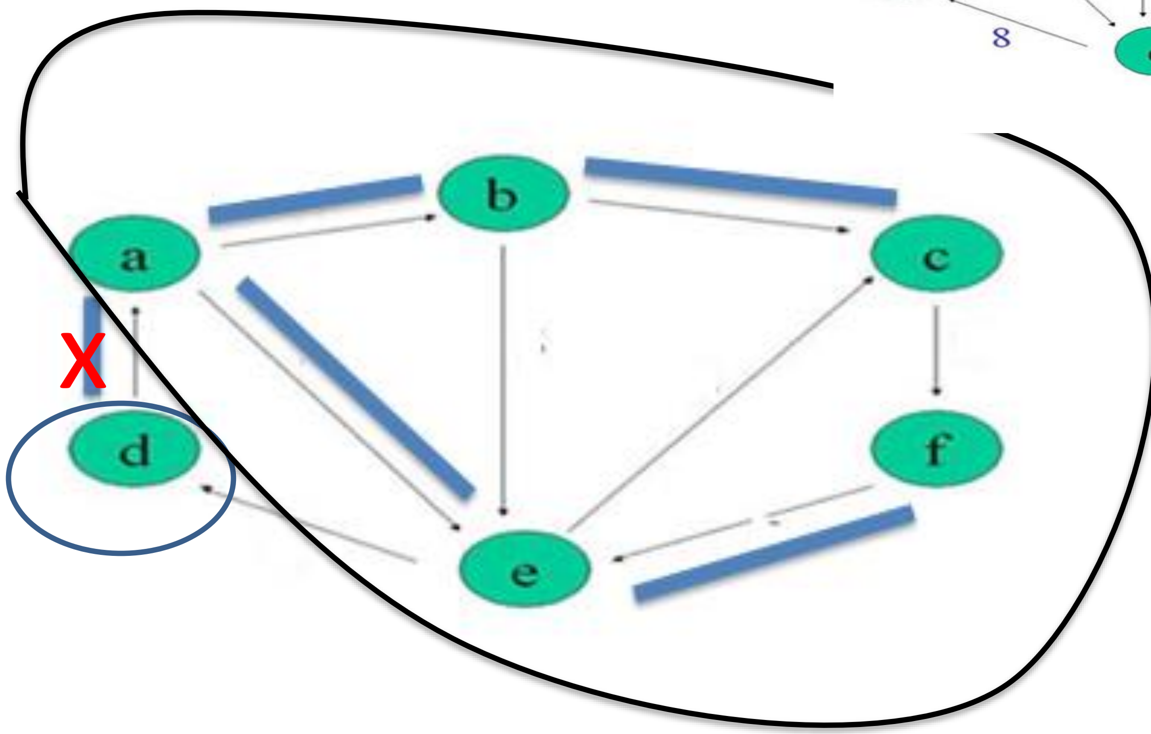
Example :



CoCycle



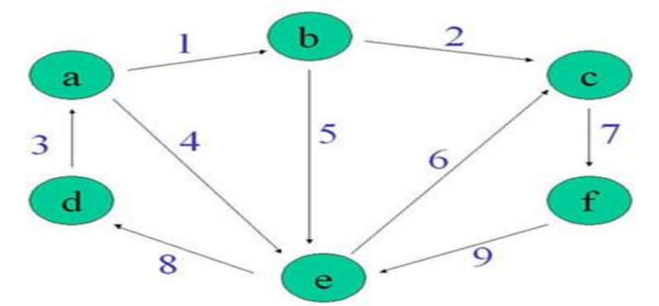
Example :

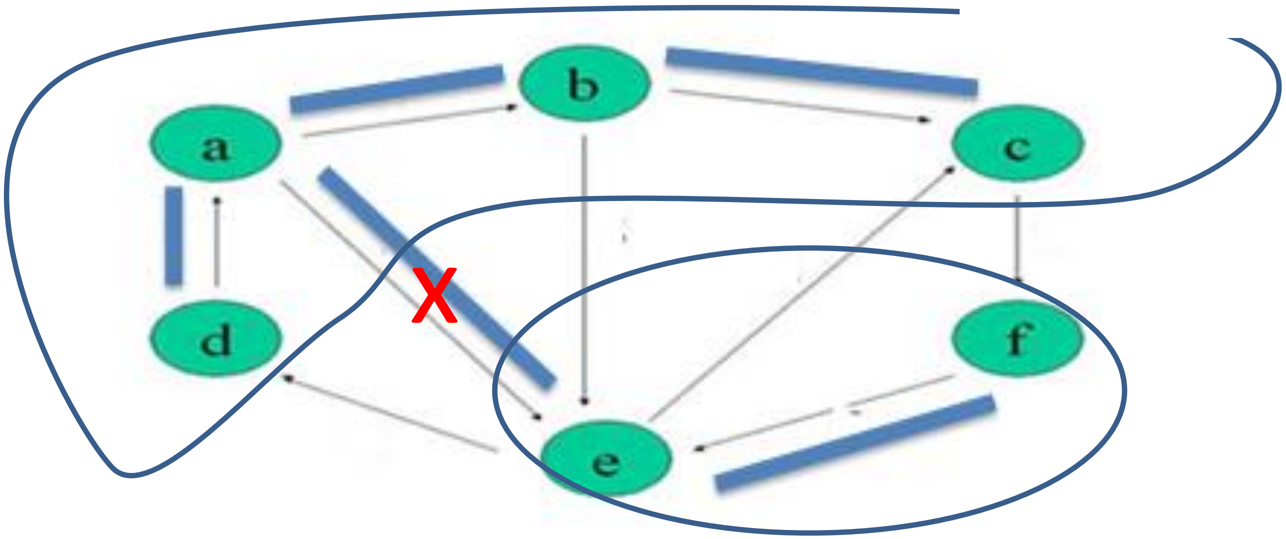


U1=(+3,-8)…………………(0,0,+1,0,0,0,0,-1,0

CoCycle

Example :

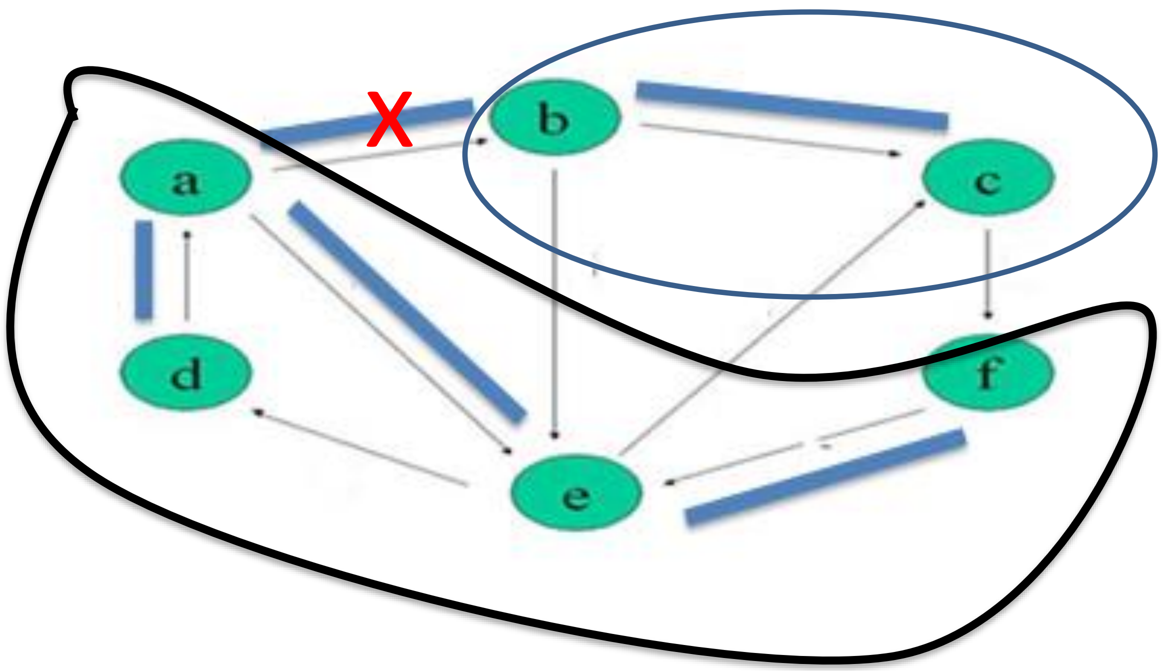




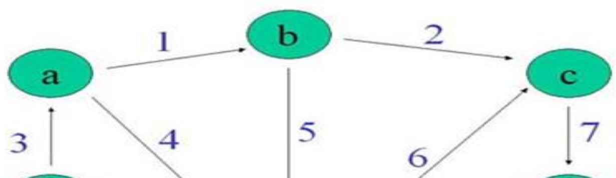
U2=(+8,-4,-5,+6,-7)……………………….(0,0,0,-1,-1,+1,-1,0)

CoCycle

Example :

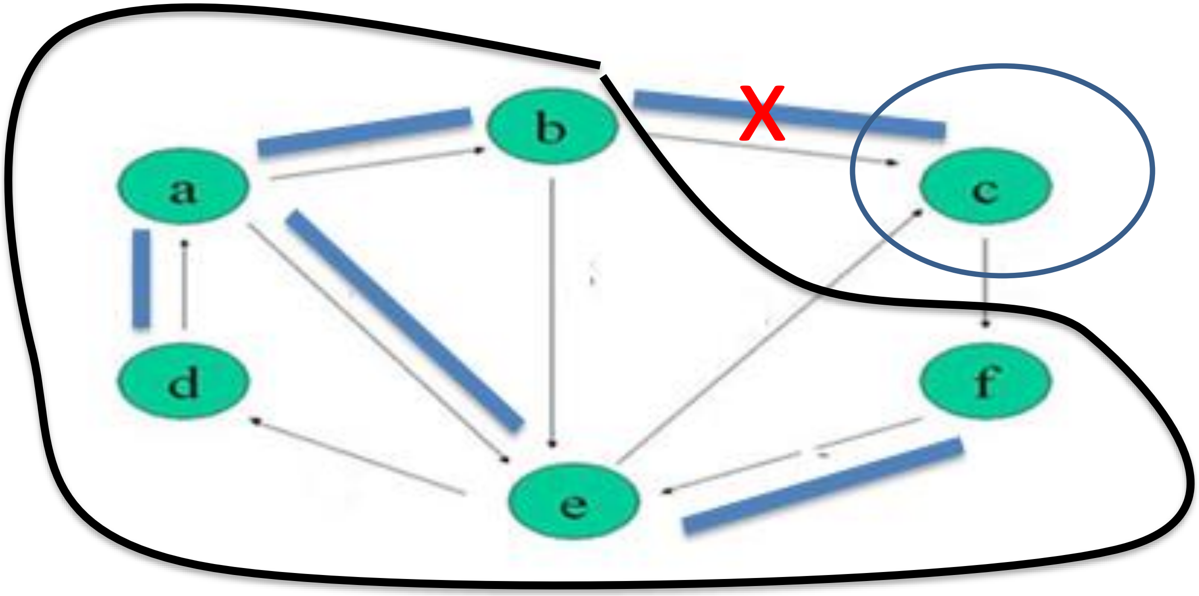


U3=(-1,+5,-6,+7)……………………….(-1,0,0,0,+1,-1,+1,0,0)



CoCycle

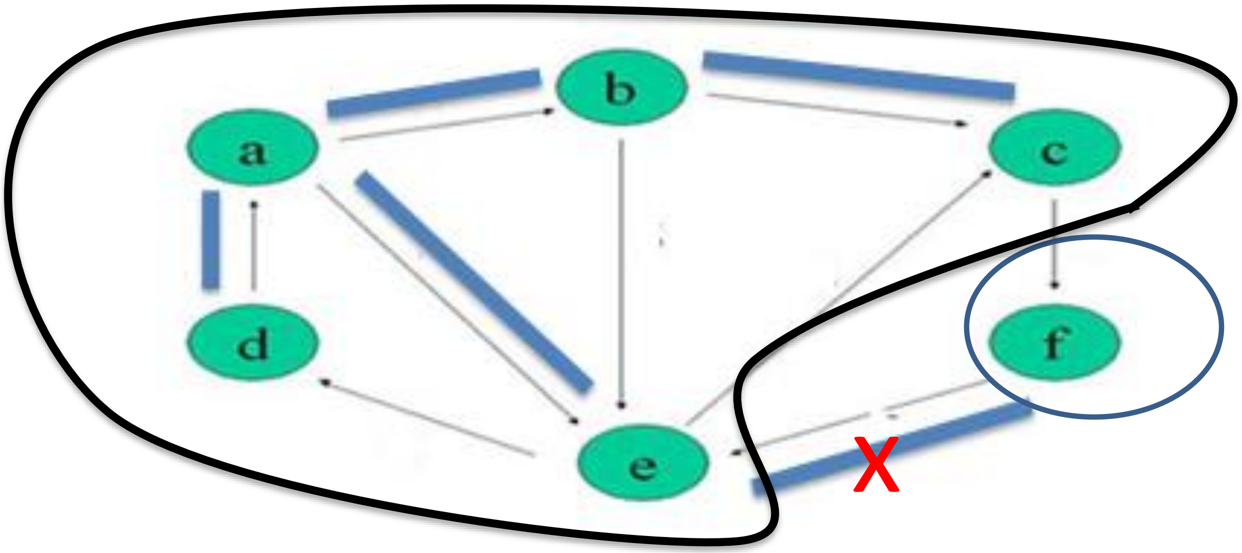
Example :



U4=(-2,-6,+7)……………………(0,-1,0,0,0,-1,+1,0,0)

CoCycle

Example :



U5=(-7,+9)……………………………(0,0,0,0,0,0,-1,0,+1)

CoCycle

- **Note:** the two vector spaces (cycle and cocycle are orthogonal)

$$c \bullet w = 0$$

For each cycle c and for each cocycle w

Example:

Cocycle W={a,b,e,c,f}
(-1,0,0,0,+1,-1,+1,0,0)
Cycle C={a,b,e,a}
(+1,0,0,-1,+1,0,0,0,0)
C*w=0

CoCycle

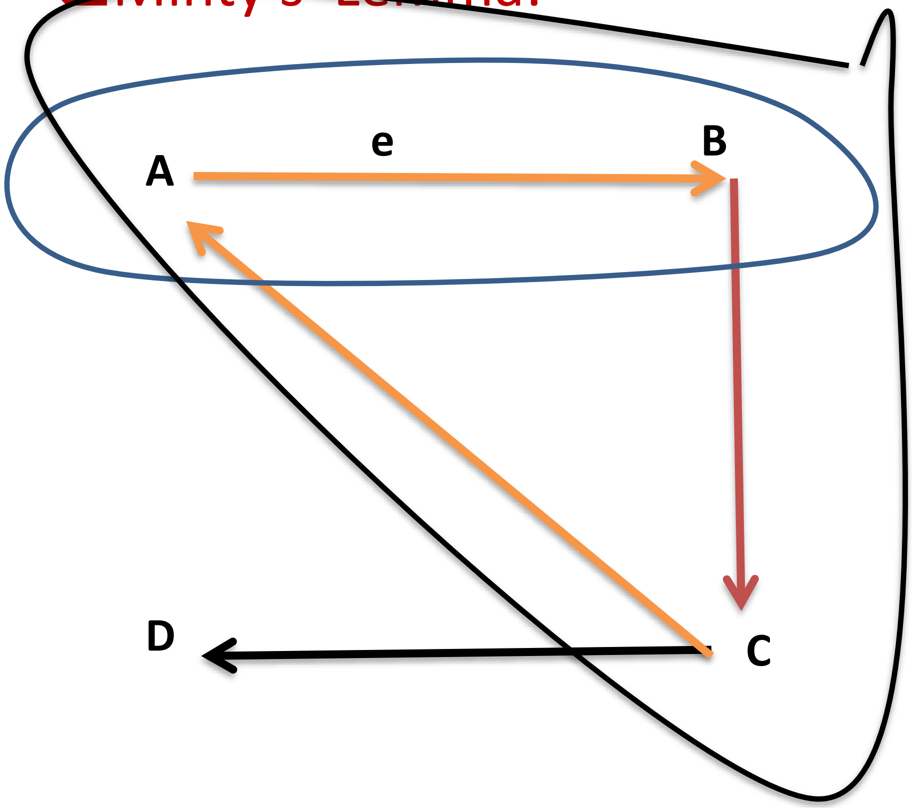
❑ Minty's Lemma:

Let G=(V,A) be a directed graph and a is an edge in A. Consider any coloring of the edges of A such that a is BLACK and the other edges are BLUE, RED or BLACK. Then we necessarily have exactly one of the following two cases:

- there exists a cycle passing through a , colored only in RED and BLACK, and of which all BLACK edges are in the same direction as a .
- there exists a cocycle containing a , colored only in BLUE and BLACK, and all edges of which are BLACK cross the border in the same direction as a .

CoCycle

- Minty's Lemma:



Seek cycle Red/Green contains e

RED,GREEN,RED

Seek COcycle Green /Black contains e

Impossible

Duality cycle/Cocycle
one possibility