Monte Carlo Methods

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Applications:

- **Optimization**: Finding the best solution in complex scenarios.
- **Numerical Integration**: Solving difficult integrals, particularly in high dimensions.
- **Probability Sampling**: Generating random samples from distributions.
- **Uncertainty Modeling**: For example, assessing risks in nuclear power plants.

Fields of Use:

- Sciences: Physics, chemistry, biology.
- Mathematics and Engineering: Statistics, cryptography, AI.
- Social Sciences: Sociology, psychology, political science.
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Limitations:

- The more precise results required, the more time and resources the calculation demands.
- Problems with many variables are very difficult to solve.
- Results depend on high-quality random number generators.
- It is not always easy to verify if the results are correct.

Origins of the Monte Carlo Method

- **Buffon's Needles (18$^{\text{th}}$ century)**: First idea of a random method to estimate π.
- **1930s**: Enrico Fermi experiments with simulations for the study of neutron diffusion but does not publish his work.
- **1946**: Stanisław Ulam proposes a modern method while playing solitaire, inspired by the idea of repeating random experiments.
- **Collaboration with John von Neumann**: They develop the first Monte Carlo calculations to solve problems related to nuclear weapons.
- **The Name "Monte Carlo"**: A reference to the famous casino, suggested by Nicholas Metropolis.

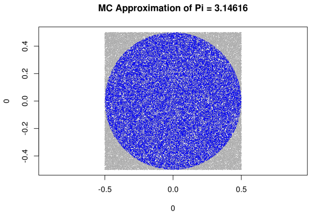
Evolution and Applications

- **1940s-50s**: Monte Carlo calculations used for the design of the H-bomb (ENIAC computer).
- **1950s-60s**: Development in the fields of physics and chemistry.
- **Recent Progress**:
  - **1990s**: Sequential Monte Carlo methods used in signal processing and Bayesian inference.
  - **Modern Applications**:
    - Quantum simulation (diffusion methods).
    - Evolutionary computing (genetic algorithms).
- **Impact**: A key method for solving complex problems across many scientific fields.

Estimating π with Monte Carlo

Principle:

- We simulate dart throws into a square centered at the origin, with dimensions $[-1, 1]$ on each axis.
- A dart is inside the inscribed circle if its distance from the origin satisfies $x^2 + y^2 < 1$.
- The ratio $\frac{\text{darts in circle}}{\text{total darts}}$ provides an estimation of $\pi/4$.



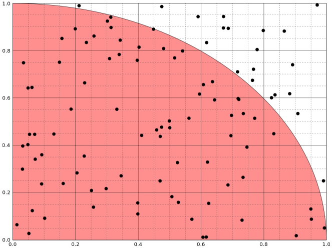
Link to Buffon's Experiment

Concept:

- This method is linked to an old mathematical experiment: Buffon's needle.
- Here, we randomly choose a point (x, y) in a square of side 1.
- If $x^2 + y^2 < 1$, the point is within the inscribed circle.

Probability:

- The probability that the point is in the circle is $\pi/4$.
- The larger the number of simulated points, the more precise the estimation of π.



Estimating the Area of a Lake with Monte Carlo

Principle:

- We consider a rectangular or square area of known sides containing a lake of unknown surface area.
- We simulate N random shots (e.g., cannon fire) over this area.
- X shots land on the ground, so $N - X$ shots land in the lake.

Calculation:

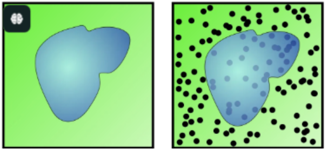
- Land Area $=$ length $\times$ width
- Probability of a shot landing in the lake $= \frac{N-X}{N}$
- Lake Area $= \frac{(N-X)}{N} \times$ Land Area

Numerical Example and Estimation Quality

Example:

- The land has a surface area of 1000 m^2.
- 500 shots are fired randomly, and 100 projectiles land in the lake.
- Estimated lake area:

$$\text{Lake Area} = (100 \times 1000)/(500) = 200\ m^2$$



Estimation Quality:

- The estimation improves with a higher number of shots.
- The quality of the random generator is crucial:
  - A biased generator (like a cannon always aiming at the same spot) distorts the results.
  - A good generator must ensure uniform coverage of the area.

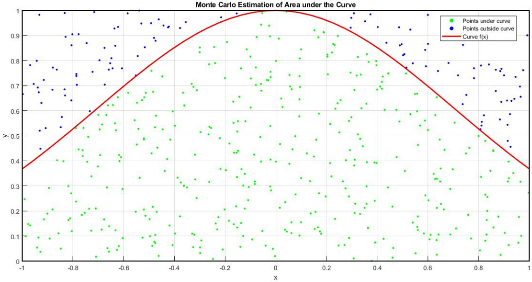
Estimating an Integral with Monte Carlo

Principle:

- Let $I = \int_a^b f(x)\,dx$ be the integral to estimate.
- Example: Calculate $\int_{-1}^{1} e^{-x^2}\,dx$.
- Method:
  - Generate N random points uniformly within a rectangle enclosing the curve.
  - Count n points that lie beneath the graph of $f(x)$.

Calculation:

$$\text{Area under the curve} = \frac{n}{N} \times \text{rectangle area}$$

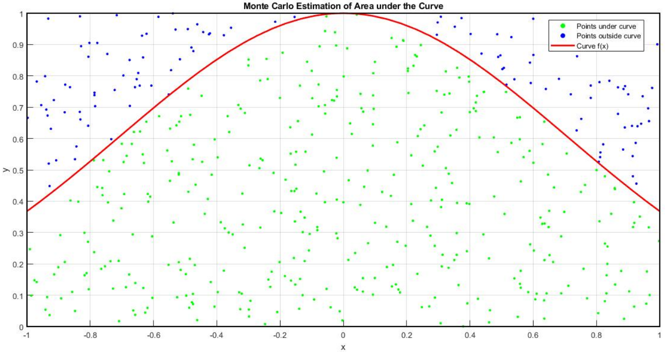


3. Calculating an Integral

$$I = \int_a^b f(x)dx$$

Calculating $\int_{-1}^{+1} e^{-x^2} dx$: We generate N random points in the rectangle (see diagram), then calculate the number n of points belonging to the area under the graph of $f(x) = e^{-x^2}, x \in [-1, 1]$.

$$\Rightarrow \text{Area} = \frac{n}{N} \times \text{rectangle\_area}$$



Conclusion

- Monte Carlo methods use randomness to solve complex problems.
- They have evolved from games of chance to modern applications in science and technology.
- An essential method for modeling and simulation in many fields.