Mohamed BOUDIAF University M'sila | Faculty of M & C.S
3rd year Bachelor's **A.M** | Academic year 2025-2026
Module: Stochastic Process | Dr. E.M.B BOUHADJAR
Date: 09-05-2026 | Time: 11:00–13:00

Exam
Prohibited items : correction fluid, pencils, red pens, and the sharing of tools.

Exercise 1 :

(4 pts) We define a certain class of signals $Y := (Y_t)_{t\geq 0}$ by :

$$Y_t = rX_t \cos(a_p t + \phi)$$

where $X := (X_t)_{t\geq 0}$ is a stationary random signal modulating a sinusoidal carrier $r\cos(a_p t + \phi)$. The following assumptions are given :

— The mean of X is zero : $E[X_t] = 0$.
— The correlation of X is denoted by $R_X(\tau)$ (i.e., $R_X(\tau) = E[X_t X_{t+\tau}]$).
— r and a_p (carrier frequency) are real constants.
— ϕ is a random variable uniformly distributed on $[0, 2\pi]$.
— ϕ and X are independent.

Calculate the mean value and the autocorrelation function of signal Y. Conclude regarding the stationarity of Y.

Note : The following trigonometric identity may be useful :

$$\cos u \cos v = \tfrac{1}{2}\big[\cos(u - v) + \cos(u + v)\big]$$

Exercise 2 :

(3 pts) Let $k \in \mathbb{N}^*$ and $(X_n)_{0\leq n\leq k}$ be a stochastic process with values in a discrete state space E. Assume that the process X has **independent increments**. Show that $(X_n)_{0\leq n\leq k}$ is a **Markov chain** by verifying that for all $n \leq k$ and for all $x_0, x_1, \ldots, x_n \in E$ such that $\mathbb{P}(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}) > 0$:

$$\mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1})$$

Exercise 3 :

(5 pts) Consider five equidistant points on a circle. At each moment, a walker jumps from one point to one of its neighbors, with a probability of 1/2 for each neighbor. Determine the graph and the transition matrix of the resulting chain. Analyze this chain by determining its communication classes, the classification of states, and its global properties (irreducibility, recurrence, aperiodicity... etc.) ?

Exercise 4 :

(5 pts) A virus can exist in N different strains. At each step, the virus either stays in its current strain or mutates to a different one :

— The probability of **staying** in the same strain is $1 - \alpha$.
— The probability of **mutation** is α. When it mutates, it chooses one of the other $N - 1$ strains with equal probability.

1. Write the transition probability matrix P and draw the transition graph. Analyze the properties of this chain (is it irreducible ? aperiodic ? etc.).
2. Using the symmetry property, find the stationary distribution π.
3. What is the average number of steps for the virus to return to its starting strain ?

Exercise 5 :

(3 pts) Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Let $N(t)$ be the combined process defined by $N(t) = N_1(t) + N_2(t)$.

a. Find the probability that $N(1) = 2$ and $N(2) = 5$.
b. Given that $N(1) = 2$, find the probability that $N_1(1) = 1$.

Solutions

Solution Exercise 1

1. Mean Value $E[Y_t]$

Using the independence of X_t and ϕ, we can separate the expectation of the product into a product of expectations :

$$E[Y_t] = E[rX_t\cos(a_pt+\phi)] = rE[X_t]E[\cos(a_pt+\phi)]$$

Since the mean of X is zero ($E[X_t]=0$), we immediately obtain :

$$\mathbf{E[Y_t] = 0} \quad (1\ pt)$$

2. Correlation Function $R_Y(t,t+\tau)$

By definition, $R_Y(t,t+\tau) = E[Y_tY_{t+\tau}]$. Substituting the expression for Y_t :

$$R_Y(t,t+\tau) = E\left[(rX_t\cos(a_pt+\phi))\,(rX_{t+\tau}\cos(a_p(t+\tau)+\phi))\right]$$

Using the independence between X and ϕ :

$$R_Y(t,t+\tau) = r^2E[X_tX_{t+\tau}]E[\cos(a_pt+\phi)\cos(a_pt+a_p\tau+\phi)]$$

— $E[X_tX_{t+\tau}] = R_X(\tau)$ because X is stationary.
— For the second term, we use the trigonometric identity $\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$:

$$\cos(a_pt+\phi)\cos(a_pt+a_p\tau+\phi) = \frac{1}{2}\left[\cos(a_p\tau)+\cos(2a_pt+a_p\tau+2\phi)\right]$$

The expectation with respect to ϕ gives :

$$E_\phi[...] = \frac{1}{2}\cos(a_p\tau) + \frac{1}{2}E_\phi[\cos(2a_pt+a_p\tau+2\phi)]$$

Since $\phi\sim\mathcal{U}[0,2\pi]$, the integral of a cosine over two full periods (2ϕ varying from 0 to 4π) is zero :

$$\int_0^{2\pi}\cos(2a_pt+a_p\tau+2\varphi)\frac{1}{2\pi}d\varphi = 0$$

Hence :

$$\mathbf{R_Y}(\tau) = \frac{\mathbf{r^2}}{\mathbf{2}}\mathbf{R_X}(\tau)\cos(\mathbf{a_p}\tau) \quad (2.5\ pts)$$

3. Conclusion (0.5 pt) The mean of Y is constant (0) and its correlation depends only on the time lag τ. Therefore, the process Y is **weakly stationary**.

Solution Exercise 2

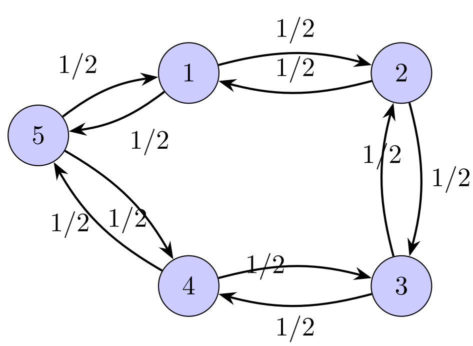
$$
\begin{aligned}
&\mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}, \dots, X_0 = x_0) \\
&= \frac{\mathbb{P}(X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0)}{\mathbb{P}(X_{n-1} = x_{n-1}, \dots, X_0 = x_0)} \\
&= \frac{\mathbb{P}(X_n - X_{n-1} = x_n - x_{n-1}, X_{n-1} - X_{n-2} = x_{n-1} - x_{n-2}, \dots, X_0 = x_0)}{\mathbb{P}(X_{n-1} = x_{n-1}, \dots, X_0 = x_0)} \\
&= \frac{\mathbb{P}(X_n - X_{n-1} = x_n - x_{n-1})\mathbb{P}(X_{n-1} - X_{n-2} = \dots, X_0 = x_0)}{\mathbb{P}(X_{n-1} = x_{n-1}, \dots, X_0 = x_0)} \quad (1\ pt) \\
&\text{by the independent increments property} \quad (1\ pt) \\
&= \frac{\mathbb{P}(X_n - X_{n-1} = x_n - x_{n-1})\mathbb{P}(X_{n-1} = x_{n-1}, \dots, X_0 = x_0)}{\mathbb{P}(X_{n-1} = x_{n-1}, \dots, X_0 = x_0)} \\
&= \mathbb{P}(X_n - X_{n-1} = x_n - x_{n-1}) \\
&= \mathbb{P}(X_n - X_{n-1} = x_n - x_{n-1}) \frac{\mathbb{P}(X_{n-1} = x_{n-1})}{\mathbb{P}(X_{n-1} = x_{n-1})} \\
&= \frac{\mathbb{P}(X_n - X_{n-1} = x_n - x_{n-1}, X_{n-1} = x_{n-1})}{\mathbb{P}(X_{n-1} = x_{n-1})} \\
&\text{by the independent increments property} \quad (1\ pt) \\
&= \frac{\mathbb{P}(X_n = x_n, X_{n-1} = x_{n-1})}{\mathbb{P}(X_{n-1} = x_{n-1})} \\
&= \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1})
\end{aligned}
$$

Thus the process $(X_n)_{n\in\mathbb{N}}$ is a Markov process.

Solution Exercise 3

The set of states is $E = \{1, 2, 3, 4, 5\}$. We have $p_{i,i+1} = p_{i,i-1} = 1/2$ for $i \in \{2, 3, 4\}$, $p_{1,2} = p_{1,5} = 1/2$ and $p_{5,1} = p_{5,4} = 1/2$, resulting in the matrix :

$$
P = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{pmatrix} \quad (1.5\ pts)
$$



(1.5 pts)

The chain is irreducible (all states communicate) and finite, therefore it is positive recurrent. Since all states are in the same class, they share the same period. We have $d(1) = \gcd\{n \geq 1; p_{1,1}^{(n)} > 0\}$. Since $p_{1,1}^{(2)} \geq p_{1,2}p_{2,1} = 1/4 > 0$ and $p_{1,1}^{(5)} \geq p_{1,2}p_{2,3}p_{3,4}p_{4,5}p_{5,1} = 1/2^5 > 0$, the only common divisor of 2 and 5 is 1. Thus $d = d(1) = 1$: the chain is aperiodic and consequently ergodic (2 pts).

Solution Exercise 4

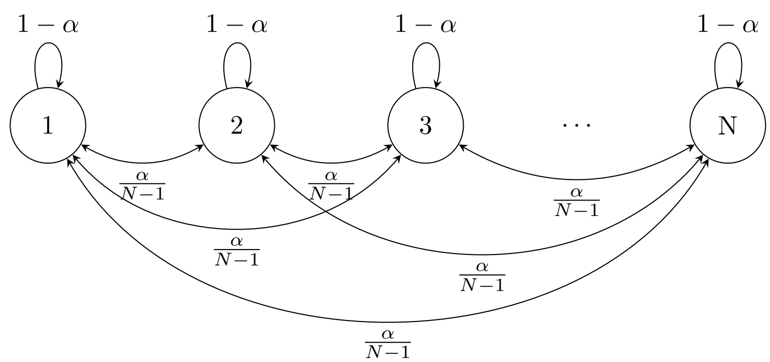
1. Transition Matrix, Graph, and Analysis

Transition Matrix P **:** The state space is $S = \{1, 2, \ldots, N\}$. The transition probabilities are given by :

$$P_{ij} = \begin{cases} 1-\alpha & \text{if } i = j \\ \frac{\alpha}{N-1} & \text{if } i \neq j \end{cases}$$

The matrix P is an $N \times N$ matrix where diagonal elements are $1-\alpha$ and all off-diagonal elements are $\frac{\alpha}{N-1}$.

$$P = \begin{bmatrix} 1-\alpha & \frac{\alpha}{N-1} & \frac{\alpha}{N-1} & \cdots & \frac{\alpha}{N-1} \\ \frac{\alpha}{N-1} & 1-\alpha & \frac{\alpha}{N-1} & \cdots & \frac{\alpha}{N-1} \\ \vdots & & \ddots & & \vdots \\ \frac{\alpha}{N-1} & \frac{\alpha}{N-1} & \frac{\alpha}{N-1} & \ddots & 1-\alpha \end{bmatrix} \quad (1.5\ pts)$$



(1.5 pts)

Transition Graph :

The graph consists of N nodes. Each node has a self-loop with weight $1-\alpha$, and every node is connected to every other node with an edge of weight $\frac{\alpha}{N-1}$.

Analysis (0.5 pt) :

— **Irreducible :** The chain is irreducible because for any two states i, j, there is a direct path (probability > 0) between them.

— **Aperiodic :** Since $1-\alpha > 0$, the self-loops $P_{ii} > 0$ ensure the chain is aperiodic.

— **Ergodic :** Being irreducible, aperiodic, and finite, the chain is ergodic and converges to a unique stationary distribution.

2. Stationary Distribution π

Since the transition matrix is symmetric ($P_{ij} = P_{ji}$), the matrix is doubly stochastic (rows and columns both sum to 1). For any doubly stochastic and irreducible matrix, the stationary distribution is uniform :

$$\pi = \left(\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right) \quad (1\ pt)$$

Proof :

$$\sum_{i=1}^{N} \pi_i P_{ij} = \sum_{i=1}^{N} \frac{1}{N} P_{ij} = \frac{1}{N} \sum_{i=1}^{N} P_{ij}$$

Because the matrix is symmetric, $\sum_{i=1}^{N} P_{ij} = \sum_{j=1}^{N} P_{ij} = 1$. Thus, $\pi_j = 1/N$.

3. Mean Recurrence Time

The average number of steps to return to the starting strain is the mean recurrence time $E[T_i]$. According to the fundamental theorem of Markov chains :

$$E[T_i] = \frac{1}{\pi_i}$$

Substituting $\pi_i = \frac{1}{N}$:

$$E[T_i] = \frac{1}{1/N} = N \quad (0.5\ pt)$$

The virus will return to its original strain in N generations on average.

Solution of Exercise 5

The process $N(t)$ is a Poisson process with rate $\lambda = 1 + 2 = 3$.

a. We know that, N is stationary increments, thus :
Thus,

$$\begin{aligned}\mathbb{P}(N(1)=2, N(2)=5) &= \mathbb{P}(N(1)=2, N(2)-N(1)=3)\\ &= \mathbb{P}(N(1)=2)\mathbb{P}(N(2)-N(1)=3)\\ &= \left[\frac{e^{-3}3^2}{2!}\right]\cdot\left[\frac{e^{-3}3^3}{3!}\right] \approx 0.05. \quad (1.5\ pts)\end{aligned}$$

b. Compute $P(N_1(1) = 1 \mid N(1) = 2)$:

$$P(N_1(1) = 1 \mid N(1) = 2) = \frac{P(N_1(1) = 1, N(1) = 2)}{P(N(1) = 2)}.$$

Now,

$$P(N_1(1) = 1, N(1) = 2) = P(N_1(1) = 1, N_2(1) = 1),$$

and since $N_1(t)$ and $N_2(t)$ are independent :

$$P(N_1(1) = 1, N_2(1) = 1) = P(N_1(1) = 1)\, P(N_2(1) = 1).$$

Therefore,

$$P(N_1(1) = 1 \mid N(1) = 2) = \frac{P(N_1(1) = 1)\, P(N_2(1) = 1)}{P(N(1) = 2)}.$$

Compute each term :

$$\begin{aligned}P(N_1(1) = 1) &= e^{-1}\cdot 1,\\ P(N_2(1) = 1) &= e^{-2}\cdot 2,\\ P(N(1) = 2) &= \frac{e^{-3}3^2}{2!}.\end{aligned}$$

Thus,

$$P(N_1(1) = 1 \mid N(1) = 2) = \frac{\left[e^{-1}\cdot 2e^{-2}\right]}{\left[\frac{e^{-3}3^2}{2!}\right]} = \frac{4}{9}. \quad (1.5\ pts)$$