Chapter 2

Complement on matrices

2.1 Subordinate matrix norms

Let A be a square matrix, we see in chapter I the definition of matrix norms.

1. $\|A\|_1 = \sup_{v\in\mathcal{C}, v\neq 0} \frac{\|Av\|_1}{\|v\|_1} = \max_j \sum_{i=1}^n |a_{ij}|$.

2. $\|A\|_2 = \sup_{v\in\mathcal{C}, v\neq 0} \frac{\|Av\|_2}{\|v\|_2} = \sqrt{\rho(A^*A)} = \sqrt{\rho(AA^*)} = \|A^*\|_2$.

3. $\|A\|_\infty = \sup_{v\in\mathcal{C}, v\neq 0} \frac{\|Av\|_\infty}{\|v\|_\infty} = \max_i \sum_{j=1}^n |a_{ij}|$.

Remarks 2.1

1. *The norm $\|A\|_2$ is invariant for unitary translate,*

$$UU^* = I \Rightarrow \|A\|_2 = \|A\|_2 = \|AU\|_2 = \|UA\|_2 = \|U^*AU\|_2.$$

2. *If A is unitary or orthogonal (then normal) $\|A\|_2 = \sqrt{\rho(A^*A)} = \sqrt{\rho(I)} = 1$.*

3. *If A is normal $AA^* = A^*A \Rightarrow \|A\|_2 = \rho(A)$.*

4. *If A is Hermitian or symmetric (then normal) $\|A\|_2 = \rho(A)$.*

Definition 2.1 *Let A be a square matrix, we define the norm*

$$\|A\| = \sup_{v\in\mathcal{C}, v\neq 0} \frac{\|Av\|}{\|v\|} = \sup_{v\in\mathcal{C}, \|v\|\leq 1} \|Av\| = \sup_{v\in\mathcal{C}, \|v\|=1} \|Av\|.$$

This norm call matrix norm subordinate to the given vector norm (This is obviously a special case of the usual definition of the norm of a linear application).

Remark 2.1 *There are matrix norms that are not subordinate to any vector norm.*

Theorem 2.1

1. *Let A be any square matrix and $\|.\|$ be a subordinate norm matrix or not. Then*

$$\rho(A) \leq \|A\|.$$

2. *Given a matrix A and a number $\varepsilon > 0$, there exists at least one subordinate matrix norm, such that*

$$\|A\| \leq \rho(A) + \varepsilon.$$

Theorem 2.2

1. *Let $\|.\|$ be a subordinate norm matrix and B a matrix verifying $\|B\| < 1$. Then the matrix $(I+B)$ is invertible and*
$$(I+B)^{-1} \leq \frac{1}{1-\|B\|}$$
.

2. *If a matrix of the form $(I+B)$ is singular then necessarily $\|B\| > 1$, for any matrix norm subordinate or not.*

2.2 Sequences of vectors and matrices

2.2.1 Sequences of vectors

Let X be a vector space equipped with a norm $\|.\|$.
A sequence of elements $x_0, x_1, ..., x_n$ of a set X will be denoted $(x_n)_{n\geq 0}$ or simply (x_n).

Definition 2.2 *We say that a sequence x_n of elements of X converges to an element x, or that x is the limit of the sequence x_n, if*
$$\lim_{n\leftarrow\infty} \|x_n - x\| = 0,$$
and we write
$$x = \lim_{n\leftarrow\infty} x_n.$$

Remark 2.2

1. *If the space is finite-dimensional, the equivalence of norms shows that the convergence of a sequence is independent of the chosen norm.*

2. *The particular choice of the norm $\|.\|_\infty$ shows that the convergence of a sequence of vectors is equivalent to the convergence of the n-sequences (n the dimension of the space) of scalars formed by the components of the vectors.*

2.2.2 Sequences of matrices

The following result provides necessary and sufficient conditions for the particular sequence formed by the successive powers of a given square matrix to converge to the zero matrix. From these conditions will arise the fundamental criterion for the convergence of iterative methods for solving linear systems.

Theorem 2.3 *Let B be square matrix. The following conditions are equivalents*

1. $\lim_{k\leftarrow\infty} B^k = 0$.

2. $\lim_{k\leftarrow\infty} B^k v = 0$ *for all vector v.*

3. $\rho(B) < 1$.

4. $\|B\| < 1$, *for at least one subordinate matrix norm $\|.\|$.*

Theorem 2.4 *Let B be square matrix and $\|.\|$ be an arbitrary matrix norm. Then*
$$\lim_{k\leftarrow\infty} \|B^k\|^{\frac{1}{k}} = \rho(B).$$

2.2.3 Block matrix decomposition

In general, a block matrix is a matrix that is partitioned into smaller sub-matrices, called blocks

Definition 2.3 *A block matrix M is a matrix whose entries M_{ij} are matrices instead of being scalars*

Why use block matrices

Block matrices are incredibly useful for serval reasons:

1. Simplifying structure (eg. diagonal, triangular, sparse,....).
2. Computational efficiency.
3. Theoretical proofs.
4. Handing large systems.
5. ..

Properties and operations

In this section, we give some properties of block-defined matrices.

- **Addition**: If two matrices A and B are partitioned with same block structure, we can add the block by block. if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, then

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$$

- **Scalar multiplication**: Multiplying a block matrix by scalar λ multiplies every block,

$$\lambda A = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} \\ \lambda A_{21} & \lambda A_{22} \end{pmatrix}.$$

- **Multiplication**: The multiplication of two block matrices is performed as if the blocks were scalars, provided the partitions are compatible. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix, partition then as follows $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, then

$$AB = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}.$$

Example 2.1 *1. Let M, such that*

$$M = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{R}^{2\times 2}$.

2. Let N, such that

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} I_2 \\ B \end{pmatrix}.$$

where $B = (3, 3) \in \mathbb{R}^{2\times 2}$.

3. The block product calculation is written

$$MN = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_2 \\ B \end{pmatrix} = \begin{pmatrix} AI_2 & 0B \\ 0I_1 & 1B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Schur complement

Proposition 2.1 *Consider the block matrix M such that*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

where the dimensions of blocks $A, B, C,$ and D are $n \times n, n \times m, m \times n, m \times m$.
Suppose that A is invertible. then

$$M = \begin{pmatrix} I_n & 0 \\ CA^{-1} & I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I_n & A^{-1}B \\ 0 & I_m \end{pmatrix}.$$

Moreover $\det(M) = \det(A)\det(D - CA^{-1}B)$.

Remark 2.3 *The matrix $D - CA^{-1}B$ is called the Schur complement of the matrix A in M.*

TD Series: Linear Systems - Sensitivities

Exercise 01: Let the following matrices:

$$A = \begin{pmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 6 & -1 \\ 3 & 1 & 0 \\ 2 & 4 & -7 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 7 & 3 \\ 4 & -2 & -2 \\ -2 & -1 & 1 \end{pmatrix}$$

1. Compute the condition number of matrix A and matrix B ($\text{cond}_1(\cdot), \text{cond}_2(\cdot)$) and $\text{cond}_\infty(\cdot)$.
2. What can be said about these matrices?

Exercise 02: We want to study a linear system $Ax = b$, where

$$A = \begin{pmatrix} 400 & -201 \\ -800 & 401 \end{pmatrix}, \quad b = \begin{pmatrix} 200 \\ -200 \end{pmatrix}$$

1. Compute the condition number of matrix A for the norms $\|\cdot\|_1, \|\cdot\|_\infty$. What do you think?
2. Compute the exact solution of the system $Ax = b$.
3. We perturb the matrix A, i.e.,
$$A = \begin{pmatrix} 401 & -201 \\ -800 & 401 \end{pmatrix}$$
   (a) Compute the solution of the system: $(A + \delta A)\bar{x} = b$.
4. Compute the residual vector $r = b - A\bar{x}$, then compute $\|r\|_\infty$
5. Compute the energy norm $\|\cdot\|_A$ for the vectors $\bar{x}$ and r.

Exercise 03

The goal of this exercise is to solve the linear system by the QR factorization method where matrices Q is orthogonal and R is upper triangular using Householder transformation (Method). Let the following linear system $AX = b$, such that

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$$

1. Determine matrices H_1 and H_2
2. Determine matrix R.
3. Determine matrix Q.
4. Determine the solution of the system.

Exercise 04

We want to solve the following linear system $AX = b$, such that

$$A = \begin{pmatrix} 1001 & 1000 \\ 1000 & 1001 \end{pmatrix}, \quad b = \begin{pmatrix} 2001 \\ 2001 \end{pmatrix}$$

1. Compute the condition number of matrix A, $k(A)$.
2. Find the solution of this system.
3. We perturb the right-hand side of the system $\delta b = (0, 1)$. Find the solution of this perturbed system.

Exercise 05
Let the following linear system $AX = b$, such that A =

1. Determine matrices H_1 and H_2
2. Determine matrix R.
3. Determine matrix Q.
4. Determine the solution of the system.

Exercise 06
In order to compare the Jacobi, Gauss-Seidel and SOR methods for solving the linear system $Ax = b$. Given the following linear system:

1. Compute the first 7 iterations for the Jacobi method taking an initial value $x^{(0)} = (1, 1, 1)^t$
2. Same question for the Gauss-Seidel method
3. Same question for the SOR method with $\omega = 1.25$
4. What can be concluded?