Chapter 1

Review of matrices

1.1 Notions of vectors and matrices

In this part we give definitions and some concepts about vectors and matrices.
Let $\mathbb{R}^n, \mathbb{R}^p$ be two vector spaces of finite dimensions.

1.1.1 Vectors

Definition 1.1 *: An element of the vector space $\mathbb{R}^n$ over $\mathbb{R}$ is called a vector v. It is an ordered set of elements of $\mathbb{R}$, as we denote $v = (v_1, v_2, \dots, v_n)^t$ or*

$$v = \begin{pmatrix} v_1 \\ . \\ . \\ v_n \end{pmatrix}$$

Remark 1.1 *: Since $\mathbb{R}^n$ is a vector space on $\mathbb{R}$, it has the classic properties of a vector space: Let $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then*

1. $(\alpha\beta)x = \alpha(\beta)x$.
2. $(\alpha + \beta)x = \alpha x + \beta x$.
3. $\alpha(x + y) = \alpha x + \alpha y$.
4. $1x = x$.

Definition 1.2
A dot product on E is an application of $E \times E$ on $\mathbb{R}$:

$$< .,. >: E \times E \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto < x, y >_E$$

with the following properties:

1. $< x, x > \geq 0$.
2. $< x, x >= 0 \Rightarrow x = 0$.
3. $< x, y >=< y, x >$.
4. $< x, y + z >=< x, y > + < x, z >$.
5. $< x, \alpha y >= \alpha < x, y >, \ \forall \alpha \in \mathbb{R}$.

Orthogonality of vectors and subspaces

- The angle θ between two vectors u and v given by $\cos(\theta) = \frac{u^t v}{\|u\|\|v\|}$.
- Two vectors u and v are orthogonal if $\theta = 90°$ that is $u^t v = 0$.
- The symbol $\perp$ is used to denote orthogonality.

1.1.2 Matrices

Basic concepts

A collection of n vectors in $\mathbb{R}^n$ arranged in a rectangular array of n rows and m columns is called matrix.
A matrix A, there for has the form

$$\begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{pmatrix}$$

Definition 1.3 *(Transpose and conjugate matrices)*
Let A is matrix such that $A \in \mathbb{C}^{n\times n}$.

1. *We say A^t is the transpose of A.*
2. *We say A^* is the conjugate of A and $A^* = \overline{(A^t)}$.*

Definition 1.4 *(Symmetric and Hermitian matrices)*
Let A is matrix such that $A \in \mathbb{C}^{n\times n}$.

1. *A is symmetric if $A = A^t$.*
2. *A is skew symmetric if $A = -A^t$.*
3. *A is Hermitian if $A^* = A$.*
4. *A is skew Hermitian if $A^* = -A$.*
5. *A is normal if $AA^* = A^*A$.*

Inverse matrix

We want to determine an inverse with respect to matrix multiplication.

Definition 1.5
A matrix $A \in \mathbb{C}^{n\times n}$ is nonsingular (or invertible) if A has an inverse, that is if there is a matrix A^{-1} so that

$$AA^{-1} = I = A^{-1}A.$$

If A does not have an inverse it is singular.

Remark 1.2 *The inverse is unique.*

Properties of inverse matrix

Let A and B are two invertible matrices, then we have.

1. $(AB)^{-1} = B^{-1}A^{-1}$.
2. $(A^t)^{-1} = (A^{-1})^t$.
3. $(A^*)^{-1} = (A^{-1})^*$.

Definition 1.6 *(Unitary and orthogonal matrices)*
Let A is matrix such that $A \in \mathbb{C}^{n\times n}$.

1. *A is unitary if $AA^* = A^*A = I$.*
2. *A is orthogonal if $AA^t = A^tA = I$.*

Remarks 1.1 *: Let A is matrix such that $A \in \mathbb{C}^{n\times n}$.*

1. *Diagonal matrices : $a_{ij} = 0$ for $j \neq i$, $A = diag(a_{11}, a_{22}, ..., a_{nn})$.*
2. *Upper triangular matrices : $a_{ij} = 0$ for $i \geq j$.*
3. *Lower triangular matrices : $a_{ij} = 0$ for $i \leq j$*
4. *Upper bi-diagonal : $a_{ij} = 0$ for $j \neq i$ or $j \neq i+1$*
5. *Lower bi-diagonal : $a_{ij} = 0$ for $j \neq i$ or $j \neq i-1$*
6. *Tri-diagonal : $a_{ij} = 0$ for any pair i, j such that $|j-i| \geq 1$, $A = tridiag$*

Trace and determinant of matrix

Definition 1.7 *Let A is matrix such that $A \in \mathbb{C}^{n\times n}$.*

1. *The trace of matrix A is defined by $tr(A) = \sum_{i=1}^{n} a_{ii}$.*
2. *The determinant of matrix A is defined by*

$$det(A) = \sum_{\sigma \in S_n} \varepsilon_\sigma a_{\sigma(1)1} a_{\sigma(2)2} ... a_{\sigma(n)n},$$

   where σ_ε denotes the signature of the permutation.

Proposition 1.1 *Let A, B be two square matrices.*

1. *$tr(A) = \sum_{i=1}^{n} \lambda_i(A)$*
2. *$det(A) = \prod_{i=1}^{n} \lambda_i(A)$*
3. *$tr(AB) = tr(BA)$*
4. *$det(AB) = det(BA) = det(A).det(B)$*

1.2 Vector norms and matrix norms

1.2.1 Vector norms

Let V an vector space over the field $\mathbb{K}$.
A Norm on V is an application : $\|.\| : V \to \mathbb{R}$ that verifies the following properties :

1. $\|v\| = 0 \Leftrightarrow v = 0$ and $\|v\| \geq 0$ for all $v \in V$.
2. $\|\alpha v\| = |\alpha|\|v\|$, for all $\alpha \in \mathbb{K}$ and $v \in V$.
3. $\|u + v\| \leq \|u\| + \|v\|$, for all $u, v \in V$.

Definition 1.8 *Let V be a finite-dimensional vector space.*

1. $\|v\|_1 = \sum_{i=1}^{n} |v_i| = |v_1| + |v_2| + ... + |v_n|$.
2. $\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2\right)^2 = \left(|v_1|^2 + |v_2|^2 + ... + |v_n|^2\right)$.
3. $\|v\|_\infty = \max_i |v_i|$.

Theorem 1.1 *Let V be a finite-dimensional vector space, for all real number $p \geq 1$ the application $\|.\|_p$ defined by*

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}$$

is a norm.

1.2.2 Matrix norms

Let A is matrix such that $A \in \mathbb{C}^{n\times n}$, we define the following special set of norms

$$\|A\|_{pq} = \max_{x\in\mathbb{C}^n, x\neq 0} \frac{\|Ax\|_p}{\|x\|_q}$$

Definition 1.9 *A matrix norm is an application $\|.\| : \mathcal{A}_n \leftarrow \mathbb{R}$ that verifies the following properties*

1. $\|A\| = 0 \Leftrightarrow A = 0$ *and* $\|A\| \geq 0$ *for all* $A \in \mathcal{A}_n$.
2. $\|\alpha A\| = |\alpha|\|A\|$, *for all* $\alpha \in \mathbb{K}$ *and* $A \in \mathcal{A}_n$.
3. $\|A + B\| \leq \|A\| + \|B\|$, *for all* $A, B \in \mathcal{A}_n$.
4. $\|AB\| \leq \|A\|.\|B\|$, *for all* $A, B \in \mathcal{A}_n$.

Remarks 1.2

1. *A matrix then being considered as a vector with n^2 components.*
2. *Given a vector norm $\|.\|$ on $\mathbb{C}$. The application $\|.\| : \mathcal{A}_n \leftarrow \mathbb{R}$ defined by*

$$\|A\| = \sup_{v\in\mathbb{C}, v\neq 0} \frac{\|Av\|}{\|v\|} = \sup_{v\in\mathbb{C}, \|v\|\leq 1} \|Av\| = \sup_{v\in\mathbb{C}, \|v\|=1} \|Av\|$$

   is a matrix norm

Theorem 1.2 *Let A be a square matrix.*

1. $\|A\|_1 = \sup_{v\in\mathcal{C}, v\neq 0} \frac{\|Av\|_1}{\|v\|_1} = \max_j \sum_{i=1}^{n} |a_{ij}|$.
2. $\|A\|_2 = \sup_{v\in\mathcal{C}, v\neq 0} \frac{\|Av\|_2}{\|v\|_2} = \sqrt{\rho(A^*A)} = \sqrt{\rho(AA^*)} = \|A^*\|_2$.
3. $\|A\|_\infty = \sup_{v\in\mathcal{C}, v\neq 0} \frac{\|Av\|_\infty}{\|v\|_\infty} = \max_i \sum_{j=1}^{n} |a_{ij}|$.
4. $\|A\|_F = \left(\sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$, $\|A\|_F = \left[tr(A^H A)\right]^{\frac{1}{2}} = \left[tr(AA^H)\right]^{\frac{1}{2}}$, *Frobenius norm*

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Exercise 01

1. Let the vectors $v_i, i = 1, 2, 3$ such that $v_1 = (1, 0, 0)^t, v_2 = (1, 3, 5)^t$ and $v_3 = (11, 5, 10)^t$
   (a) Compute the norms $\|v_i\|_1, \|v_i\|_2$ and $\|v_i\|_\infty$
   (b) Compute the dot product of v_1 and v_2, v_1 and v_3, and v_2 and v_3.
2. Consider the following matrices:

$$A = \begin{pmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 6 & -1 \\ 3 & 1 & 0 \\ 2 & 4 & -7 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 7 & 3 \\ 4 & -2 & -2 \\ -2 & -1 & 1 \end{pmatrix}$$

   (a) Compute the norms $\|A\|_1, \|A\|_2$ and $\|A\|_\infty$, and similarly for matrices B and C.
   (b) Compute the Frobenius norm $\|I\|_F$

Exercise 02
Let A and B two matrices. Show that:

1. Diagonal matrices are symmetric.
2. Diagonal matrices are Hermitian.
3. Diagonal matrices commute, i.e., if A and B are two diagonal matrices, then AB is diagonal and $AB = BA$.
4. Let D be a diagonal matrix. If $D = (I + A)^{-1}A$ then A is diagonal.
5. Let $A \in \mathbb{C}^{m\times n}$ be a matrix. Prove that $A + A^t$ is symmetric and $A + A^*$ is Hermitian.

Exercise 03
Let matrix Z such that

$$Z = \begin{pmatrix} 1+2i & 3-5i & 7 \\ 8+i & 6i & 2-i \end{pmatrix}.$$

And let $U = A + iB$ and $U^* = A - iB$ where A and B are real matrices.

1. Compute Z^t, Z^* and Z^{**}.
2. Show that $tr(U^*) \neq tr(U)$ in general.
3. Are the following matrices: Unitary, Hermitian:

$$A = \frac{1}{2}\begin{pmatrix} i & -\sqrt{3} \\ \sqrt{3} & -i \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix}$$

4. Are the following matrices: Orthogonal, Normal:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 04

Let Z be a matrix defined in the form

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

1. Case: Matrices A and D are not of the same size.
   (a) Show that $Tr(Z) = tr(A) + tr(D)$
2. Case: Matrices A and D are square matrices.
   (a) Show that, if Z is symmetric then A and D are also symmetric.
   (b) Show that, if Z is diagonal then A and D are also diagonal.
   (c) Show that, if Z is upper triangular then A and D are also upper triangular.