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2024/2025
1^{st}YEAR MASTER
SEMESTER 02

Probability and Stochastic Processes Exam Correction

Exercice 0.1. (5 points) The given density function is:

$$f_X(x) = \begin{cases} k\frac{8000}{(x+50)^3}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Part 1: Determine the value of k, compute $E[X]$ and Var(X)

1. To find k, we use the property of a probability density function:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

Substituting $f_X(x)$:

$$\int_0^{\infty} k\frac{8000}{(x+50)^3}\, dx = 1.$$

Let $u = x + 50$, so $du = dx$. When $x = 0$, $u = 50$, and as $x \to \infty$, $u \to \infty$. The integral becomes:

$$\int_{50}^{\infty} k\frac{8000}{u^3}\, du = 1.$$

Evaluating the integral:

$$k \cdot 8000 \int_{50}^{\infty} u^{-3}\, du = k \cdot 8000 \left[-\frac{1}{2u^2}\right]_{50}^{\infty} = -0 + k \cdot 8000 \left(\frac{1}{2 \cdot 50^2}\right) = 1$$

Simplify:

$$k \cdot 8000 \cdot \frac{1}{5000} = 1 \implies k = \frac{5}{8}. \quad \text{.........} \ 1$$

2. To find $E[X]$, we compute:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{0} x.0\, dx + \int_0^{\infty} x \cdot \frac{5}{8} \cdot \frac{8000}{(x+50)^3}\, dx.$$

Using $u = x + 50$, $x = u - 50$, and $du = dx$, the integral becomes:

$$E[X] = 5000 \int_{50}^{\infty} \frac{u-50}{u^3}\, du = 5000 \left(\int_{50}^{\infty} u^{-2}\, du - 50 \int_{50}^{\infty} u^{-3}\, du\right).$$

Evaluating the integral:

$$E[X] = 5000 \left(\left[-\frac{1}{u}\right]_{50}^{\infty} - \left[-\frac{1}{2u^2}\right]_{50}^{\infty}\right) = 5000 \left(\frac{1}{50} - \frac{1}{2 \cdot 50}\right) = 50. \quad \text{....} \ 1$$

3. To find Var(X), compute:

$$\text{Var}(X) = E[X^2] - (E[X])^2. \quad \text{.....} \ 1$$

Similarly, calculate $E[X^2]$ and substitute.

Part 2: Probabilities

1.

$$P(X \geq 200) = \int_{200}^{\infty} f_X(x)\,dx = \frac{5}{8} \cdot 8000 \int_{200}^{\infty} \frac{1}{(x+50)^3}\,dx.$$

Change of variable: let $u = x + 50$, so $du = dx$. The limits of integration change as follows:

$$x = 200 \implies u = 250, \quad x = \infty \implies u = \infty.$$

Thus, the integral becomes:

$$P(X \geq 200) = 5000 \int_{250}^{\infty} \frac{1}{u^3}\,du = 5000 \left[\frac{-1}{2u^2}\right]_{250}^{\infty} = 0 - 5000\left(\frac{-1}{2 \cdot 250^2}\right) = \frac{1}{25} = 0,4.$$ 1

Substitute $f_X(x)$ and compute similarly as above.

2. For $P(80 \leq X \leq 120)$:

$$P(80 \leq X \leq 120) = \int_{80}^{120} f_X(x)\,dx = 5000 \int_{130}^{170} \frac{1}{u^3}\,du,$$

where $u = x + 50$, so the limits of integration become 130 and 170.

$$P(80 \leq X \leq 120) = 5000\left(\frac{-1}{2 \cdot 170^2} - \frac{-1}{2 \cdot 130^2}\right) = 0,0614,$$ 2

Exercice 0.2. (5 points)

$$X \sim \text{Bernoulli}(p), \quad P(X=1) = p, \quad P(X=0) = 1-p,$$ 0,5

$$Y \sim \text{Bernoulli}(q), \quad P(Y=1) = q, \quad P(Y=0) = 1-q.$$ 0,5

Since $Z = X + Y$, the possible values of Z are:

$$Z \in \{0, 1, 2\}.$$ 1

1. **For $Z = 0$:**

$$P(Z=0) = P(X=0 \cap Y=0) = P(X=0) \cdot P(Y=0) = (1-p)(1-q).$$ 0,75

2. **For $Z = 1$:** $Z = 1$ occurs in two cases: $X = 1, Y = 0$ or $X = 0, Y = 1$. Using independence:

$$P(Z=1) = P(X=1 \cap Y=0) + P(X=0 \cap Y=1),$$

$$P(Z=1) = P(X=1)P(Y=0) + P(X=0)P(Y=1) = p(1-q) + (1-p)q,$$

$$P(Z=1) = p + q - 2pq.$$ 0,75

3. **For $Z = 2$:**

$$P(Z=2) = P(X=1 \cap Y=1) = P(X=1) \cdot P(Y=1) = p \cdot q.$$

The distribution of Z is given by:

$$P(Z=0) = (1-p)(1-q),$$
$$P(Z=1) = p + q - 2pq,$$
$$P(Z=2) = pq.$$ 0,75

Thus, Z follows a **Binomial distribution** with parameters $n = 2$ and $p' = p + q - pq$:

$$Z \sim \text{Binomial}(n = 2, p' = p + q - pq).$$ 0,75

Exercice 0.3. (5 points) (a) The events $X_8 = 6$ occur with $r = 7$ positive (+1) steps and $l = 1$ negative (-1) steps, but they could be in any order. Hence, 1

$$P(X_8 = 6) = C_8^7 (0.6)^7 (0.4)^1 = 8 \times 0.6^7 \times 0.4 = 0.0896.$$ 1,5

(b) For $X_8 = -4$, we have $r = 2$ positive (+1) steps and $l = 6$ negative (-1) steps. Thus: 1

$$P(X_8 = -4) = C_8^2 (0.6)^2 (0.4)^6 = \frac{8!}{2!6!} \times 0.6^2 \times 0.4^6 = 0.0413.$$ 1,5

Exercice 0.4. (5 points)

1. Filling in the Transition Matrix and Finding α

The transition matrix is:

$$P = \begin{pmatrix} ? & \frac{4}{3}\alpha & \frac{2}{3}\alpha \\ \alpha & ? & \alpha \\ 0 & \frac{8}{3}\alpha & ? \end{pmatrix}.$$

For P to be a valid transition matrix, the sum of each row must equal 1.

Row 1:

$$? + \frac{4}{3}\alpha + \frac{2}{3}\alpha = 1 \implies ? = 1 - \left(\frac{4}{3}\alpha + \frac{2}{3}\alpha\right) = 1 - 2\alpha.$$

Row 2:

$$\alpha + ? + \alpha = 1 \implies ? = 1 - 2\alpha.$$

Row 3:

$$0 + \frac{8}{3}\alpha + ? = 1 \implies ? = 1 - \frac{8}{3}\alpha.$$

The matrix becomes:

$$P = \begin{pmatrix} 1-2\alpha & \frac{4}{3}\alpha & \frac{2}{3}\alpha \\ \alpha & 1-2\alpha & \alpha \\ 0 & \frac{8}{3}\alpha & 1-\frac{8}{3}\alpha \end{pmatrix}.$$ 1

For all entries to be non-negative, α must satisfy:

$$1 - 2\alpha \geq 0, \quad 1 - \frac{8}{3}\alpha \geq 0, \quad \text{and } \alpha \geq 0.$$

Solving these inequalities gives:

$$0 \leq \alpha \leq \frac{3}{8}.$$ 1

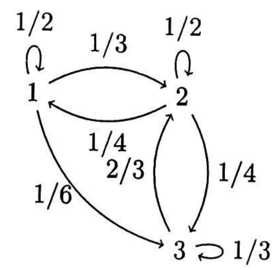
2. For $\alpha = \frac{1}{4}$:

(a) Drawing the Markov Chain

Using $\alpha = \frac{1}{4}$, the transition matrix becomes:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

The Markov chain diagram is as follows:



1

(b) Steady-State Probabilities

Let the steady-state probabilities be $\pi = (\pi_1, \pi_2, \pi_3)$, satisfying $\pi P = \pi$ and $\pi_1 + \pi_2 + \pi_3 = 1$. Solving the system:

$$\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2,$$

$$\pi_2 = \frac{1}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{3}\pi_3,$$

$$\pi_3 = \frac{1}{6}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{3}\pi_3,$$

$$\pi_1 + \pi_2 + \pi_3 = 1.$$

The solution is:

$$\pi_1 = \frac{6}{25}, \quad \pi_2 = \frac{12}{25}, \quad \pi_3 = \frac{7}{25}.$$ 1

(c) Computing $P(X_1 = 3)$, $P(X_2 = 2)$, $P(X_3 = 1)$

Using $P(X_0 = 1) = P(X_0 = 2) = \frac{1}{4}$, $P(X_0 = 3) = \frac{1}{2}$:

1

$$P(X_1 = 3) = \frac{1}{4}P(1 \to 3) + \frac{1}{4}P(2 \to 3) + \frac{1}{2}P(3 \to 3).$$

Substitute values from P:

$$P(X_1 = 3) = \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} = \frac{11}{48}.$$

Similarly, compute $P(X_2 = 2)$ and $P(X_3 = 1)$ iteratively using matrix multiplication.