Stochastic process

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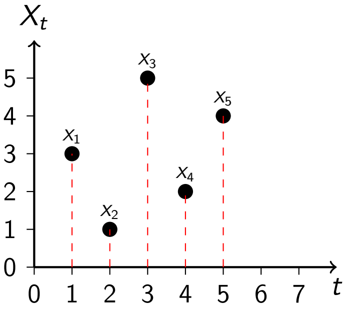
Stochastic process

- A stochastic process is a collection of random variables $\{X_t\}_{t \in T}$ indexed by time $t \in T$.
- If the time variable t takes positive integer values $T = I\!N$ (or $T = I\!N^*$), the stochastic process is said to be discrete.

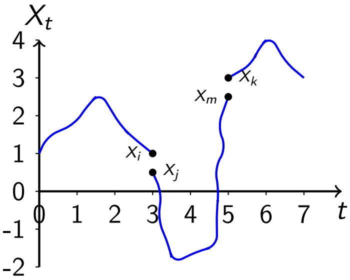
$$X_0, X_1, X_3, X_4, \ldots \tag{1}$$



- if the time variable takes postivie real values ($T = [0, \infty)$) the stochastic process is said to be continuous.

$$(X_t)_{t \geq 0} \tag{2}$$

- When T is a singleton (say $T = \{k\}$), the process $\{X_t\}_{t \in T} \equiv X_k$ is really just a single random variable
- When T is finite (e.g., $T = \{1, 2, ..., n\}$), we get a random vector



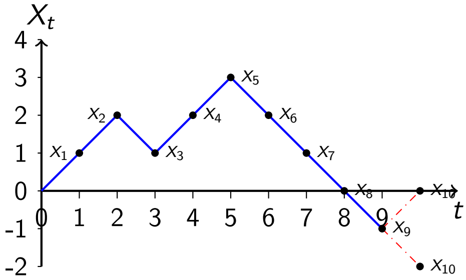
In this lecture, we will focus on discrete-time stochastic processes. One of the most important ones we will discuss is the simple random walk.

Simple random walk

Let $Y_1; Y_2 \dots$ be independent and identically distributed (i.i.d.) random variables such that $Y_i = \pm 1$ with equal probability (each with probability $\frac{1}{2}$). Then, we define for each time t as follows; $X_0 = 0$ and

$$X_k = \sum_{i=!}^{k} Y_i = Y_1 + Y_2 + \cdots + Y_k; \ \forall k \geq 1 \tag{3}$$

Then, the sequence of random variables $X_0, X_1, X_2, \dots$ is called a one-dimensional simple random walk.



- By the central limit theorem, we know that for larg eenough n;thed istribution of $\frac{1}{\sqrt{n}}X_n$ converges to the normal distribution with mean 0 and variance 1. This already tells us some information about the random walk. We state some further properties of the random walk.

Proposition

- $E[X_k] = 0$ for all k.
- (Independent increment) For all $0 = k_0 \leq k_1 \leq \cdots \leq k_r$, the random variables $X_{k_{i+1}} - X_{k_i}$ for $0 \leq i \leq r-1$ are mutually independent.
- (Stationary) For all $h \geq 1$ and $k \geq 0$, the distribution of $X_{k+h} - X_k$ is the same as the distributionof X_h.

Proof

- Using linearity of expectation:

$$E[X_k] = E\left[\sum_{i=1}^{k} Y_i\right] = \sum_{i=1}^{k} E[Y_i] = \sum_{i=1}^{k} 0 = 0$$

- Consider the increments:

$$X_{k_{i+1}} - X_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} Y_j$$

  Each increment is a sum over a disjoint subset of the Y_j's. Since the Y_j are independent, the sums are also independent.

- We compute:

$$X_{k+h} - X_k = \sum_{i=k+1}^{k+h} Y_i$$

  This is a sum of h i.i.d. random variables, and hence its distribution does not depend on k, only on h. Therefore:

$$X_{k+h} - X_k \overset{d}{=} \sum_{i=1}^{h} Y_i = X_h$$

The exact probability distribution of a random walk

As before, we assume that the walk is such that $X_0 = 0$, with steps to the right or left occurring with probabilities p and $q = 1 - p$, respectively. The probability distribution of the random variable X_n, the position after n steps, is a more difficult problem. The position X_n, after n steps, can be written as:

$$X_n = R_n - L_n,$$

where R_n is the random variable representing the number of right (positive) steps $(+1)$ and L_n is the random variable representing the number of left (negative) steps (-1). Furthermore,

$$N = R_n + L_n,$$

where N is the random variable of the total number of steps. Hence,

$$R_n = \frac{1}{2}(N + X_n).$$

Now, let $v_{n,x}$ be the probability that the walk is at position x after n steps. Thus,

$$v_{n,x} = P(X_n = x).$$

The type of distribution can be deduced by the following combinatorial argument. To reach position x after $n \geq |x|$ steps requires $r = \frac{1}{2}(n+x)$ $(+1)$ steps (and consequently $l = n - r = \frac{1}{2}(n-x)$ (-1) steps). Right (r) and left (l) steps must be integers, so it is implicit that if x is an odd (even) integer, then n must also be odd (even). We now ask: in how many ways can $r = \frac{1}{2}(n+x)$ steps be chosen from n?
The answer is:

$$h_{n,x} = \frac{n!}{r!l!} = \frac{n!}{r!(n-r)!} = C_n^r.$$

The $r = \frac{1}{2}(n+x)$ steps occur with probability p^r, and the $l = \frac{1}{2}(n-x)$ steps occur with probability q^l.

Hence, the probability that the walk is at position x after n steps is:

$$v_{n,x} = h_{n,x} p^r q^l = C_n^r \binom{n}{r} p^r q^l = C_n^{\frac{1}{2}(n+x)} p^{\frac{1}{2}(n+x)} q^{\frac{1}{2}(n-x)}. \quad (2)$$

We observe that (2) defines a binomial distribution with index n and probability p.

Example

Find the probability that a random walk of 8 steps with probability $p = 0.6$ ends at:

(a) position $x = 6$,

(b) position $x = -4$.

(a) The events $X_8 = 6$ occur with $r = 7$ positive $(+1)$ steps and $l = 1$ negative (-1) steps, but they could be in any order. Hence, by equation (2):

$$P(X_8 = 6) = C_8^7 (0.6)^7 (0.4)^1 = 8 \times 0.6^7 \times 0.4 = 0.0896.$$

(b) For $X_8 = -4$, we have $r = 2$ positive $(+1)$ steps and $l = 6$ negative (-1) steps. Thus:

$$P(X_8 = -4) = C_8^2 (0.6)^2 (0.4)^6 = \frac{8!}{2!6!} \times 0.6^2 \times 0.4^6 = 0.0413.$$

Markov Chain

One important property of the simple random walk is that the effect of the past on the future is summarized only by the current state, rather than the whole history. In other words, the distribution of X_{k+1} depended only on the value of X_k, not on the whole set of values of $X_0; X_1; \ldots; X_k$. A stochastic proces swith such property is called a Markov chain. More formally let $X_0; X_1; \ldots$ be a discrete-time stochastic process where each X_i takes value in some discrete set S (note that this is not the case in the simple random walk). The set S is called the state space. We say that the stochastic process has the Markov property if

$$P(X_{n+1} = i/X_n; X_{n-1}; \ldots; X_0) = P(X_{n+1} = i/X_n), \qquad (4)$$

for all $n \geq 0$and $i \in S$. We will discuss the case when S is a finite set. In this case, we let S=[m] for some positive integer m. A stochastic process with the Markov property is called a Markov chain.

Note that a finite Markov chain can be described in terms of the transition probabilities

$$p_{ij} = P(X_{n+1} = j / X_n = i) \quad i; j \in S \tag{5}$$

One can easily see that

$$\sum_{i \in S} p_{ij} = 1, \quad i \in S \tag{6}$$

All the elements of a Markov chain model can be encoded in a transition probability matrix

$$P_{ij} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{d1} & p_{d2} & \cdots & p_{dn} \end{bmatrix}$$

Note that the sum of each column is equal to one.

Example

- A machine can be either working or broken on a given day. If it is working, it will break down in the next day with probability 0.01, and will continue working with probability 0.99. If it breaks down on a given day, it will be repaired and be working in the next day with probability 0.8, and will continue to be broken down with probability 0.2. We can model this machine by a Markov chain with two states: working, and broken down. The transition probability matrix is given by

$$P = \begin{array}{c c} & \begin{array}{cc} W & B \end{array} \\ \begin{array}{c} W \\ B \end{array} & \begin{pmatrix} 0.99 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

- A simple random walk is an example of a Markov chain. However, there is no transition probability matrix associated with the simple random walk since the sample space is of infinite cardinatlity.

Let $r_{ij}(n) = P(X_n = j / X_0 = i)$ be the $n - th$ step transition probabilities. These probabilities satisfy the recurrence relation

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$

where $r_{ij}(1) = p_{ij}$. Hence the n-step transition probability matrix can easily be shown to be P^n.
A stationary distribution of a Markov chain is a probability distribution over the state space S (where $P(X_0 = j) = \pi_j$) such that

$$\pi_j(n) = \sum_{k=1}^{m} \pi_k p_{kj} \quad \forall j \in S$$

Example

Let S $= \mathbb{Z}_n$ and $X_0 = 0$. Consider the Markov chain $X_0, X_1, X_2, \ldots$ such that $X_{n+1} = X_n + 1$ with proability $\frac{1}{2}$ and $X_{n+1} = X_n - 1$ with probability $\frac{1}{2}$.

- Then the stationary distribution of this Markov chain is $\pi_i = \frac{1}{n}$ for all i Note that the vector $(\pi_1, \pi_2, \ldots, \pi_{;})$ is an eigenvector of P with eigenvalue 1. Hence the following theorem can be deduced from the Perron-Frobenius theorem. Theorem If $p_{ij} > 0$ for all $i, j \in S$, then there exists a unique stationary distribuion of the system. Moreover,

$$\lim_{n \to +\infty} r_{ij}(n) = \pi_j \quad \forall i, j \in S$$

A corresponding theorem is not true if we consider infinite state spaces.

Martingale

Definition

A discrete-time stochastic process $\{X_0, X_1, \dots\}$ is a martingale if

$$X_t = E[X_{t+1}/\mathcal{F}_t]$$

for all $t \geq 0$, where $\mathcal{F}_t = X_0, \dots, X_t$ (hence we are conditioning on the initial segment of the process).

This says that our expectated gain in the process is zero at all times. We can also view this definition as a Mathematical formalization of a game of chance being fair.

Proposition

For all $t \geq s$, we have $X_s = E[X_t/\mathcal{F}_s]$

Proof

This easily follows from induction.

Example

1. Random walk is amartingale.
2. Let $Y_1; Y_2; \dots$ be i.i.d. random variables such that $Y_i = 2$ with probability $\frac{1}{2}$ and $Y_i = \frac{1}{3}$ with probability $\frac{2}{3}$ description

Optional stopping theorem

Definition

(Stopping time) Given a stochastic process $\{X_0; X_1; \dots\}$, a non-negative integer-valued random variable τ is called a stopping time if for every integer $k \geq 0$, the event $\tau \leq k$ depends only on the events $X_0; X_1; \dots X_k$

Example

- In the coin toss game, consider a gambler who bets 1$ all the time. Let τ be the r^{st} time at which the balance of the gambler becomes $100. Then τ is a stopping time.

- Consider the same gambler as in (i). Let τ be the time of the first peak (local maximum) of the balance of the gambler. Then τ is not a stopping time.

Theorem

Suppose that $\{X_0; X_1; X_2 \dots\}$ *is a martingale sequence and* τ *is a stopping time such that* $\tau \leq T$ *for some constant* T*. Then* $E[X_\tau] = E[X_0]$*.*