Pair of Random Variables

GASMI Abdelkader

April 19, 2025

Pair of Discrete Real Random Variables

Let X and Y be two discrete real random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, such that:

$$X(\Omega) = \{x_i : i \in I\} \quad \text{and} \quad Y(\Omega) = \{y_j : j \in J\}$$

Distributions Associated with a Pair (X, Y)

Joint Distribution The joint probability distribution of the pair (X, Y) is given by the probability:

$$p_{ij} = P(X = x_i \text{ and } Y = y_j)$$

Marginal Distributions The marginal probability distributions are given by:

$$P_{\text{marg}}(X = x_i) = \sum_{j \in J} p_{ij}$$

and

$$P_{\text{marg}}(Y = y_j) = \sum_{i \in I} p_{ij}$$

Conditional Distributions

Conditional Distribution of X given $Y = y_j$:

$$P(X = x_i \mid Y = y_j) = \frac{p_{ij}}{P_{\text{marg}}(Y = y_j)}$$

Conditional Distribution of Y given $X = x_i$:

$$P(Y = y_j \mid X = x_i) = \frac{p_{ij}}{P_{\text{marg}}(X = x_i)}$$

Remark

$$\sum_{(i,j) \in I \times J} p_{ij} = 1$$

Conditional Moments

Conditional Mathematical Expectation:

The conditional mathematical expectation of the random variable X given $Y = y_j$ is defined as:

$$\mathbb{E}[X \mid Y = y_j] = \sum_{i \in I} x_i P(X = x_i \mid Y = y_j) = \frac{\sum_{i \in I} x_i p_{ij}}{\sum_k p_{kj}} \tag{1.5}$$

Conditional Variance:

The conditional variance of the random variable X given $Y = y_j$ is defined as:

$$\mathrm{Var}[X \mid Y = y_j] = \mathbb{E}[(X - \mathbb{E}[X \mid Y = y_j])^2 \mid Y = y_j]$$

Joint Cumulative Distribution Function

Let (X, Y) be a pair of discrete real random variables. The joint cumulative distribution function is defined as:

$$\forall (x, y) \in \mathbb{R}^2,\ F_{X,Y}(x, y) = P(X \leq x \text{ and } Y \leq y)$$

Marginal Cumulative Distribution Functions

The marginal cumulative distribution function of X is defined as:

$$\forall x \in \mathbb{R},\ F_X(x) = \sum_{x_i \leq x} P(X = x_i)$$

Similarly, for Y:

$$\forall y \in \mathbb{R},\ F_Y(y) = \sum_{y_j \leq y} P(Y = y_j)$$

Covariance

The covariance of X and Y is given by:

$$\operatorname{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Definition: X and Y are said to be uncorrelated if $\operatorname{cov}(X, Y) = 0$.

Linear Correlation Coefficient

Let X and Y be random variables with finite covariance and non-zero variances. Their linear correlation coefficient is defined as:

$$\rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sigma_X \sigma_Y}$$

Independence

Two discrete random variables X and Y defined on the same probability space are independent if:

$$\forall (x_i, y_j) \in X(\omega) \times Y(\omega),\ P(X = x_i \text{ and } Y = y_j) = P(X = x_i) \cdot P(Y = y_j)$$

Range and Interpretation of Linear Correlation Coefficient ρ

- **Perfect Negative Correlation ($\rho = -1$):**
  - There is a perfect linear relationship between X and Y with a negative slope.
  - As X increases, Y decreases proportionally.
- **Strong Negative Correlation ($-1 < \rho < 0$):**
  - There is a strong negative linear relationship between X and Y.
  - As X increases, Y tends to decrease.
- **No Correlation ($\rho = 0$):**
  - There is no linear relationship between X and Y.
- **Strong Positive Correlation ($0 < \rho < 1$):**
  - There is a strong positive linear relationship between X and Y.
  - As X increases, Y tends to increase.
- **Perfect Positive Correlation ($\rho = 1$):**
  - There is a perfect linear relationship between X and Y with a positive slope.
  - As X increases, Y increases proportionally.

Practical Guidelines for Interpretation

- $|\rho| \approx 0.0 - 0.3$: Weak linear relationship.
- $|\rho| \approx 0.3 - 0.7$: Moderate linear relationship.
- $|\rho| \approx 0.7 - 1.0$: Strong linear relationship.

Remark

If X and Y are independent, then:

$$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$$

Properties of Sum of Discrete Random Variables

Let X and Y be two discrete random variables with values in $\mathbb{N}$ ($X(\omega) \in \mathbb{N}$ and $Y(\omega) \in \mathbb{N}$). The sum $X + Y$ is a random variable with values in $\mathbb{N}$, and its distribution is given by:

$$\forall n \in \mathbb{N},\ P(X + Y = n) = \sum_{i+j=n} P(X = i \text{ and } Y = j)$$

If X and Y are independent random variables, the distribution of $X + Y$ can be derived from the marginal distributions of X and Y:

$$\forall n \in \mathbb{N},\ P(X + Y = n) = \sum_{i+j=n} P(X = i) \cdot P(Y = j)$$

Remark

If X and Y are independent, then they are uncorrelated. However, the converse is generally not true.

Joint Cumulative Distribution Function

Let X and Y be continuous random variables. The joint cumulative distribution function is defined as:

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

Joint and Marginal Densities

If the joint cumulative distribution function $F_{X,Y}$ of (X, Y) is differentiable with respect to x and y, the joint density is:

$$f(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}$$

The marginal densities are given by:

$$f_{\text{marg},X}(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_{\text{marg},Y}(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Conditional Densities

The conditional density of X given $Y = y$ is:

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_{\text{marg},Y}(y)}$$

Similarly, the conditional density of Y given $X = x$ is:

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_{\text{marg},X}(x)}$$

Conditional Expectation

The conditional expectation of X given $Y = y$ is:

$$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x \mid y)\, dx$$

Similarly, the conditional expectation of Y given $X = x$ is:

$$\mathbb{E}[Y \mid X = x] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y \mid x)\, dy$$

Independence of Continuous Random Variables

A pair of continuous random variables (X, Y) with joint density $f_{X,Y}$ is independent if:

$$f_{X,Y}(x, y) = f_{\text{marg},X}(x) \cdot f_{\text{marg},Y}(y)$$

Remark

If X and Y are independent, then:

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$$

Sum of Continuous Random Variables

Let (X, Y) be continuous random variables with joint density $f_{X,Y}$. The density of the sum $X + Y$ is given by:

$$f_{X+Y}(s) = \int_{-\infty}^{\infty} f_{X,Y}(s-t,t)\,dt = \int_{-\infty}^{\infty} f_{X,Y}(t,s-t)\,dt$$

If X and Y are independent:

$$f_{X+Y}(s) = \int_{-\infty}^{\infty} f_{\text{marg},X}(s-t) \cdot f_{\text{marg},Y}(t)\,dt$$

Example: Continuous Random Variables

Description:

- Let X be a continuous random variable uniformly distributed over $[-1, 1]$.
- Define $Y = X^2$, a non-negative continuous random variable.

Key Computations:

- $\mathbb{E}[X] = 0$ (by symmetry of X over $[-1, 1]$).
- $\mathbb{E}[Y] = \int_{-1}^{1} X^2 \cdot f_X(x)\,dx = \frac{1}{3}$.
- $\mathbb{E}[XY] = \int_{-1}^{1} X \cdot X^2 \cdot f_X(x)\,dx = \int_{-1}^{1} X^3 \cdot \frac{1}{2}\,dx = 0$.
- $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$.

Conclusion:

- X and Y are uncorrelated ($\mathrm{Cov}(X, Y) = 0$).
- X and Y are not independent, as Y depends on X ($Y = X^2$).

Example: Discrete Random Variables

Description:

- Let X be a discrete random variable taking values $-1, 0, 1$, each with probability $\frac{1}{3}$.
- Define $Y = X^2$, which takes values:

$$Y = \begin{cases} 1 & \text{if } X = -1 \text{ or } X = 1, \\ 0 & \text{if } X = 0. \end{cases}$$

Key Computations:

- $\mathbb{E}[X] = 0$ (by symmetry).
- $\mathbb{E}[Y] = 1 \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{2}{3}$.
- $\mathbb{E}[XY] = (-1) \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$.
- $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$.

Conclusion:

- X and Y are uncorrelated ($\mathrm{Cov}(X, Y) = 0$).
- X and Y are not independent, as Y is determined by X ($Y = X^2$).