Random Variables-Introduction, Distributions, Density and Mass Functions

GASMI Abdelkader

March 2, 2025

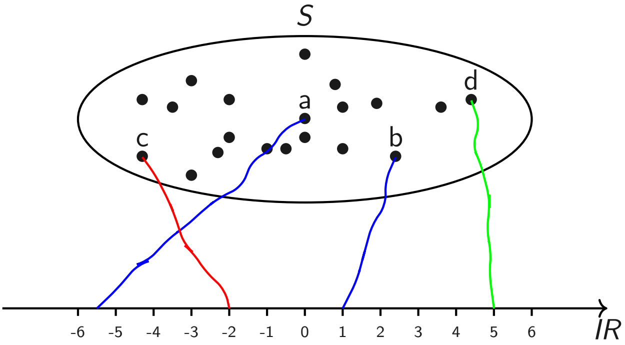
Introduction

- Statistical analysis involves focusing on numerical aspects of data, such as sample proportions, means, and standard deviations, regardless of whether the experiment yields qualitative or quantitative outcomes.
- Random variables serve as a bridge between experimental outcomes and numerical functions, allowing us to quantify and analyze data effectively.
- Two main types of random variables exist: discrete and continuous.

Random Variables

Definition

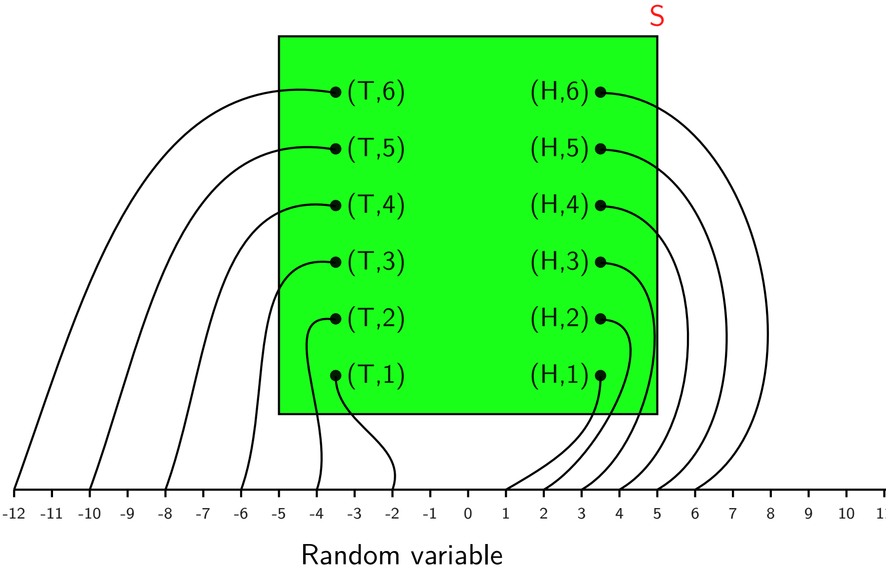
let S be a sample space of a considered random experience, a random variable (rv) is any rule that associates a number with each outcome in S.



Random variable

- In mathematical language a random variable is a real function that maps the set of all experimental outcomes of a sample space S into a set of real numbers IR.
- We shall represent a **random variable** by a **capital letter** (such as X, Y, or W) and any particular **value of the random variable** by a **lower case letter** (such as x, y, or w)
- Given an experiment defined by a sample space S with elements s, we assign to every sa real number $X(s)$ according to some rule and call $X(s)$ a random variable.
- There are three types of random variables:
  - **Discrete Random Variable** (random variables have discrete values; the sample space can be discrete, continuous or even mixture of discrete and continuous)
  - **Continuous Random Variable** (continuous range of values, it cannot be produced from a discrete sample space or a mixed sample space).
  - **Mixed Random Variables** (less important type of random variables)

- Example: (Discrete Random Variable):
  - An experiment consists of **rolling a die** and **flipping a coin**. The sample space is shown in Fig. below and the random variable X maps the sample space of 12 elements into 12 values of X.
  - Function X chosen such that:
    - A coin Head (H) outcome corresponds to positive values of X that are equal to the numbers that show up on the die.
    - A coin is Tail (T) outcome corresponds to the negative values of X that are equal in magnitude to twice the number that shows up on the die


S
(T,6)
(T,5)
(T,4)
(T,3)
(T,2)
(T,1)
(H,6)
(H,5)
(H,4)
(H,3)
(H,2)
(H,1)
-12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11
Random variable

- Conditions for a **Function** to be a **Random Variable**:
  - It not be multivalued
  - The set $\{X \leq x\}$ shall be an event for any real number x
  - $p(x \leq x)$ is equal to the sum of the probabilities of all the elementary events corresponding to $\{x \leq x\}$.
  - The probabilities of events $\{x = +\infty\}$ and $\{x = -\infty\}$ be 0:

$$p(x = +\infty) = p(x = -\infty) = 0$$

Probability Distributions

- A probability distribution consists of the values of a random variable and their corresponding probabilities.
- There are **two kinds** of probability distributions: discrete and continuous.
- If X is discrete, then the values $p(X = a_1)$, $p(X = a_2)$, ... tell us everything we need to know about X.
- Let X be a discrete random variable, and suppose that the possible values that it can assume are given by x_1, x_2, x_3, ... , arranged in some order. Suppose also that these values are assumed with probabilities given by

$$p(X = x_k) = f(x_k), \qquad k = 1, 2, 3. \dots$$

- Probability function, also referred to as probability mass function (PMF), given by

$$p(X = x) = f(x)$$

1. In general, $f(x)$ is a probability function if

$$\boxed{\begin{array}{l} 1.1\ \ f(x) \geq 0 \\ 1.2\ \ \sum_x f(x) = 1 \end{array}}$$

   1.1 The $f(x)$ is a function with nonnegative values
   1.2 The sum of the probabilities of a probability distribution must be 1.
      - Example When a die is rolled

Value, x	1	2	3	4	5	6
Probability, p(X=x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

      - Example Construct a discrete probability distribution for the number of heads when three coins are tossed.

- Solution
  1. Recall that the sample space for tossing three coins is

     TTT, TTH, THT, HTT, HHT, HTH, THH, and HHH.
  2. The outcomes can be arranged according to the number of heads, as shown.
     - 0 heads TTT
     - 1 head TTH, THT, HTT
     - 2 heads THH, HTH, HHT
     - 3 heads HHH
  3. Finally, the outcomes and corresponding probabilities can be written in a table, as shown.

Outcome, x	0	1	2	3
Probability, p(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

- Roll two dice, let Y be the maximum of their outcomes.
- Roll two dice, let X be the sum of their outcomes.

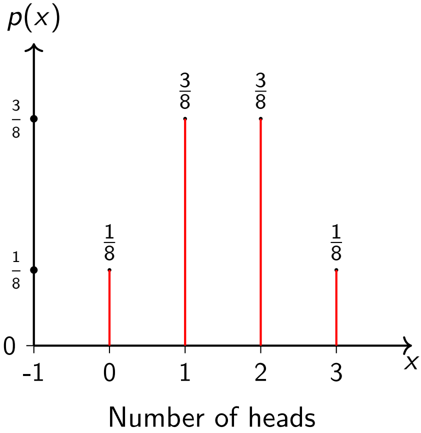
1. p(Y=1)=$p(\{(1,1)\}) = \frac{1}{36}$
2. p(Y=2)=$p(\{(1,2),(2,1),(2,2)\}) = \frac{3}{36} = \frac{1}{12}$
3. p(Y=3)=$p(\{(1,3),(2,3),(3,1),(3,2),(3,3)\}) = \frac{5}{36}$

$$\begin{aligned} p(Y=4) &= p(\{(1,4),(2,4),(3,4),(4,1),(4,2),(4,3)(4,4)\}) \\ &= \frac{7}{36} \\ p(Y=5) &= p(\{(1,5),(2,5),(3,5),(4,5),(5,1),(5,2),(5,3),(5,4), \\ &\quad (5,5)\}) = \frac{9}{36} \\ p(Y=6) &= p(\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3), \\ &\quad (6,4),(6,5),(6,6)\}) = \frac{11}{36} \end{aligned}$$

- Roll two dice, let X be the sum of their outcomes.

1. $\mathrm{p(X{=}2)}{=}p(\{(1,1)\}) = \frac{1}{36}$
2. $\mathrm{p(X{=}3)}{=}p(\{(1,2),(2,1)\}) = \frac{2}{36} = \frac{1}{18}$
3. $\mathrm{p(X{=}4)}{=}p(\{(1,3),(2,3),(3,1),(2,2)\}) = \frac{4}{36}$
4. $\mathrm{p(X{=}5)}{=}p(\{(1,4),(2,3),(3,2),(4,1)\}) = \frac{4}{36}$
5. $\mathrm{p(X{=}6)}{=}p(\{(1,5),(2,4),(3,3),(4,2),(5,1)\}) = \frac{5}{36}$
6. $\mathrm{p(X{=}7){=}p}(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36}$
7. $\mathrm{p(X{=}8){=}p}(\{(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{4}{36}$
8. $\mathrm{p(X{=}9){=}p}(\{(3,6),(4,5),(5,4),(6,3)\}) = \frac{3}{36}$
9. $\mathrm{p(X{=}10){=}p}(\{(4,6),(5,5),(6,4)\}) = \frac{3}{36}$
10. $\mathrm{p(X{=}11){=}p}(\{(5,6),(5,6)\}) = \frac{2}{36}$
11. $\mathrm{p(X{=}12){=}p}(\{,(6,6)\}) = \frac{1}{36}$

- A discrete probability distribution can also be shown graphically by labeling the x axis with the values of the outcomes and letting the values on the y axis represent the probabilities for the outcomes.



Cumulative Distribution Function (CDF)

- The probability of the event $\{X \leq x\}$ must depend on x. Denote
$$p(X \leq x) = F_X(x)$$
Where x is any real number
We call this function, denoted $F_X(x)$ the cumulative probability distribution function (CDF) of the random variable X (or just the distribution function of X)
- Properties Distribution Functions:

1. $F_X(-\infty)=p(X \leq -\infty) = p(\emptyset) = 0$
2. $F_X(+\infty)=p(X \leq +\infty) = p(S) = 1$
3. $0 \leq F_X(x) \leq 1$
4. $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$
5. $p(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

- Distribution Function for Discrete Random Variable:
  - The distribution function of a discrete random variable X can be obtained from its probability function by noting that, for all $x \in (-\infty, +\infty)$

$$F_X(x) = p(X \leq x) = \sum_{Xlex} p(X \leq x)$$

  - If X takes on only a finite number of values $x_1, x_2, \ldots, x_n$, then the distribution function is given by

$$F_X(x) = \begin{cases} 0 & if - \infty < x < x_1 \\ f(x_1) & if x_1 < x < x_2 \\ f(x_1) + f(x_2) & if x_2 < x < x_3 \\ \vdots & \\ f(x_1) + f(x_2) + \cdots + f(x_n) & if x_n < x < \infty \end{cases}$$

  - The distribution function of a discrete random variable X is given by:

$$F_X(x) = \sum_{i=0}^{n} p(X \leq x) u(x - x_i)$$

Probability Density and Mass Functions

Probability Density Function (PDF)

Definition (Continuous Random Variables)

A non-discrete random variable X is said to be absolutely continuous, or simply continuous, if its distribution function may be represented as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \quad (-\infty < x < \infty)$$

where the function $f(x)$ has the properties

$$f(x) \geq 0$$

$$\int_{-\infty}^{+\infty} f(t)dt = 1$$

A function $f(x)$ is more often called a probability density function or simply density function

Definition

Interval probability that X lies between two different values, say, a and b, is given by

$$p(q < X < b) = \int_q^b f(t)dt$$

Property

The probability density function (or density function) (PDF) is defined as the derivative of the distribution function:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

1. $f_X(x) \geq 0 \quad \forall x$
2. $\int_{-\infty}^{+\infty} f_X(x)dx = 1$
3. $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$
4. $p(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x)dx$

Mass function

The probability density function (or density function) (PDF) is defined as the derivative of the distribution function:

Definition (Mass function)

The density function of a discrete random variable X (mass function) is given by:

$$f_X(x) = \sum_{i=1}^{N} p(X = x_i)\delta(x - x_i)$$

Example

$$f_X(x) = \frac{1}{4}\delta(x) + \frac{2}{4}\delta(x-1) + \frac{1}{4}\delta(x-2)$$

Where $\delta(.)$ is the unit impulse defined by:

$$\delta(x) = \frac{du(x)}{dx}$$

Example

Let X have the discrete values in the set $\{-1, -0.5, 0.7, 1.5, 3\}$. The corresponding probabilities are $\{0.1, 0.2, 0.1, 0.4, 0.2\}$: The distribution function:

$$F_X(x) = 0.1u(x+1) + 0.2u(x+0.5) + 0.1u(x-0.7) + 0.4u(x-1.5) \\ + 0.2u(x-3)$$

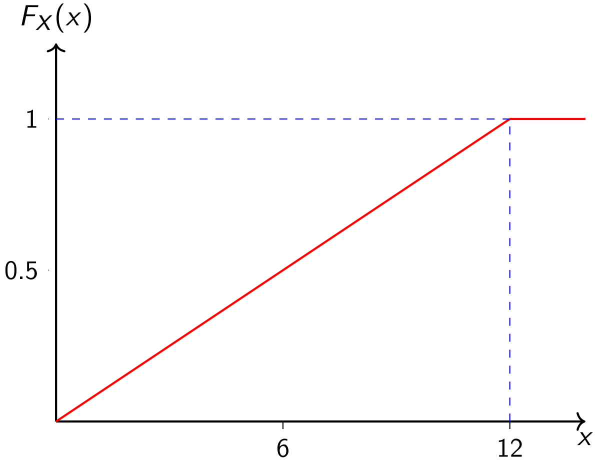
1. $p(X \leq 1.5) = 0.1 + 0.2 + 0.1 + 0.4 = 0.8$
2. $p(X \leq 2) = 0.1 + 0.2 + 0.1 + 0.4 = 0.8$

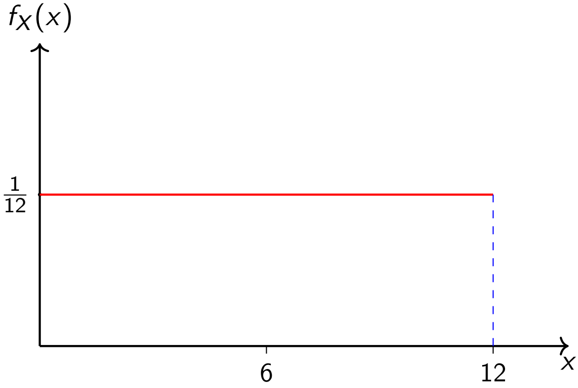
The density function:

$$f_X(x) = 0.1\delta(x+1) + 0.2\delta(x+0.5) + 0.1\delta(x-0.7) + 0.4\delta(x-1.5) \\ + 0.2\delta(x-3)$$

Example

The corresponding distribution and the density functions for the wheel of chance experiment are shown in Fig.





1. $p(X \leq 6) = F_X(x) = 0.5$
2. Or $= \int_{-\infty}^{6} f_X(x)dx = 0.5$

The corresponding distribution and the density functions for the wheel of chance experiment are shown in Fig.

Example

1. Find the constant c such that the function

$$f_X(x) = \begin{cases} c.(4x - 2x^2), & 0 < x < 2 \\ 0, & otherwise \end{cases}$$

   Is a density function,

2. Compute $p(1 < X < 2)$.
3. Find the distribution function for the random variable
4. Use the result of (3) to find $p(1 < x < 2)$

Mathematical Expectation

- The Expected Value of a Random Variable
- Expectation is the name given to the process of averaging when
- random variable is involved. The Mean value, the Statistical average, or the Expected Value of a random variable X are different terms for the Expectation and denoted as $E[X]$or $\bar{X}$
- Mathematical expectation can be thought of more or less as an average over the long run.
- The expectation of X is very often called the mean of X and is denoted by $\mu_{??}$, or simply μ and it is often called a measure of central tendency.

Definition (Expected Value)

- If X is a discrete random variable and $f(x)$ is the value of its probability mass function at x, the expected value of X is.

$$E[X] = \sum_{x} xf(x)$$

Example

In example 4, if we calculate the mathematical expectation, we find:

$$
\begin{aligned}
E(X) = \int_{-\infty}^{+\infty} xf(x)dx &= \int_{-\infty}^{0} x \cdot 0dx + \frac{3}{8}\int_{0}^{2} x(4x - 2x^2)dx + \int_{2}^{+\infty} \\
&= \frac{3}{8}\int_{0}^{2} (4x^2 - 2x^3)dx \\
&= \frac{3}{8}(\frac{4}{3}x^3 - \frac{1}{2}x^4)\Big|_0^2 = 1
\end{aligned}
$$

Property

The mathematical expectation of a random variable X satisfies

1. $E(\alpha) = \alpha, \forall\alpha$ *a constant of* $\mathbf{R}$
2. $E(X + \alpha) = E(X) + \alpha, \forall\alpha$ *a constant of* $\mathbf{R}$
3. $E(X_1 + X_2) = E(X_1) + E(X_2)$

Variance and standard deviation of a random variable

Definition

- We call variance of a discrete random variable X the quantity given by the formula:

$$Var(X) = \sum_k (x_k - E(X))^2 P(X = x_k), \tag{1}$$

provided that this series is convergent

- We call variance of a continuous random variable X the quantity given by the formula:

$$Var(X) = \int_{-\infty}^{+\infty} (x - E(X))^2 f(x)dx, \tag{2}$$

provided that this integral absolutely converges

Example

In the previous example if we calculate the variance we find:

$$\begin{aligned}
Var(X) &= \int_{-\infty}^{+\infty} (x - E(X))^2 f(x)dx \\
&= \int_{-\infty}^{0} (x-1)^2 \cdot 0dx + \frac{3}{8}\int_0^2 (x-1)^2(4x - 2x^2)dx + \int_2^{+\infty} (x \\
&= \frac{3}{8}\int_0^2 (-2x^4 + 8x^3 - 10x^2 + 4x)dx \\
&= \frac{3}{8}(-\frac{2}{5}x^5 + 2x^4 - \frac{10}{3}x^3 + 4x^2)\Big|_0^2 = \frac{1}{5}
\end{aligned}$$

Property

The following properties are verified

- $Var(X) == E(X^2) - E(X)^2$
- $Var(\alpha X + \beta) = \alpha^2 Var(X), \forall \alpha$ *and* β *constants of* $\mathbf{R}$

Definition

Let X be a real random variable and $r \in \mathbf{N}^\star$, we call the moment of order r of X the quantity:

$$E(X^r) = \sum_k (x_k)^r P(X = x_k), \tag{3}$$

and the centered moment of order r of X is the quantity:

$$E(X - E(X))^r = \sum_k (x_k - E(X))^r P(X = x_k). \tag{4}$$

Remarque

The variance is the centered moment of the second order.

Definition

The standard deviation noted σ of a random variable X is defined as the root of its variance:

$$\sigma(X) = \sqrt{Var(X)}. \tag{5}$$