Classification of States

To better understand Markov chains, we need to introduce some definitions. The first definition concerns the accessibility of states from each other: If it is possible to go from state i to state j, we say that state j is accessible from state i. In particular, we can provide the following definitions.

Definitions

We say that state j is **accessible** from state i, written as $i \to j$, if $p_{ij}^{(n)} > 0$ for some n. We assume every state is accessible from itself since $p_{ii}^{(0)} = 1$.

Two states i and j are said to **communicate**, written as $i \leftrightarrow j$, if they are accessible from each other. In other words,

$$i \leftrightarrow j \iff i \to j \text{ and } j \to i.$$

Communication is an equivalence relation. This implies:

- Every state communicates with itself, $i \leftrightarrow i$;
- If $i \leftrightarrow j$, then $j \leftrightarrow i$;
- If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

Thus, the states of a Markov chain can be partitioned into **communicating classes**, where only members of the same class communicate with each other. That is, two states i and j belong to the same class if and only if $i \leftrightarrow j$.

Example

Consider the Markov chain shown in Figure 11.9. It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$. Find the equivalence classes for this Markov chain. **Solution:**

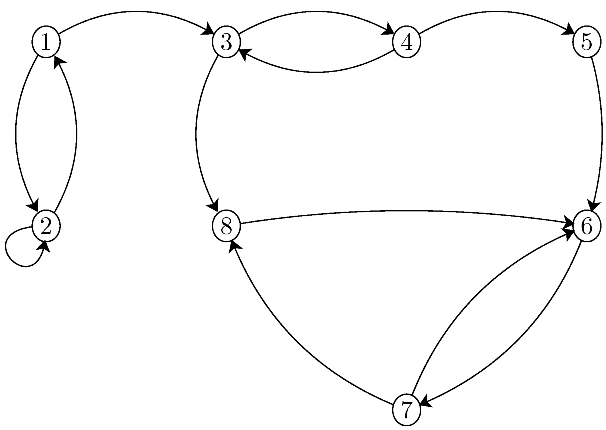

Figure 1: A state transition diagram.

Communicating Classes

There are four communicating classes in this Markov chain. Looking at Figure 1, we observe the following:

- States 1 and 2 communicate with each other, but they do not communicate with any other nodes in the graph.
- States 3 and 4 communicate with each other, but they do not communicate with any other nodes in the graph.
- State 5 does not communicate with any other states, so it forms a class by itself.
- States 6, 7, and 8 communicate with each other and construct another class.

Thus, the classes are:

$$\begin{aligned}
&\text{Class } 1 = \{\text{state } 1, \text{ state } 2\},\\
&\text{Class } 2 = \{\text{state } 3, \text{ state } 4\},\\
&\text{Class } 3 = \{\text{state } 5\},\\
&\text{Class } 4 = \{\text{state } 6, \text{ state } 7, \text{ state } 8\}.
\end{aligned}$$

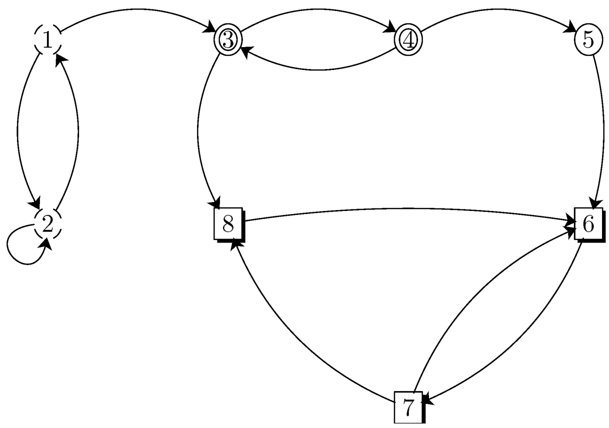

Figure 2: Equivalence classes.

A Markov chain is said to be **irreducible** if it has only one communicating class. As we will see shortly, irreducibility simplifies the analysis of limiting behavior. Looking at Figure 2, we notice that there are two kinds of classes:

- If at any time the Markov chain enters Class 4, it will always remain in that class.
- For other classes, this is not true. For example, if $X_0 = 1$, then the Markov chain might stay in Class 1 for a while, but at some point, it will leave that class and never return to it again.

The states in Class 4 are called **recurrent states**, while the other states in this chain are called **transient states**.

Definition of Recurrent and Transient States

In general:

- A state is said to be **recurrent** if, any time we leave that state, we will return to that state in the future with probability one.
- A state is called **transient** if the probability of returning to it is less than one.

Here is the formal definition: A state is **recurrent** if, any time the chain leaves that state, it will return to that state in the future with probability 1. Otherwise, it is called **transient**. Formally:

$$f_{ii} = P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i).$$

State i is recurrent if $f_{ii} = 1$, and transient if $f_{ii} < 1$.

Periodicity

The period of a state i is the largest integer d such that $p_{ii}^{(n)} = 0$ whenever n is not divisible by d. Formally, let:

$$d(i) = \gcd\{n > 0 : p_{ii}^{(n)} > 0\}.$$

- If $d(i) > 1$, state i is **periodic**.
- If $d(i) = 1$, state i is **aperiodic**.

Why is periodicity important?

Periodicity is crucial for analyzing limiting distributions. An irreducible Markov chain is aperiodic if all states are aperiodic. To check aperiodicity:

- If there exists l, m such that $\gcd(l, m) = 1$ and $p_{ii}^{(l)} > 0$, $p_{ii}^{(m)} > 0$, then state i is aperiodic.
- The chain is aperiodic if there exists n such that all elements of P^n are strictly positive.

Example

Consider the Markov chain in Figure 11.11:

- Is Class 1 = {state 1, state 2} aperiodic?
- Is Class 2 = {state 3, state 4} aperiodic?
- Is Class 4 = {state 6, state 7, state 8} aperiodic?

Solution: Analyze the gcd of transition steps to determine aperiodicity.

First Visit to a State

The first time the Markov chain visits a particular state is an important concept in analyzing the behavior of Markov chains. Specifically, consider the time it takes for the chain to visit state j for the first time after starting at state i.

We define:

$$T_j = \min\{n \geq 1 : X_n = j\},$$

where T_j represents the **first passage time** to state j. If the chain starts at state i, the probability of visiting state j for the first time at step n is given by:

$$P(T_j = n \mid X_0 = i).$$

This probability can be used to compute various metrics, such as the expected time to first reach state j, which is known as the **mean first passage time**:

$$E[T_j \mid X_0 = i].$$

If state j is **recurrent**, the chain is guaranteed to visit state j eventually, and $P(T_j < \infty \mid X_0 = i) = 1$. If state j is **transient**, the probability of ever visiting state j is less than one, i.e.,

$$P(T_j < \infty \mid X_0 = i) < 1.$$