Probability Vectors and Stochastic Matrices

Definition

A vector $U = (u_1, u_2, \ldots, u_n)$ is called a *probability vector* if:

- The components are non-negative.
- The sum of the components is 1.

Examples of Probability Vectors

Example 7.5: Consider the vectors:

$$U = \left(\frac{2}{3}, 0, -\frac{1}{3}, \frac{1}{3}\right), \quad v = \left(\frac{1}{2}, \frac{1}{4}, 0, \frac{3}{4}\right), \quad w = \left(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}\right).$$

- U is not a probability vector since its third component is negative.
- v is not a probability vector since the sum of its components is greater than 1.
- w is a probability vector.

Constructing Probability Vectors

Example 7.6: The nonzero vector $v = (2, 3, 5, 0, 1)$ is not a probability vector because:

$$2 + 3 + 5 + 0 + 1 = 11.$$

However, multiplying by the reciprocal of the sum gives:

$$\lambda v = \left(\frac{2}{11}, \frac{3}{11}, \frac{5}{11}, 0, \frac{1}{11}\right).$$

Definition

A square matrix $P = (p_{ij})$ is a *stochastic matrix* if:

- Each entry is non-negative.
- The sum of the entries in each row is 1.

Examples of Stochastic Matrices

Example 7.7: Consider the matrices:

$$\text{(i)} \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & -0.2 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} 0.5 & 0.5 \\ 0.6 & 0.8 \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{pmatrix}.$$

- (i) is not stochastic since it has a negative entry.
- (ii) is not stochastic since the sum of the second row is not 1.
- (iii) is stochastic.

Properties of Stochastic Matrices

Theorem 7.2: If A and B are stochastic matrices, then:

- The product AB is stochastic.
- All powers A^n are stochastic matrices.

Definition

A stochastic matrix P is *regular* if all entries of some power P^m are positive.

Examples of Regular Stochastic Matrices

Example 7.8:

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

is regular since:

$$A^2 = \begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix}.$$

Non-Regular Stochastic Matrices

Example 7.9:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here, A^m retains 1 and 0, so A is not regular.

Properties of Regular Stochastic Matrices

Theorem 7.3: Let P be a regular stochastic matrix. Then:

1. P has a unique fixed probability vector t, and the components of t are positive.
2. The sequence $P, P^2, P^3, \ldots$ approaches a matrix T whose rows are all t.
3. For any probability vector p, the sequence $pP, pP^2, pP^3, \ldots$ approaches t.

Example: Finding Fixed Points

Example 7.10:

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix}.$$

The unique fixed point is:

$$t = \left(\frac{1}{3}, \frac{2}{3} \right).$$