Probability: Fundamental Definitions and Concepts

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Contents

1 Introduction

Probability, as a fundamental concept in mathematics and statistics, is often defined and discussed utilizing three common approaches

- ✍ Classical Approach
  - ✧ This approach deals with situations where all possible outcomes are equally likely.
  - ✧ Probability is calculated by dividing the number of favorable outcomes by the total number of possible outcomes.
  - ✧ It is applicable to simple and well-defined experiments, such as coin tossing, dice rolling, or drawing balls from an urn.
- ✍ Relative Frequency Approach or empirical probability
  - ✧ This approach relies on empirical observation and experimentation.
  - ✧ Probability is determined by observing the frequency of occurrence of an event over a large number of trials.
  - ✧ It involves conducting experiments or observations to collect data and then calculating the proportion of times the event of interest occurs relative to the total number of trials.
- ✍ Subjective Approach

- ✧ This approach considers probability as a measure of an individual's belief or degree of confidence in the occurrence of an event.
- ✧ Probability is assigned based on personal judgment, experience, expertise, or available information.
- ✧ It is subjective in nature and can vary from person to person depending on their knowledge, perceptions, and biases.
- ✧ Subjective probability is commonly used in decision-making, risk assessment, and forecasting, where precise data may be lacking, and personal judgment plays a significant role.

2 Probability Introduced through Sets and Relative Frequency

2.1 Probability Experiments

Definition 2.1. The chance process is any experiment or process whose outcome cannot be predicted, even if all of its possible results are known.

Definition 2.2. A probability experiment is a chance process that leads to well defined outcomes or results.

2.2 An outcome

Definition 2.3. An outcome of a probability experiment is the result of a single trial of a probability experiment.

- Each outcome of a probability experiment occurs at random. This means you cannot predict with certainty which outcome will occur when the experiment is conducted.
- Each outcome of the experiment is equally likely unless otherwise stated. That means that each outcome has the same probability of occurring.
- Sample points: These outcomes are called sample points.

2.3 Sample space

Definition 2.4. The set of all possible outcomes (realizations) of a probability experiment is called a sample space and denoted by S.

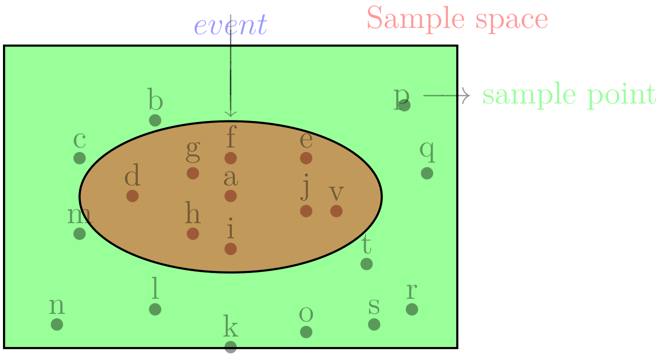
Example. Sample spaces for various probability experiments

Experiment	Sample Space
Toss one coin	$\{H, T\}$
Roll a die	$\{1, 2, 3, 4, 5, 6\}$
Toss two coins	$\{(H, H), (H, T), (T, H), (T, T)\}$
H:heads T:tails	

2.4 Events

Definition 2.5. An event then usually consists of one or more outcomes of the sample space. Certain subsets of S are referred to as events.

- ☞ An event with one outcome is called a simple event.
- ☞ When an event consists of two or more outcomes, it is called a compound event.
- ☞ Mutually exclusive: If two events have no common outcomes they are called mutually exclusive.
- ☞ Events A and B are mutually exclusive, if only one of them can occur, that is,



Example. ✍ Rolling dice

- ☞ Sample space is $S = \{1, 2, 3, 4, 5, 6\}$
- ☞ Events are the subsets of S, e.g.,
  - ☛ A = "outcome is even"$= \{2, 4, 6\}$.
  - ☛ B = "outcome is > 4"$= \{5, 6\}$.

✍ Rolling two dice

- ☞ Sample space is:

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

- ☞ Events are all subsets of S, e.g.,
  - ☞ A = "outcomes are the same" $= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$.
  - ☞ B = "outcome of the 1. roll is 1" $= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$.

2.5 Discrete Sample Space

Example. ✍ Experiment of rolling a single die and observing the number that shows up. There are six elements $S = \{1, 2, 3, 4, 5, 6\} \longleftarrow$ Discrete Sample Space.

2.6 Continuous Sample Space

Example. ✍ The position of a robot in a room.
$S = \{(x, y),\ 0 \le x \le L,\ 0 \le y \le W\} \longleftarrow$ continuous sample space (L and W are the length and width of the room.)

2.7 Probability Definition and Axioms

Definition 2.6. S be the sample space of a random Experiment. Suppose that to each event A of S, a number denoted by p(A) is associated with A. If p satisfies the following axioms, then it is called a probability and the number p(A) is said to be the probability of A.

- Axiom 1: $p(A) \leq 1$
- Axiom 2: $p(S) = 1$
- Axiom 3: If $\{A_1, A_2, A_3, ..., A_N\}$ is sequence of mutualy exclusive events, then;

$$p\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} A_n$$

Note rhat;

(a) Likely equally: p(A)=p(B)

(b) The probability of the empty set $\emptyset$ is 0. That is $p(\emptyset)$=0

- Sample spaces are used in classical probability to determine the numerical probability that an event will occur. The formula for determining the probability of an event E is

$$\boxed{p(E) = \frac{\text{Number of outcomes contained in E}}{\text{Total number of outcomes in the sample space}}}$$

2.8 Mathematical Model of Experiments

A real experiment is defined mathematically by three things:

- Assignment of the sample space S
- Definition of events of interest
- Making probability assignments to the events such that the axioms are satisfied.

Example. A die is tossed; find the probability of each event

- Getting a two
- Getting an even number
- Getting a number less than 5

Solution: The sample space is $\{1, 2, 3, 4, 5, 6\}$, so there are six outcomes in the sample space.A die is tossed; find the probability of each event:

- $P(2) = \frac{1}{6}$.
- $P(\text{even number}) = \frac{3}{6} = \frac{1}{2}$
- $P(\text{number less than } 5) = \frac{4}{6} = \frac{2}{3}$

?

Example. The experiment of tossing a coin once:

- The sample space is $S = \{H, T\}$, then $p(S) = 1$
- Let events $A = \{H\}$ and $B = \{T\}$, then: $p(A) = p(B)\frac{1}{2}$.

Example. The experiment of rolling a single dice:

- The sample space is $S = \{1, 2, 3, 4, 5, 6\}$
- Let events $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, then: $p(A) = \frac{3}{6} = 0.5$, $p(B)\frac{2}{6}$ qnd $p(A \cup B) = p(A) + p(B) = \frac{5}{6}$.

Example. An experiment consists of observing the sum of the numbers showing up when two dice are thrown. If three events define by::

- $A = \{sum = 7\}$, $B = \{8 < sum \leq 11\}$ and $C = \{10 < sum\}$, then
- The sample space is $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

☞ $p(A) = \dfrac{6}{36}$, $p(B) = \dfrac{9}{36}$ and $p(c) = \dfrac{3}{36}$.

Definition 2.7. The Relative frequency probability p of event A is defined as:

$$p(A) = \lim_{n\to+\infty} \frac{n_A}{n}.$$

The ratio $\dfrac{n_A}{n}$ is the relative freauency (or avreage number of success) for the event A

✍ Probabilities can be computed for situations that do not use sample spaces. In such cases, frequency distributions are used and the probability is called empirical probability.

☞ Empirical probability is sometimes called relative frequency probability textcolorblueA is defined as:

$$p(E) = \frac{\text{freauency of E}}{\text{sum of the freauencies}}$$

Example. Suppose a class of students consists of 4 first-year students, 8 Second year, 6 third year, and 7 master's degrees.The information can be summarized in a frequency distribution as follows:

Rank	Frequency
First-year	4
Second year	6
Third year	7
Master's degrees	8
Total	25

✍ Using the frequency distribution shown previously, find the probability of selecting a Master's degree student at random.

✍ Solution: Since there are 8 Master's degree and a total of 25 students:

$$p(Master) = \frac{8}{25}$$

2.9 Combining events | Summary

Iterpretation	Set theory expression
Certain event	S
Impossible event	$\emptyset$
A and B occur	$A \cap B$
A or B occur	$A \cup B$
A does not occur	A^c
B occur but A does not	$B \setminus A$
A and B are mutually exlusive	$A \cap B = 0$

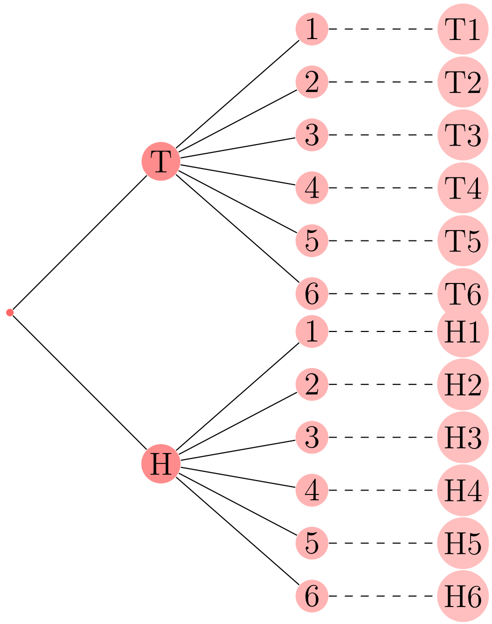
2.10 Finding Sample spaces | Summary

✍ Two specific devices will be used to find sample spaces for probability experiments. They are tree diagrams and tables.

✍ A tree diagram consists of branches corresponding to the outcomes of two or more probability experiments that are done in sequence.

☛ In order to construct a tree diagram, use branches corresponding to the outcomes of the first experiment. Then from each branch of the first experiment draw branches that represent the outcomes of the second experiment. You can continue the process for further experiments.

☛ Example A coin is tossed and a die is rolled. Draw a tree diagram and find the sample space.



- Hence there are twelve outcomes. They are H1, H2, H3, H4, H5, H6, T1,T2, T3, T4, T5, and T6.
- Once the sample space has been found, probabilities for events can be computed.
- Example:A coin is tossed and a die is rolled. Find the probability of getting:
  - A head on the coin and a 3 on the die.
  - A head on the coin.
  - A 4 on the die.
- Solution:A coin is tossed and a die is rolled. Find the probability of getting:
  - Since there are 12 outcomes in the sample space and only one way to get a head on the coin and a three on the die, $p(A) = \frac{1}{12}$.
  - Since there are six ways to get a head on the coin, namely H1, H2, H3,H4, H5, and H6, P(head on the coin) $=\frac{6}{12} = \frac{1}{2}$.
  - Since there are two ways to get a 4 on the die, namely H4 and T4, P(4 on the die) $=\frac{2}{12} = \frac{1}{6}$.
- Probability Rules | Summary:A coin is tossed and a die is rolled. Find the probability of getting:
  - Rule 1: The probability of any event will always be a number from zero to one. This can be denoted mathematically as $0 \leq p(A) \leq 1$. Probabilities cannot be negative nor can they be greater than one, .
  - Rule 2: When an event cannot occur, the probability will be zero[Example: A die is rolled; find the probability of getting a 7]
  - Rule 3: When an event is certain to occur, the probability is 1 [A die is rolled; find the probability of getting a number less than 7.]
  - Rule 4: The sum of the probabilities of all of the outcomes in the sample space is 1.
  - Rule 5: The probability that an event will not occur is equal to 1 minus the probability that the event will occur.

3 Joint and Conditional Probability

- In many engineering applications we often perform an experiment that consists of many experiments, examples are:
  - Observation the input and output digits of a binary communication system.
  - Observation of the trajectories of several objects in space.

✍ Suppose: E is the random experiment contains two subexperiments E1 and E2.

✎ If

(a) S_1sample space for E1 with $\{a_1, a_2, ..., a_{n1}\}$ outcomes.

(b) S_2 sample space for E2 with $\{b_1, b_2, ..., b_{n2}\}$ outcomes.

✎ Then

- the sample space of the combined experiment is the Cartesian product:

$$S_1 \times S_2 = \{(a_i, b_j) : i = 1, 2, ..., n1; j = 1, 2, ..., n2\}$$

✎ If events A_1, A_2,..., A_n are defined for S_1

✎ If events B_1, B_2,..., B_m are defined for S_2

✎ Then events A_i, B_j are the events of the total experiment (see the table below).

✎ We can define probability measures on S_1, S_2 and S as the following:

✏ Assume that

- N : Number of outcomes for S;
- N_A : Number of outcomes for A_i
- N_B : Number of outcomes for B_j
- N_{AB} : Number of outcomes for $A_i \cap B_j$

✍ Joint Probability: the probability

$$p(A \cap B) = \frac{N_{AB}}{N}$$

✌ The probability $p(A \cap B)$ is called the joint probability for two events A and B.

✌ It can be shown that:

$$p(A \cap B) = p(A) + p(B) - p(A \cup B)$$

✌ Equivalently:

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

✌ For mutually exclusive events:

$$p(A \cup B) = p(A) + p(B) \text{ where } (A \cap B) = \emptyset$$

✍ Marginal Probability : if events A_1, A_2,..., A_n are mutually exclusive and exhaustive, then

✌ Marginal Probability for B_j.

$$p(B_j) = \sum_{i=1}^{n} p(A_i \cap B_j)$$

✌ Marginal Probability for A_i.

$$p(A_i) = \sum_{j=1}^{m} p(A_i \cap B_j)$$

✌ For mutually exclusive events:

$$p(A \cup B) = p(A) + p(B) \text{ where } (A \cap B) = \emptyset$$

✌ It can be shown that:

$$p(B_j) = \sum_{i=1}^{n} p(A_i \cap B_j)$$

	A_1	A_2	...	A_n	Marginal Probability for B_j
B_1	$p(A_1 \cap B_1)$	$p(A_2 \cap B_1)$	...	$p(A_n \cap B_1)$	$p(B_1)$
B_2	$p(A_1 \cap B_2)$	$p(A_2 \cap B_2)$	...	$p(A_n \cap B_2)$	$p(B_2)$
...	...	...	...	...	...
B_m	p$(A_1 \cap B_m)$	$p(A_2 \cap B_m)$	...	$p(A_n \cap B_m)$	$p(B_m)$
Marginal Probability for A_1	$p(A_2)$	$p(A_2)$	...	$p(A_n)$	1

3.1 Conditional Probability

✍ Conditional Probability: The probability $p(A/B)$ is called the conditional probability for two events A and B

- Given some event B with nonzero probability; $p(B) \neq 0$. The conditional probability of A given B is:

$$p(A/B) = \frac{p(A \cap B)}{p(B)}$$

- An expression for the Conditional Probability in terms of joint and marginal probabilities.

$$p(B/A) = \frac{p(A \cap B)}{p(A)} = \frac{\frac{N_{AB}}{N}}{\frac{N_A}{N}} = \frac{N_{AB}}{N_A},\ p(A) \neq 0$$

$$Conditional Probability = \frac{Joint\ probability}{Marginal\ probability}$$

- Since B is known to have occurred, it becomes the new sample space replacing the original.
- For mutually exclusive events A and B::

$$A \cap B = \emptyset\ and\ p(B/A) = 0$$

Example. ✍ The HBO cable network took a survey of 500 subscribers to determine people?s favorite show, the data are shown in the following table: (for details: `https://youtu.be/SrEmzdOT65s`to see the YouTube video)

	Male	Female
Game of Thrones	80	120
West World	100	25
Others	50	125

✍ Compute the total sum for each row and column, we obtain the following table:

	Male	Female	Total
Game of Thrones	80	120	200
West World	100	25	125
Others	50	125	175
Total	23050	270	500

✍ Each square can be called joint event, because the data written in the square depended on two different variables.

✍ Obtain the probability distribution table by dividing each value by the total number of subscribers ?observations?.

`https://youtu.be/SrEmzdOT65s`