Basic Concepts of Set Theory

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Contents

1 Introduction

Probability has a fort relationship with set operations, so let's start by defining the important notation and terminology.

2 Set Definitions

Definition 2.1. A set is a collection of objects called elements of the set.

Example. Set of computer science students, set of numbers.

Definition 2.2.
- **A set of sets** sometimes called class of sets.
- A set is usually denoted by a **capital letter** while an element is represented by a **lower-case letter**.
  - If a is an element of set A, then we write $a \in A$
  - If a is not an element of set A, then we write $a \notin A$
- A set is specified by the content of two braces: $\{.\}$
- Two methods exist for specifying content:
  - The tabular method.
  - The rule method.

Example. The set of all integers between 5 and 10 would be:
1. In tabular method such as:
   (a) $A = \{6, 7, 8, 9\}$.
   (b) $S = \{book, cellphone, mp3, paper, laptop\}$
1. In rule method such as:
   (a) $\mathrm{A} = \{integers\, between\, 5\, and\, 10\,\}$

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1. A set with a large or infinite number of elements are best described by a statement or rule method. for example:
   (a) $\{Integers\, from\, 5\, to\, 10000\, inclusive\, \}$
   (b) $S = \{x|x\, is\, a\, city\, with\, a\, population\, over\, 1\, million\}$

Definition 2.3. A set is called countable if its elements can be put in one-toone correspondence with natural numbers.

Example.
1. $A = \{1, 3, 5, 7\}$
2. $S = \{H, T\}$
3. $B = \{1, 2, 3, ...\}$

Definition 2.4. A set is called uncountable, if its elements not countable

Example.
1. $A = \{0.5 < x < 8.5\}$
2. $S = \{-0.5 \leq y \leq 12\}$
3. $B = \{t/t \geq 0\}$

Definition 2.5 (Subset)**.** The set A is called a subset of B if every element in A is also an element in B (A contained in B), we write

$$A \subseteq B \tag{1}$$

If at least one element exists in B which is not in A, then A is a proper subset of B, denoted by

$$A \subset B \tag{2}$$

- The null set is clearly a subset of all other sets.

Definition 2.6 (Disjoint Set)**.** Two sets A and B is called disjoint or mutually exclusive if they have no common elements :

$$A \cap B = AB = \emptyset \tag{3}$$

Example.
- $A = \{1, 3, 5, 7\}$
- $B = \{1, 2, 3; ...\}$
- $C = \{0.5 < x \leq 8.5\}$
- $D = \{0.0\}$
- $E = \{2, 4, 6, 8, 10, 12, 14\}$
- $F = \{-0.5 < x \leq 12.5\}$

1. A: tabular-specified countable and finite.
2. B: is also tabular-specified and countable but infinite.
3. C: rule- specified, uncountable and infinite
4. D and E are mutually exclusive.
5. F is uncountable and infinite
6. Set A is contained in set B, C and F.
7. $D \subset F$, $C \subset F$, $E \subset B$
8. Sets A, D and E are mutually exclusive.

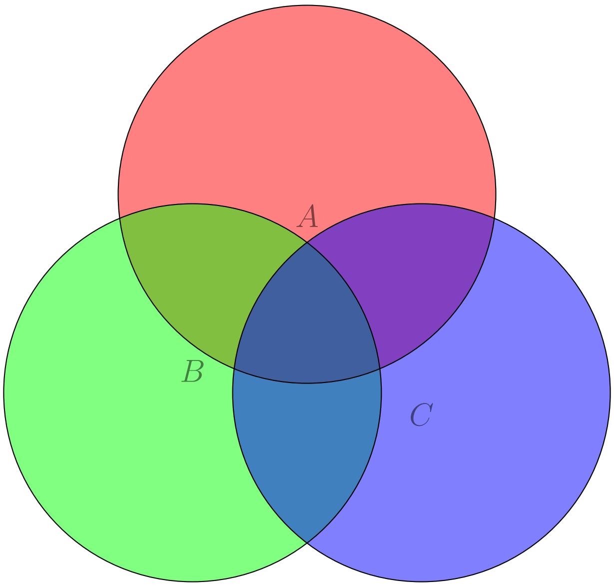
- The largest set of objects under discussion in a given situation is called the universal set, denoted S.

> **Example.** In the problem of rolling a die, we are interested in the numbers that show on the upper face. The universal set is:
> $$S = \{1, 2, 3, 4, 5, 6\} \tag{4}$$

- For any universal set with N elements, there are 2^N possible subsets of S.

3 Set Operations

- Geometrical representation of the sets using Venn diagram. The relationship between subsets and the universal set can be illustrated graphically using Venn diagram.
  - Sets are represented by closed-plane figures.



Definition 3.1 (Equality)**.** Two sets A and B are equal if and only if they have the same elements. We write $A = B$
$$A \subseteq B \, and \, B \subseteq A \tag{5}$$

Definition 3.2 (Difference)**.** The difference of two sets A and B, is the set containing all elements of A that are not present in B. We write
$$A \setminus B \tag{6}$$

> **Example.** If $A = \{0.6 \leq x \leq 1.6\}$ and $B = \{1.0 \leq y \leq 2.5\}$
> Then $A \setminus B = \{0.6 \leq t \leq 1\}$ and $B \setminus A = \{1.6 \leq t \leq 2.5\}$

✍ Note that $A \setminus B \neq B \setminus A$

Definition 3.3 (Union)**.** The union of two sets A and B, written as $C = A \cup B$, is the set containing all elements of both A and B or both, Union sometimes called the sum of two sets

Definition 3.4 (Intersection)**.** The intersection of two sets A and B, written as $C = A \cap B$, is the set of all elements common to both A and B or both, Intersection sometimes called the product of two sets

✍ For mutually exclusive sets A and B, $A \cap B = \emptyset$

✍ In general case, the union and intersection of of N sets $A_n, n = 1, 2, 3, ?., N$, become:
$$A_1 \cup A_2 \cup ... \cup A_N = \cup_{n=1}^{N} A_n \tag{7}$$

$$A_1 \cap A_2 \cap ... \cap A_N = \cap_{n=1}^{N} A_n \tag{8}$$

Definition 3.5 (Complement)**.** The complement of set A, denoted by $\bar{A}$, is the set of all elements not in A.

✍ Note that:

$$\bar{S} = \emptyset, \ \emptyset = \bar{S}, \ A \cup \bar{A} = S, \ and \ A \cap \bar{A} = \emptyset \tag{9}$$

Example. Given the four sets:

$$S = \{1 \leq integer \leq 12\}$$
$$A = \{1, 3, 5, 12\}$$
$$B = \{2, 6, 7, 8, 9, 10, 11\}$$
$$C = \{1, 3, 4, 6, 7, 8\}$$

Then

$$A \cup B = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12\}$$
$$A \cup C = \{1, 3, 4, 5, 6, 7, 8, 12\}$$
$$B \cup C = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11\}$$
$$A \cap B = \emptyset, A \cap C = \{1, 3\}, B \cap C = \{6, 7, 8\}$$
$$\bar{A} = \{2, 4, 6, 7, 8, 9, 10, 11\}$$
$$\bar{B} = \{1, 3, 4, 5, 12\}$$
$$\bar{C} = \{2, 5, 9, 10, 11, 12\}$$

4 Algebra of sets and duality

Property 1. Commutative Law

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Property 2. Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Property 3. Associative Law

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$$

Property 4. De Morgan's Law

$$\overline{A \cup B} = \bar{B} \cap \bar{A}$$
$$\overline{A \cap B} = \bar{B} \cup \bar{A}$$

- Replace unions by intersections, intersections by unions, by use of a venn

Property 5. Duality Principle In any an identity we replace unions by intersections, intersections by unions, S by $\emptyset$, and $\emptyset$ by S, then the identity is preserved.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$