Asymmetric Encryption

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Course outline

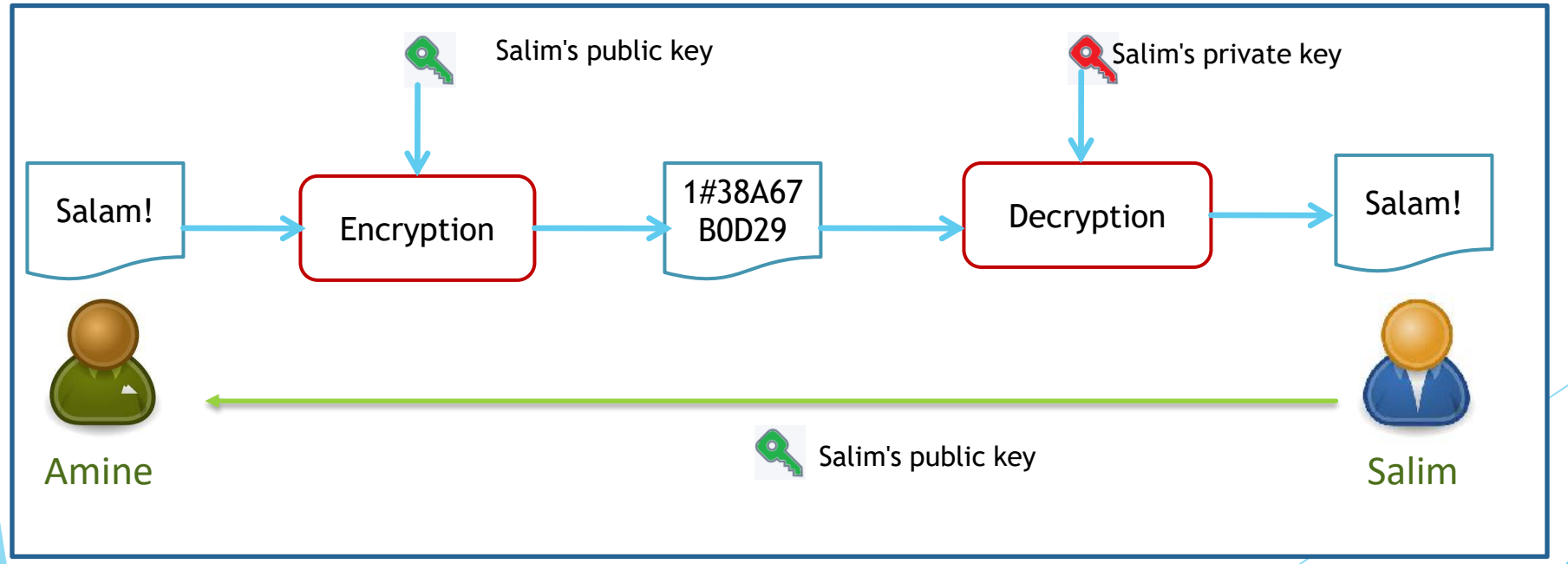
- Principles
- Mathematical concepts
- RSA Cryptosystem

Asymmetric cryptography: Principles

- Also called: public key cryptography (PKC stands for Public Key Cryptography).
- The security of PKC based on difficulty of computational problems.

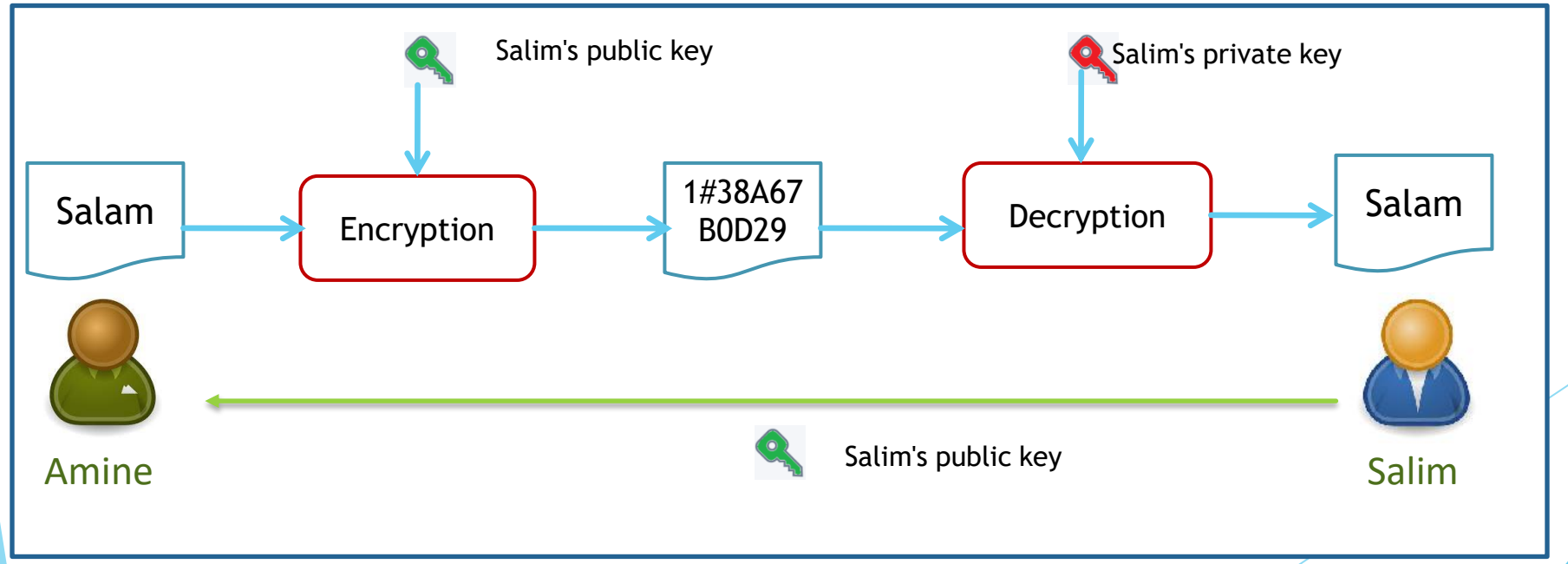
Asymmetric cryptography: Principles

- Each entity has a key pair (public key, private key).
- The two keys are linked mathematically.
- The public key is known to everyone.
- To encrypt a message, we use the recipient's public key.



Asymmetric cryptography: Principles

- The private key never transmitted over the network.
- The private key used to decrypt the ciphertext.



Mathematical concepts
Trapdoor one-way function

- The computational problems used in public-key cryptography are based on the existence of **trapdoor one-way function.**
- We assume $F(x) = y$ as one-way function.
  - *F* is easy to calculate: It is easy (polynomial time) to calculate $F(x)$ for any x.
  - *F* is difficult to reverse: It is difficult for $y \in F$ to find a x such as $F(x) = y$.
- In the function with trapdoor: Calculating the inverse will be easy if we know a additional information.The trapdoor is the *key private of the recipient*.

Asymmetric Cryptography: principles

- The asymmetric cryptography algorithms are used, for example:
  - Provide source authentication services, identity, and integrity using digital signatures;
  - Exchange keys protocols.
  - Public key infrastructures.
  - To protect data confidentiality.

Prime numbers

Prime Numbers:

- A natural number p is said to be prime if its only divisors are 1 and itself.
- All other numbers are called composite numbers. (0 and 1 are excluded).

Prime numbers

Decomposition into irreducible factors.

- Theorem: Any integer greater than 1 can be decomposed as a single product of prime numbers.

$$a = \prod_{i=1}^{k} p_i^{n_i} = p_1^{n1} p_2^{n2} \ldots p_k^{nk}$$

- This writing is called the decomposition of *a* in irreducible factors.
- Example: $12348 = 2^{2*}\ 3^{2*}\ 7^{3}$

Mathematical concepts
Primality Test

Primality Test

- Is 2019 prime number? We need to test the possibility of dividing 2019/n for :
  - Algo 1: for any integer n between 2 and 2019-1
  - Algo2:for any integer n between 2 and $\sqrt{2019}$
  - Algo3:(Sieve ofEratosten) for any prime n between 2 and$\sqrt{2019}$

Mathematical concepts
Primality Test

Fermat Theorem :

- Let p is a prime number.
- For any natural number a coprime to p, we have::

$$a^{p-1} = 1 \bmod p.$$

Mathematical concepts
Primality Test

Fermat's test:

- It's a method probabilistic to test the primality of an integer.
- We randomly chooses a number a such that $1<a\leqslant p-1$
  - If $a^{p-1}= 1 \bmod p$, we said that p is pseudo(-Fermat)-prime based a.
  - If $a^{p-1}\neq 1 \bmod p$, then p is not a prime number.

Mathematical concepts
Primality Test

Fermat's Test:

- Example: p = 341
- a=2: $2^{340} = 1 \pmod{341}$
- a=3: $3^{340} = 56 \pmod{341}$
- Therefore, Fermat's test, with p=341 and a=3, proves that p is compound.
- We have 341= 11 *31

Mathematical concepts
Primality Test

There are several primality testing algorithms:

- Miller-Rabin test
- CriteriaLucas
- …

Mathematical concepts
Euler's indicator

- Let *n* is an integer greater than 2,
- the Euler function of n, denoted $\varphi(n)$, refers the number of intgeres between1and n, and prime with n.
- If n is prime, $\varphi(n) = n - 1$.
- If n =pq such that p and q are prime numbers, $\varphi(n) = \varphi(p)\ \varphi(q) = (p-1)(q-1)$.
- $\varphi(p^k) = p^{k-1}(p-1)$

Mathematical concepts
Euler's indicator

Examples:

- $\varphi(11) = 11 - 1 = 10$
- $\varphi(15) = \varphi(5) \times \varphi(3) = (5-1)(3-1) = 8$
- $\varphi(8) = \varphi(2^3) = 2^{3-1}(2-1) = 4$

Euler Theorem:

- For any integer *n* and everything $a \in Z$, we have :

$$a^{\varphi(n)} = 1 \bmod n$$

Mathematical concepts
Factoring numbers

- Factoring integers:
  - Let two large prime numbers, p and q.
  - Calculate $p \times q$ is easier.
  - Problem: Factorizing $n = pq$.
  - **Applications:** RSA and Rabin cryptosystems.

Mathematical concepts

Factoring numbers

Test (competition)

- Find the two prime factors of the following products:
- **35**
  - 5 × 7
- **221**
  - 13 × 17
- **4453**
  - 61 × 73
- **503807**
  - 521 × 967
- **50123093**
  - 7297 × 6869

Mathematical concepts
Factoring numbers

- Find the two prime factors of the following product:

310741824049004372135075003588856793003734602284272754572016194882320644051808150455634682967172328678243791627283803341547107310850191954852900733772482278352574238645401469173660247765234660 9

=163473364580925384844313388386509085984178367003309231218111085238933310010450815121211816751157 9

×190087128166482211312685157393541397547189678996851549366663853908802710380210449895719126146557 1

- RSA keys are typically between 1024 bits (309 digits) and 2048 bits (617 digits) in size.
- In December 2018, the largest prime number was $M_{82589933}$=282589933-1 (Mersenne numbers), which has 24,862,048 digits when written in base 10.

Mathematical Concepts
Modular Expension

- Many primality testing algorithms and cryptographic algorithms use modular exponentiation in the form of powers $a^e \pmod n$ for large values of the exponent e.
- We present a method considered standard for performing modular exponentiation. It uses the principle of squaring and multiplying.

Mathematical Concepts
Modular Expension

- This technique is based on the writing of the exponent *e* in numeration binary.
- Let e=29 = $(11101)_{2.}$
- We have $a^{e= 16 + 8 + 4 + 1}$ and $a^{e}= a^{16}*a^{8}*a^{4}*a$

Mathematical Concepts

Modular Expension

Input:a, n,ewithe = $(e_{m-1}, .., e_0)_2$

Output:r =a^e (mod n)

Begin

r←$a^{e_{m-1}}$

fori ← m− 2 to 0 do

r←r*rmod n

If e_i= 1 then r←r* amod n

endfor

end

Mathematical Concepts
Modular Expension

- Example:
- $15^{29} \bmod 101 = ?$
- a=15, e=29, n=101
- $29 = (11101)_2$, the size of the binary sequence, m=5

i	-	3	2	1	0
ei	1	1	1	0	1
r	15	23	47	4	16
r=r*a	-	42	99	-	38

- $15^{29} \bmod 101 = 38$

Course outline

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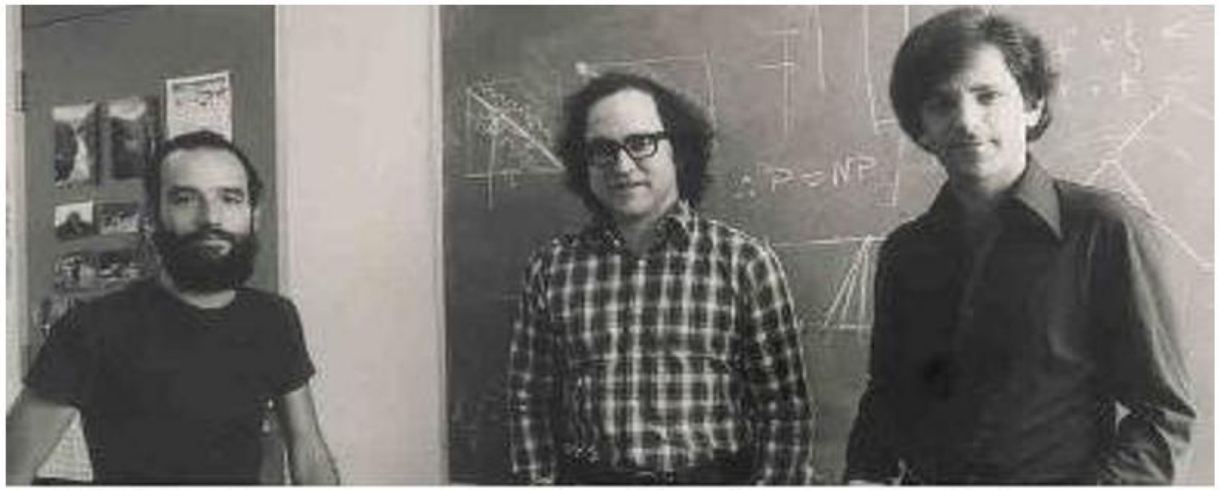
- RSA Cryptosystem

RSA Cryptosystem

- invented in 1977 by:



Adi Shamir Ron Rivest Len Adleman

RSA Cryptosystem

- The security of RSA relies on the difficulty in factoring large integers.
- Web Servers, creditcards, Internet (SSL/TLS), e-payment, …
- Microsoft, Apple Computer, Cisco Systems, Intel, …

RSA: Key Generation

RSA Module:

- p and q are two secret prime numbers of the same size.
- N=pq is the RSA module.

The indicatord'Euler:

- Φ(N) = (p - 1)(q - 1).

The keys:

- Choose e: a prime integer, 1≤ e ≤$\varphi(N)$,GCD(e, $\varphi(N)$)=1
- Calculate d, such as: 1 ≤ d ≤$\varphi(N)$,ed≡1 (mod$\varphi(N)$)

  →d = e^{-1} mod$\varphi(N)$

- Public key:e, n
- Private key:d

RSA: Encryption & Decryption

B wants to send a message to A:

1) A generates the keys (e,d) and publishes the public key.
2) Encryption: B calculates $c \equiv m^e \pmod N$ and sending it to A.
3) Decyption: A calculates $c^d \equiv m \pmod N$ and retrieves the message *m*.

RSA Cryptosystem: Example

- A generates the keys:
- p = 11 and q = 5
- N = p*q =**55**
- $\varphi(55)$ =(11-1)*(5-1) =**40**
- A Choisit e:
  - **e** is prime number.
  - Let e=3 (for example, and we have $(e,\varphi(55))=1$)
- Calculate d:
  - **ed** ≡1 (mod40) → d = 3^{-1}mod 40
  - We determine d=27 (modular inverse of *e* on $Z_{\Phi(n)}$)

Public Key **(55.3)**and the private key**(27)**

RSA Cryptosystem: Example

- B want to send a message (m=51) to A:
  - B calculates $c = 51^3 \bmod 55 = 46$
  - B sends c= 46 to A.
- Decrypting the ciphertext:
  - Calculate $m = 46^{27} \bmod 55 = 51$

Recommendation for RSA

- Using strong keys, p and q have large prime numbers.
- The size of the text to be encrypted is important.
- Do not use a common module RSA (n) with multiple keys.

Factoring of the RSA module

- The module RSA-250 has a size of 829 bits, equivalent to 250 decimal digits.
- RSA-250 has been factored in 2020 by F.Boudot, P. Gaudry, A. Guillevic, N.Heninger, E.Thoméand P. Zimmermann.
- This factorization was found using the Number Field Sieve algorithm using the open-source applicationCADO-NFS.

Recommendation

- NIST (2019)

Date	Niveau de Sécurité	Algorithme symétrique	Factorisation Module	Logarithme discret Clef	Logarithme discret Groupe	Courbe elliptique	Hash (A)	Hash (B)
Legacy [1]	80	2TDEA	1024	160	1024	160	SHA-1 [2]	
2019 - 2030	112	(3TDEA) [3] AES-128	2048	224	2048	224	SHA-224 SHA-512/224 SHA3-224	
2019 - 2030 et au-delà	128	AES-128	3072	256	3072	256	SHA-256 SHA-512/256 SHA3-256	SHA-1 KMAC128
2019 - 2030 et au-delà	192	AES-192	7680	384	7680	384	SHA-384 SHA3-384	SHA-224 SHA-512/224 SHA3-224
2019 - 2030 et au-delà	256	AES-256	15360	512	15360	512	SHA-512 SHA3-512	SHA-256 SHA-512/256 SHA-384 SHA-512 SHA3-256 SHA3-384 SHA3-512 KMAC256

https://www.keylength.com/fr/4/

Other Algorithms

- **El Gamal c**ryptosystem invented in 1984

- The security of this cryptosystem based on the problem of discrete logarithm.

Other Algorithms

- **ECC (Elliptic Curve Cryptography),**invented in 1985 by Koblitz and Mille.

- The security of this cryptosystem based on the problem of ***discrete elliptic logarithm.***

Exercise

Let n = 85 and e = 7 (public key).

- Find p and q
- Calculate Φ(n) and d.
- Encrypt the message m = 12