The Analytic Hierarchy Process (AHP) for Decision Making

The analytic hierarchy process (AHP) was developed by Thomas L. Saaty. The AHP is designed to solve complex problems involving multiple criteria.

An advantage of the AHP is that it is designed to handle situations in which the subjective judgments of individuals constitute an important part of the decision process.

The process requires the decision maker to provide judgments about the relative importance of each criterion and then specify a preference for each decision alternative on each criterion.

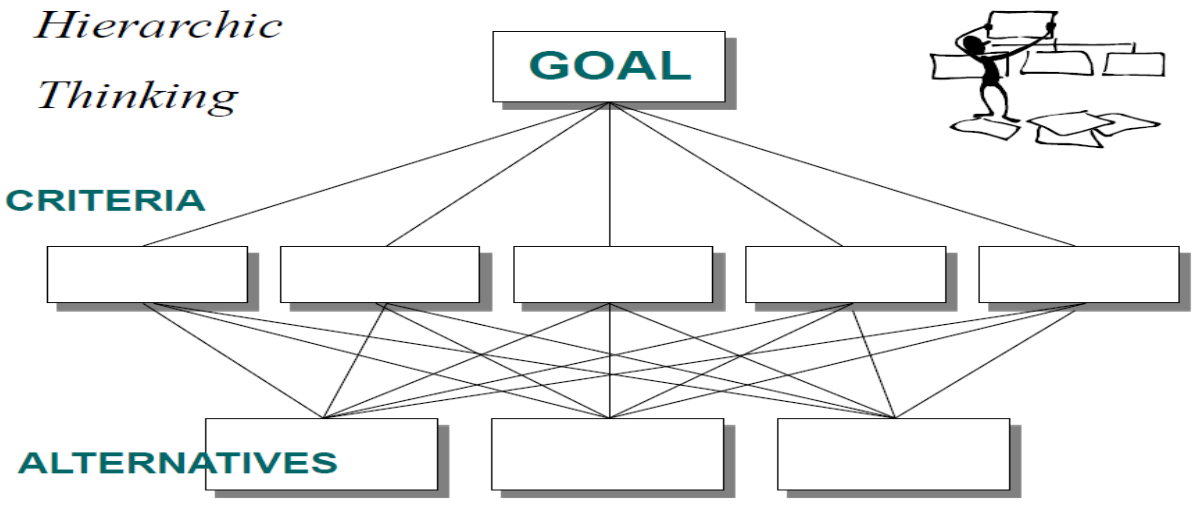
Major Steps of AHP

1) To develop a graphical representation of the problem in terms of the overall goal, the criteria, and the decision alternatives. (i.e., the hierarchy of the problem)
2) To specify his/her judgments about the relative importance of each criterion in terms of its contribution to the achievement of the overall goal.
3) To indicate a preference or priority for each decision alternative in terms of how it contributes to each criterion.
4) Given the information on relative importance and preferences, a mathematical process is used to synthesize the information (including consistency checking) and provide a priority ranking of all alternatives in terms of their overall preference.

The output of the AHP is a prioritized ranking indicating the overall preference for each of the decision alternatives.

Hierarchy Development

The first step in the AHP is to develop a graphical representation of the problem in terms of the overall **goal**, the **criteria**, and the **decision alternatives**.



Pairwise Comparisons

The AHP employs an underlying scale with values from 1 to 9 to rate the relative preferences for two items.

Preference Scale

Table 1 Original analytic hierarchy process scale

Numerical rating	Verbal judgments
9	Extremely more important or preferred
8	Very strongly to extremely more important or preferred
7	Very strongly preferred more important or preferred
6	Strongly to very strongly more important or preferred
5	Strongly preferred more important or preferred
4	Moderately to strongly more important or preferred
3	Moderately preferred more important or preferred
2	Equally to moderately more important or preferred
1	Equally preferred more important or preferred

Synthesis

The procedure to estimate the relative priority for each decision alternative in terms of the criterion is referred to as synthesization.

- Once the matrix of pairwise comparisons has been developed, priority of each of the elements (priority of each alternative on specific criterion; priority of each criterion on overall goal) being compared can be calculated.
- The exact mathematical procedure required to perform synthesization involves the computation of eigenvalues and eigenvectors.

Procedure for Synthesizing Judgments

The following three-step procedure provides a good approximation of the synthesized priorities.

Step 1: Sum the values in each column of the pairwise comparison matrix.

Step 2: Divide each element in the pairwise matrix by its column total.

The resulting matrix is referred to as the normalized pairwise comparison matrix.

Step 3: Compute the average of the elements in each row of the normalized matrix.

These averages provide an estimate of the relative priorities of the elements being compared

Consistency

An important consideration in terms of the quality of the ultimate decision relates to the consistency of judgments that the decision maker demonstrated during the series of pairwise comparisons.

The AHP provides a measure of the consistency of pairwise comparison judgments by computing a consistency ratio. A ratio exceeding 0.10 are indicative of inconsistent judgments.

Estimating Consistency Ratio -1

Step 1: Multiply each value in the first column of the pairwise comparison matrix by the relative priority of the first item considered. Same procedures for other items. Sum the values across the rows to obtain a vector of values labeled "**weighted sum**."

Step 2: Divide the elements of the vector of weighted sums obtained in Step 1 by the corresponding priority value.

Step 3: Compute the average of the values computed in step 2. This average is denoted as λmax.

Step 4: Compute the consistency index (CI):

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

Where n is the number of items being compared.

Step 5: Compute the consistency ratio (CR):

$$CR = \frac{CI}{RI}$$

Where RI is the random index, which is the consistency index of a randomly generated pairwise comparison matrix. RI depends on the number of elements being compared and takes on the following values.

Random Index

n	1	2	3	4	5	6	7	8	9	10
RI	0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

Example: Inconsistency

Given the matrix bellow

Comfort	Car A	Car B	Car C
Car A	1	2	8
Car B	1/2	1	6
Car C	1/8	1/6	1

Preferences: If, A➔B (2); B ➔C (6)

Then, A ➔C (should be 12) (actually 8)

Inconsistency

Consistency Checking

Step 1: Multiply each value in the first column of the pairwise comparison matrix by the relative priority of the first item considered. Same procedures for other items. Sum the values across the rows to obtain a vector of values labeled "weighted sum."

$$0.593\begin{bmatrix}1\\1/2\\1/8\end{bmatrix}+0.341\begin{bmatrix}2\\1\\1/6\end{bmatrix}+0.066\begin{bmatrix}8\\6\\1\end{bmatrix}=\begin{bmatrix}0.593\\0.297\\0.074\end{bmatrix}+\begin{bmatrix}0.682\\0.341\\0.057\end{bmatrix}+\begin{bmatrix}0.528\\0.396\\0.066\end{bmatrix}=\begin{bmatrix}1.803\\1.034\\0.197\end{bmatrix}$$

Step 2: Divide the elements of the vector of weighted sums by the corresponding priority value.

$$\begin{bmatrix}1.803/0.593\\1.034/0.341\\0.197/0.066\end{bmatrix}=\begin{bmatrix}3.040\\3.032\\2.985\end{bmatrix}$$

Step 3: Compute the average of the values computed in step 2

$$\lambda_{max}=\frac{3.040+3.032+2.985}{3}=3.019$$

Step 4: Compute the consistency index (CI).

$$CI=\frac{\lambda_{max}-n}{n-1}=\frac{3.019-3}{3-1}=0.010$$

Step 5: Compute the consistency ratio (CR).

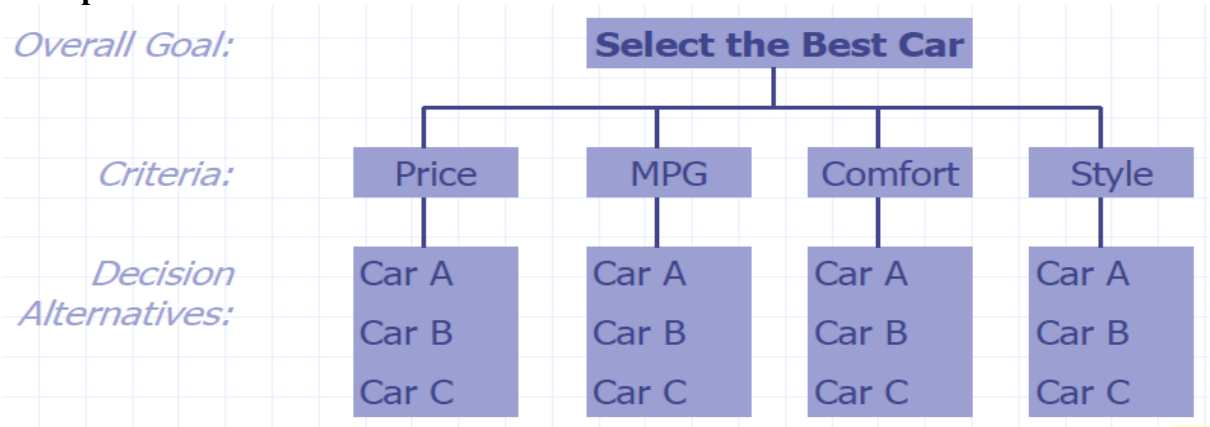
$$CR=\frac{CI}{RI}=\frac{0.010}{0.58}=0.017\leq 0.10$$

The degree of consistency exhibited in the pairwise comparison matrix for comfort is acceptable.

Development of Priority Ranking

The overall priority for each decision alternative is obtained by summing the product of the criterion priority (i.e., weight) (with respect to the overall goal) times the priority (i.e., preference) of the decision alternative with respect to that criterion. By ranking these priority values, we will have AHP ranking of the decision alternatives.

Example 1



Synthesizing Procedure

Comparison of cars with respect to comfort criteria

Step 0: Prepare pairwise comparison matrix

Comfort	Car A	Car B	Car C
Car A	1	2	8
Car B	1/2	1	6
Car C	1/8	1/6	1

Step 1: Sum the values in each column.

Comfort	Car A	Car B	Car C
Car A	1	2	8
Car B	1/2	1	6
Car C	1/8	1/6	1
Column totals	13/8	19/6	15

Step 2: Divide each element of the matrix by its column total

Comfort	Car A	Car B	Car C
Car A	8/13	12/19	8/15
Car B	4/13	6/19	6/15
Car C	1/13	1/19	1/15

All columns in the normalized pairwise comparison matrix now have a sum of 1.

Step 3: Average the elements in each row.

Comfort	Car A	Car B	Car C	Row Avg.
Car A	0.615	0.632	0.533	0.593
Car B	0.308	0.316	0.400	0.341
Car C	0.077	0.053	0.067	0.066
			Total	1.000

0.593
0.341
0.066

The result is usually represented as the (relative) priority vector.

Comparison of cars with respect to the rest of criteria's

Comfort	Car A	Car B	Car C
Car A	1	2	8
Car B	1/2	1	6
Car C	1/8	1/6	1

Price	Car A	Car B	Car C
Car A	1	1/3	¼
Car B	3	1	½
Car C	4	2	1

MPG	Car A	Car B	Car C
Car A	1	1/4	1/6
Car B	4	1	1/3
Car C	6	3	1

Style	Car A	Car B	Car C
Car A	1	1/3	4
Car B	3	1	7
Car C	1/4	1/7	1

Criterion	Price	MPG	Comfort	Style
Price	1	3	2	2
MPG	1/3	1	1/4	1/4
Comfort	1/2	4	1	1/2
Style	1/2	4	2	1

Priority Ranking

Step 0: Calculate priority vector for each matrix.

	Price	MPG	Comfort	Style
Car A	0.123	0.087	0.593	0.265
Car B	0.320	0.274	0.341	0.655
Car C	0.557	0.639	0.066	0.080

	Criterion
Price	0.398
MPG	0.085
Comfort	0.218
Style	0.299

Step 1: Sum the product of the criterion priority (with respect to the overall goal) times the priority of the decision alternative with respect to that criterion**.**

$$\text{Overall car A priority} = 0.398(0.123) + 0.085(0.087) + 0.218(0.593) + 0.299(0.265) = 0.265$$

Step 2: Rank the priority values

Alternative	Priority
Car B	0.421
Car C	0.314
Car A	0.265
Total	1.000