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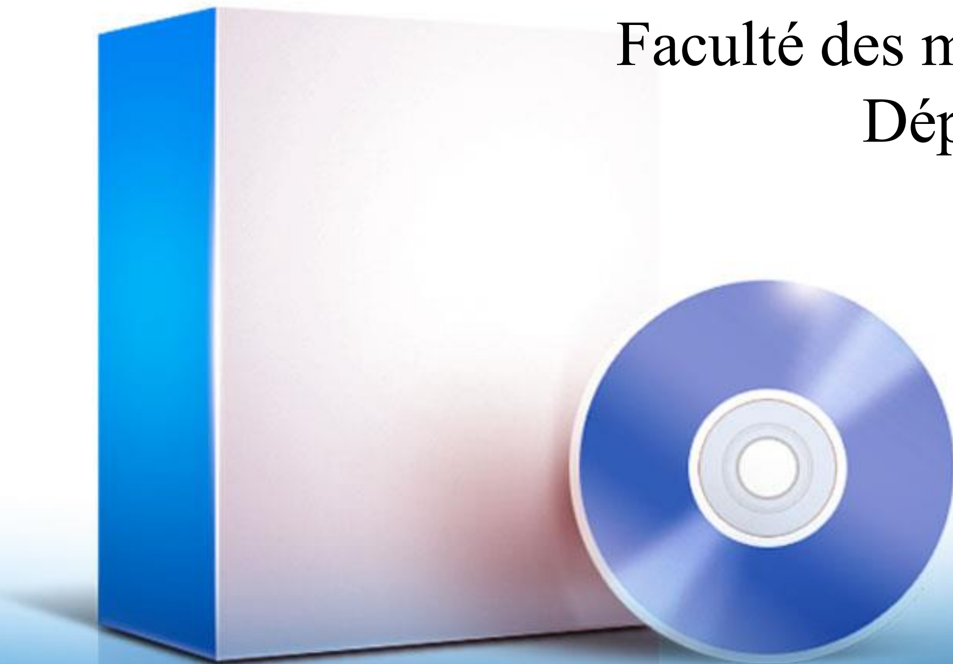

2-2-Predicate logic

[redacted]

Introduction

Consider the following example:
If x is the father of y, and if z is the father of x, then z is the grandfather of y.
If we want to express this sentence in propositional logic as:

x is the father of y ……... p

z is the father of x, ……... q

z is the grandfather of y. ……... r

$$p \wedge q \Longrightarrow r$$

The syntax of predicate logic

An expression in predicate logic is generally composed of the following elements:

Constants: {a, b, c, d ……..}

Variables: {x, y, z, w ……..}

Functions:

Predicates (**relations**): {P, Q, R, S…..}

Logical connectives: {$\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$}, the same as in propositional logic.

Example: Consider the following sentence: "Mohammed's brother really likes fruit."In this example:

The syntax of predicate logic

The brother of Samir likes the fruit very much

Function
The result of this function is of the same type as its parameter.

Constant
Because it's an instance. We didn't say "men."

Predicate

Variable

Logical quantifiers

After defining the concept of a set as a collection of elements, a need arises to express that a formula is true for a certain number of a set, or for all the elements of another. Predicate logic is an extension of propositional logic by two quantifiers:

$\forall$ The universal quantifier (for all).

$\exists$ The existential quantifier (it exists).

∀ The universal quantifier (for all)

$\forall\, x \in S : F$ which means the formula F is true for all elements of S, for example $\forall\ n \in nat : n$+1>0 asserts that the formula n+1 > 0 is true for all elements of the set of natural numbers.

Example: All Muslims love Muhammad (صلي الله عليه وسلم)

$$\forall\ x : musulman(x) \rightarrow \mathrm{love}(x, mohamed)$$

∃ The existential quantifier (it exists)

$\exists\, x \epsilon S : F$ which means that the formula F is true for some elements of the set S (at least one element), for example $\exists\, n \epsilon\, nat : n$+1=5 asserts that the formula n+1 = 5 is true for some elements of the set of natural numbers. F is true for some elements in the set S if and only if F is not false for all elements of the set S.

Example: Some Christians love Muhammad(صلي الله عليه وسلم)

$$\exists\, x : \text{Christian}(x) \rightarrow \text{love}(x, Mohamed)$$

The brother of Samir likes the fruit very much

Function — Constant — Predicate — Variable

$$\forall\, x : fruit(x) \rightarrow like(brother(Samir,)\ x)$$

$$\forall\, x : like\big(brother(Samir), fruit(x)\big)$$

Atomic terms and predicates

In predicate logic:

All constants are terms.

All variables are terms.

if $t_1, t_2, t_3 \ldots t_n$ are terms, then the function $f(t_1,t_2,t_3 \ldots t_n)$ is a term too.

If we have a predicate $P(t_1, t_2, t_3 \ldots t_n)$ such as $t_1, t_2, t_3 \ldots t_n$ are terms, then the predicate p is said to be atomic.

Example : P(a,x,f(y)), Q(x,y) …..R(f(x,y)) are all atomic predicates.

well-formed formula (WFF)

A **well-formed formula (WFF)** is a syntactically correct expression that follows all the formal grammar rules of the logical language. In simple terms, it’s a **meaningful and properly structured statement** built using:

- **Constants** (e.g., a,b,ca, b, ca,b,c)
- **Variables** (e.g., x,y,zx, y, zx,y,z)
- **Predicates** (e.g., P(x),Q(x,y)P(x), Q(x, y)P(x),Q(x,y))
- **Logical connectives** (¬, ∧, ∨, →, ↔)
- **Quantifiers** (∀, ∃)
- **Parentheses** to clarify scope and structure

well-formed formula (WFF)

Formal Definition

A **well-formed formula (WFF)** in predicate logic can be defined recursively as follows:

1-Atomic formulas:

If P is an n-ary predicate and $t_1,t_2,...,t_n$ are terms (constants, variables, or functions), then

$$P(t_1,t_2,...,t_n) \text{ is a WFF.}$$

2-Negation:

If φ a WFF, then ¬φ is also a WFF.

well-formed formula (WFF)

Formal Definition

A **well-formed formula (WFF)** in predicate logic can be defined recursively as follows:

3-Binary connectives:

If φ and ψ are WFFs, then the following are WFFs:

$$(\varphi \wedge \psi),(\varphi \vee \psi),(\varphi \rightarrow \psi), (\phi \leftrightarrow \psi)$$

4-Quantification:

If φ is a WFF and x is a variable, then

$\forall x\ \varphi$ and $\exists x\ \varphi$ are WFFs.

well-formed formula (WFF)

Example

- $P(x)$ — atomic formula (WFF)
- $\neg P(x)$ — WFF
- $(P(x) \rightarrow Q(x))$ — WFF
- $\forall x \ (P(x) \rightarrow \exists\, y\, Q(y,x))$ — WFF

- $P(\rightarrow Q(x))$

- $\forall\, x\, P(x \rightarrow y)$

Example

Question:

Translate the following sentences into predicate logic expressions:

All birds have two wings and two legs.

$$\forall\ x : \text{bird}(x) \rightarrow \text{two–wings}(x) \wedge \text{two–legs}(x)$$

Example

Question:

Each child will receive either a gold fish or a silver fish.

$$\forall\ x : \text{child}(x) \rightarrow \text{gold-fish}(x) \vee \text{silver-fish}(x)$$

The Prenex normal form:

A formula is said to be in **Prenex Normal Form (PNF)** if **all quantifiers are placed at the front** (the *prefix*) of the formula, followed by a **quantifier-free part** (the *matrix*).

Formally, a formula in PNF looks like this:

$$Q_1 x_1 \ Q_2 x_2 \ \ldots \ Q_n x_n \ [\text{quantifier-free part}]$$

where each Q_i is either $\forall$ (for all) or $\exists$ (there exists).

Proposition

Every predicate logical formula can be transformed into an equivalent formula in Prenex normal form.

Putting a formula into PNF simplifies reasoning and helps in **automated theorem proving**, **Skolemization**, and **resolution** — because the structure becomes regular and easier to manipulate.

WFF to formula in Prenex normal form:

Transforming an WFF into a formula in prenex normal form :

Step 1: Remove all implications and equivalences.

$$A \Rightarrow B \Leftrightarrow \neg A \vee B$$

$$(A \Leftrightarrow B) \Leftrightarrow (A \Rightarrow B) \wedge (B \Rightarrow A)$$

FBF en formule sous forme normal de prénexe :

Step 2: Form literals by transforming negations

$\neg(\neg A) \Leftrightarrow A$

$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$

$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$

$\neg(\exists x\, A(x)) \Leftrightarrow \forall x\, \neg A(x)$

$\neg(\forall x\, A(x)) \Leftrightarrow \exists x\, \neg A(x)$

FBF en formule sous forme normal de prénexe :

Step 3: Substitute variables to avoid conflicts.

Step 4: Place quantifiers at the beginning of the formula without changing their order of appearance.

Example

Example 1

$\exists x\, A(x) \;\Rightarrow\; \forall x\; B(x)$

Elimination of ($\Leftrightarrow$ et $\Rightarrow$) : $\neg(\exists x\; A(x)) \;\vee\; \forall x\, B(x)$

Negation Elimination : $\forall x\, \neg A(x) \;\vee\; \forall x\, B(x)$

Put ($\exists$, $\forall$) in the head : $\forall x\, (\neg A(x) \;\vee\; B(x))$

Example

Example 2

$\exists x (\neg A(x) \lor \forall x B(x))$

Substitute the variables: $\exists x (\neg A(x) \lor \forall y B(y))$

Put $(\exists, \forall)$ in the head : $\exists x \forall y (\neg A(x) \lor B(y))$ this is the Prenex normal form

Skolem Normal Form (SNF)

The **Skolem Normal Form (SNF)** is a special kind of logical formula obtained from a **Prenex Normal Form (PNF)** by **eliminating all existential quantifiers (∃)** using **Skolem functions or constants**.

After Skolemization, the formula contains **only universal quantifiers (∀)** at the front.

Eaemple

$\exists x \exists y \exists z \forall t \forall v \exists w \; p(x, y, z, t, v, w)$

First the formula is in prenex normal form, so:

$\exists x \exists y \exists z \forall t \forall v \exists w \; p(x, y, z, t, v, w)$

$\exists x$ is not preceded by a universal quantifier $\forall$, then we replace x with a constant “a”

$\exists y \exists z \forall t \forall v \exists w \; p(a, y, z, t, v, w)$

Exemple

Same thing for $\exists y \exists z$, we replace y and z by the constants b and c : $\forall t \forall v \exists w\, p(a, b, c, t, v, w)$

$\exists w$ is preceded by two universal quantifiers $\forall t \forall v$, then we replace w by the function $f(t,v)$ and we obtain:

$\forall t \forall v\, p(a, b, c, t, v, f(t,v))$ and it is a Skolem normal form and we denote it $p(a, b, c, t, v, f(t,v))$ without mentioning the quantifiers.