Chapter 4: Shortest Path Problem

4.1. Description of the problem

Let the graph G = (X, U, L) with X = {1, 2, …, n}; U = {u_1, u_2, …, u_m}

L is the set of values on the arcs called lengths.

The problem is to find a shortest path SP for one of the following cases:

- From a given vertex *x* to another given vertex *y*.
- From a given vertex *x* to all other vertices.
- From any vertex *x* to any vertex *y*.

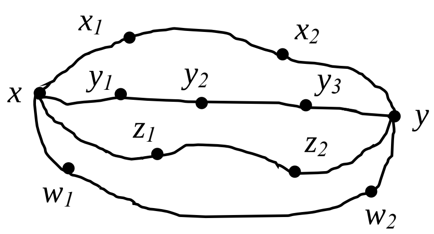

Fig 4.1: Description of the SP problem

The algorithms that will be covered in this chapter generally give the shortest paths from a vertex *x* to all other vertices (case 2)

Suppose we are looking for a SP between vertex 1 and each of vertices 2, 3, …, n

Let *i* be the current vertex, and let:

- *P(i)*: the current path between 1 and *i*, $\pi(i)$ the length of this path
- *P*(i)* a shortest path between 1 and *i*, of length $\pi^*(i)$.

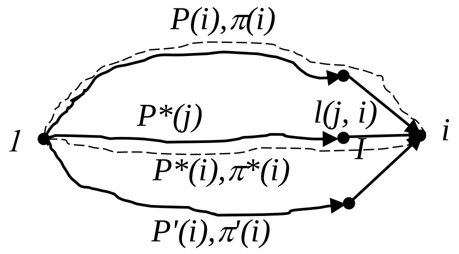

Fig 4.2: Characteristics of SP

If *j* is a predecessor of *i*, then we have:

$\pi^*(i) = Min[\ \pi(j)+l(j,\ i)]\ j \in \Gamma^-(i)$ with: $l(j,\ i)$ is the length of the arc between vertices *j* and *i*

We also have:

$\pi^*(i) \leq Min\{\ \pi(i)\ ;\pi(j) + l(j,\ i)\}\ j \in \Gamma^-(i)$

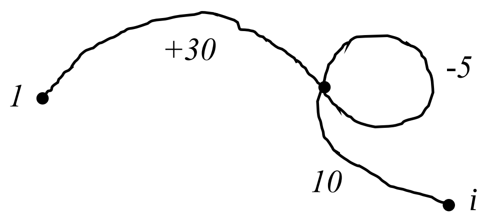

Fig 4.3: Negative Circuit Problem

A necessary condition for the existence of a SP from 1 to *i* is that there exists no path from 1 to *i* with a negative length circuit generating a path of shorter length if it is traversed a random number of times.

4.2. Some properties of shortest path

1. Any subpath of a minimum length path is itself of minimum length.

2. We can also apply this property with paths of maximum length and we say in general: any subpath of an optimum path is itself optimum.

3. The shortest path SP = $(1, x_1, x_2, \ldots, x_s)$ between 1 and x_s forms a tree with root 1 and containing all vertices x_i (i = 1, 2, …, s)

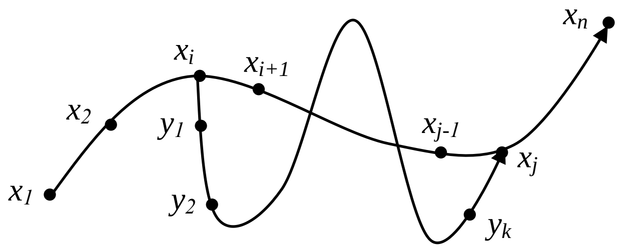

Fig 4.4: SPP and negative circuit

Example 4.1: Example of looking for a PCC between 1 and 6

$P_1(1, 6) = \{1, 2, 3, 6\} \Longrightarrow w(P_1) = 10$

$P_2(1, 6) = \{1, 3, 6\} \Longrightarrow w(P_2) = 4$ (the SP)

$P_3(1, 6) = \{1, 3, 5, 6\} \Longrightarrow w(P_3) = 13$

$P_4(1, 6) = \{1, 2, 3, 5, 6\} \Longrightarrow w(P_4) = 19$

P5(1, 6) = {1, 4, 3, 6} ==> $w(P_5) = 5$

$P_6(1, 6) = \{1, 4, 3, 5, 6\} \Longrightarrow w(P_6) = 14$

$P_7(1, 6) = \{1, 4, 5, 6\} \Longrightarrow w(P_7) = 13$

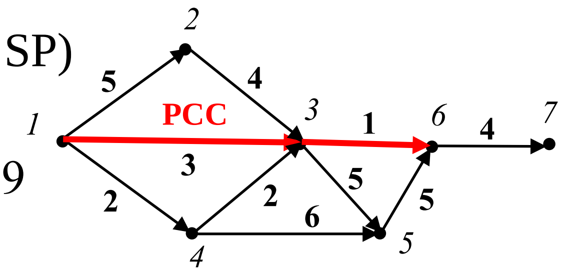

Fig 4.5: Example of SP search

4.3. Shortest path search algorithms

4.3.1. Dijkstra's algorithm:

This algorithm gives all the shortest paths from a given vertex to all other vertices in the graph

i) $S = \{s\}$;$\alpha = s$;$\pi(s) = \pi(\alpha) = 0$;$\pi(x) = +\infty \ \ \forall x \in (X - S)$; $A(x) = \phi \ \ \forall x \in (X - S)$

ii) We consider all arcs u such that: $I(u) = \alpha$ and $T(u) = x \in (X - S)$

<u>**If**</u> $\pi(\alpha)+d(u)<\pi(x)$ then put $\pi(x) = \pi(\alpha)+d(u)$; $A(x) = \{u\}$

Otherwise continue

iii) Choose a vertex $y \in (X - S)$; $\pi(y) = \text{Min}[\pi(x)] \ \forall x \in (X - S)$

<u>**If**</u> $\pi(y) = +\infty$ then stop S is not a root of G

Otherwise put $\alpha = y$ $S = S \cup \{\alpha\}$

If S = X then finish {problem is solved}

Otherwise go to (i)

Example 4.3: Application of Dijkstra's algorithm

i) $S = \{s\}$;$\alpha = s$;$\pi(s) = \pi(\alpha) = 0$;
$\pi(x) = +\infty \ \ \forall x \in (X - S)$;
$A(x) = \phi \ \ \forall x \in (X - S)$

ii) We consider all arcs u
such that: $I(u) = \alpha$ and $T(u) = x \in (X - S)$
we have: (s, 1), (s, 2)

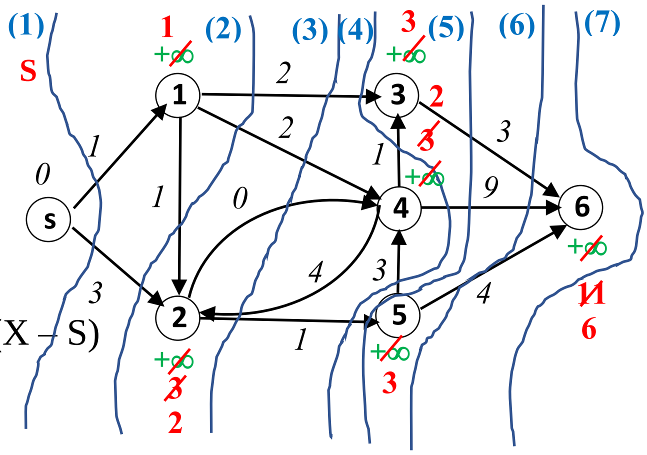

Fig 4.7: Different phases of Dijkstra's algorithm

For (s, 1):$\pi(1) = 0+1=1$; $A(1) = \{(s, 1)\}$
For (s, 2):$\pi(2) = 0+3=3$; $A(2) = \{(s, 2)\}$

iii) Choose a vertex $y \in (X - S)$; $\pi(y) = Min[\pi(x)]$ $\forall x \in (X - S)$

$y = 1$; $\alpha = 1$; $S = \{s, 1\}$

Go to (ii): We consider all arcs u such that: $I(u) = 1$ and $T(u) = x \in (X - S)$

We have: (1, 2), (1, 3), (1, 4)

For (1, 2):$\pi(2) = 1+1=2 < \pi(2) = 3$; $A(2) = \{(1, 2)\}$

For (1, 3):$\pi(3) = 1+2=3 < \pi(3) = +\infty$; $A(3) = \{(1, 3)\}$

For (1, 4):$\pi(4) = 1+2=3 < \pi(4) = +\infty$; $A(4) = \{(1, 4)\}$

iii) Choose a vertex $y \in (X - S)$;$\pi(y) = Min[\pi(x)]$ $\forall x \in (X - S)$

$y = 2$;$\alpha = y = 2$; $S = \{s, 1, 2\}$

Go to (ii): We consider all the arcs u such that: $I(u) = \alpha = 2$ and $T(u) = x \in (X - S)$

We have: (2, 4), (2, 5)

For (2, 4):$\pi(4) = 2+0=2 < \pi(4) = 3$; $A(4) = \{(2, 4)\}$

For (2, 5):$\pi(5) = 2+1=3 < \pi(4) = +\infty$; $A(5) = \{(2, 5)\}$

iii) Choose a vertex $y \in (X - S)$;$\pi(y) = Min[\pi(x)]$ $\forall x \in (X - S)$

$y = 4$;$\alpha = y = 4$; $S = \{s, 1, 2, 4\}$

Go to (ii): We have the arcs: (4, 3), (4, 6)

For (4, 3):$\pi(3) = 2+1=3 = \pi(3) = 3$; $A(3) = \{(4, 3)\}$

For (4, 6):$\pi(6) = 2+9=3 < \pi(6) = +11$; $A(6) = \{(4, 6)\}$

iii) Choose a vertex $y \in (X - S)$;$\pi(y) = Min[\pi(x)]$ $\forall x \in (X - S)$

$y = 3$;$\alpha = y = 3$; $S = \{s, 1, 2, 4, 3\}$

Go to (ii): We have the arcs: (3, 6)

$\pi(6) = 3+3=6 < \pi(6) = 11$; $A(6) = \{(3, 6)\}$

iii) Choose a vertex $y \in (X - S)$;$\pi(y) = Min[\pi(x)]$ $\forall x \in (X - S)$

$y = 5$;$\alpha = y = 5$; $S = \{s, 1, 2, 4, 3, 5\}$

Go to (ii): We have the arcs: (5, 6)

$\pi(6) = 3+4=7 > \pi(6) = 6$; $A(6) = \{(3, 6)\}$

iii) Choose a vertex $y \in (X - S)$;$\pi(y) = Min[\pi(x)]$ $\forall x \in (X - S)$

$y = 6$;$\alpha = y = 6$; $S = \{s, 1, 2, 4, 3, 5, 6\}$

We see that $S = X$ ==> Finish the algorithm and the problem is solved

4.3.2. Bellman Algorithm:

The graph must be without circuit, and l(u) arbitrary

i) $S = \{s\}$;$\pi(s) = 0$; $A = \phi$

ii) Find a vertex $x \in (X - S)$ such that all its predecessors are in S $(\Gamma^{-}(x) \in S)$

<u>If</u> (x does not exist) then end with:

$$\begin{cases} S\ n'est pas\ une\ racine\ ou\ \exists\ circuit\ dans\ G \\ S = X \end{cases}$$

<u>Otherwise</u> $(\exists x \in (X - S)$ then continue

iii) Put $\pi(x) = \mathrm{Min}[\pi(I(u))+d(u)]$ with $I(u) \in S$, $T(u)=x$

$S \cup \{x\}$; $A = A \cup \{u\}$; go to (ii)

Example 4.4: Application of Bellman's algorithm

Let the following graph G(X, U, C) be Without circuit, l(u) is arbitrary

i) $S = \{s\}$; $\pi(s) = 0$; $A = \phi$

ii) Find a vertex $x \in (X - S)$
such as $(\Gamma^{-}(x) \in S)$
x = 1 (x = 2 no, because $\Gamma^{-}(2) \notin S$)

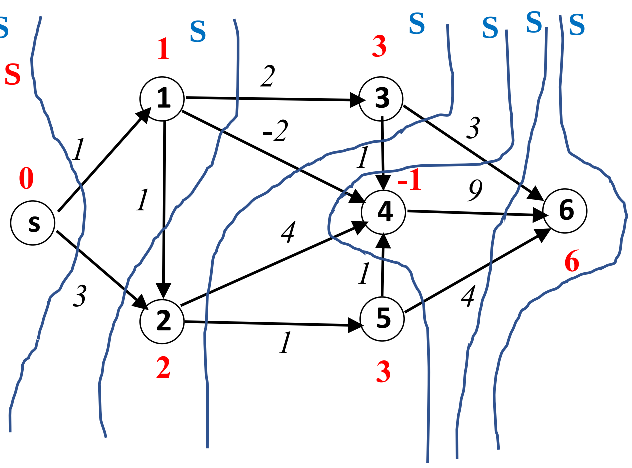

Fig 4.8: Different phases of Bellmann's algorithm

For x = 1 $\pi(1) = \mathrm{Min}[\pi(s)+1] = \mathrm{Min}[0 + 1] = 1$, $S = \{S, 1\}$; $A = \{(S, 1)\}$

Go to (ii): Find a vertex $x \in (X - S)$ such that $(\Gamma^{-}(x) \in S)$

x = 2 ; x = 3

For x = 2 $\pi(2) = \mathrm{Min}[0+3, 1+1] = 2$, $S = \{S, 1, 2\}$; $A = \{(S, 1), (1, 2)\}$

For x = 3 $\pi(3) = \mathrm{Min}[1+2] = 3$, $S = \{S, 1, 2, 3\}$; $A = \{(S, 1), (1, 2), (1, 3)\}$

Go to (ii): Find a vertex $x \in (X - S)$ such that $(\Gamma^{-}(x) \in S)$

$x = 5$
$\pi(5) = Min[2+1] = 3$, $S = \{S, 1, 2, 3, 5\}$; $A = \{(S, 1), (1, 2), (1, 3), (2, 5)\}$
(4 no, 6 no)

Go to (ii): Find a vertex $x \in (X - S)$ such that $(\Gamma^{-}(x) \in S)$
$x = 4$; $\pi(4) = Min[1-2, 3+1, 2+4]=-1$, $S=\{S, 1, 2, 3, 5, 4\}$; $A=\{(S, 1), (1, 2), (1, 3), (2, 5), (1, 4)\}$

Go to (ii): Find a vertex $x \in (X - S)$ such that $(\Gamma^{-}(x) \in S)$
$x = 6$; $\pi(6) = Min[3+3, -1+9, 3+4] = 6$, $S = \{S, 1, 2, 3, 5, 4, 6\}$; $A = \{(S, 1), (1, 2), (1, 3), (2, 5), (1, 4), (3, 6)\}$

Go to (ii): Find a vertex $x \in (X - S)$ such that $(\Gamma^{-}(x) \in S)$
$S = X$ Finish