

MOHAMED BOUDIAF UNIVERSITY OF M'SILA

DEPARTMENT OF COMPUTER SCIENCE

The Foundations of Graph Theory

Chapter 3: Coloring problem in graphs

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Coloring problem in graphs

Problem Statement

Let $G(X, U)$ be an undirected graph.

There are three types of coloring in G .

1. Vertices' Coloring.
2. Edges' Coloring.
3. Faces' Coloring.

Definition

- Coloring the vertices/edges/faces of a graph G consists of assigning a color to each of the vertices (edges/faces) so that no two adjacent vertices (edges/faces) have the same color.
- A graph is said to be **p -chromatic** if its vertices admit a coloring in p colors.
- We call **chromatic number $\gamma(G)$** (chromatic index **$q(G)$** for edge coloring) the minimum number of different colors needed to perform a vertex coloring (edges/faces) of G .

Vertices' Coloring

$G(X, U)$ an undirected graph

- Let I be a subset of vertices $I \subset X$, I is said **stable set** if it only includes non-adjacent vertices two by two (any two vertices of I are not adjacent).
- Vertices with the same color in a vertex coloring form a stable set (non-adjacent).
- So, a vertices' coloring of G is just a partition of G into stable sets.
- We define $\alpha(G)$ **the stability number** which is **the maximum cardinality of a stable set**. So if $\gamma(G)$ is the chromatic number, and since every set of vertices of the same color has a cardinality less than or equal to $\alpha(G)$ so we have:

$$\alpha(G) \cdot \gamma(G) \geq N(G)$$

With:

$N(G) = |X|$ is the number of vertices of the graph (the order)

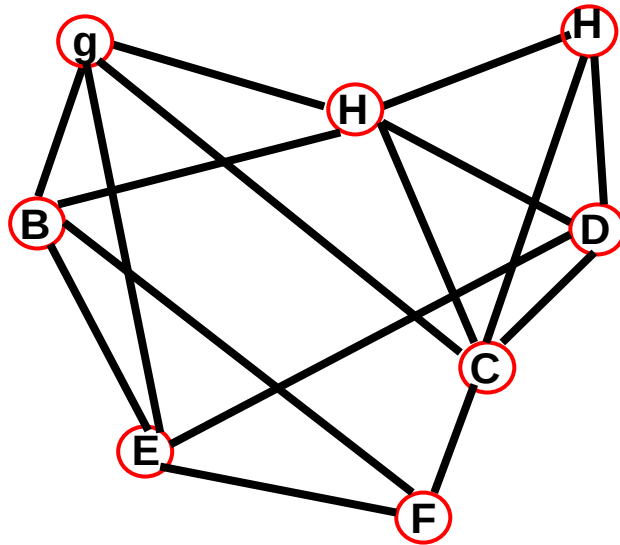
We can deduce:

- $\gamma(G) \geq \lceil N(G)/\alpha(G) \rceil$
- Note that the determination of the chromatic number $\gamma(G)$, as well as obtaining a minimal coloring of the vertices of G is a complex problem and requires the use of heuristic coloring algorithms.

- We should also note that $\gamma(G)$ for a graph with n vertices and m edges is defined under the following conditions:
 - ✓ $\gamma(G) \leq (n - d_{\min})$, with: $d_{\min}$ the minimum degree of the vertices in G .
 - ✓ $\gamma(G) \geq \text{the cardinal of the largest clique in } G$
 - ✓ $\gamma(G) \geq n^2 / (n^2 - 2m)$
 - ✓ $\gamma(G) \leq (n+1 - \alpha(G))$ with $\alpha(G)$ the stability number of G .
 - ✓ $\gamma(G) \leq d_{\max} + 1$ with $d_{\max}$ the maximum degree of vertices in G .

OBS: The above properties are demonstrated in the literature.

Coloring algorithm (Welsh and Powell)



1. Put the vertices of the graph G in decreasing order of degree $x_1, x_2, \dots, x_n$ with $d(x_i) \geq d(x_j)$ for $1 \leq j \leq n$

Summit	Degree
A	5
C	5
B	4
D	4
E	4
G	4
F	3
H	3

2. Assign color c_1 to x_1 and the next vertex in the list that is not adjacent to x_1 and not already colored, and so on with vertices in the list that are not adjacent to vertices already colored with color c_1 .

3. Assign color c_2 to the first vertex not yet colored, as well as to subsequent vertices that are not adjacent to vertices already colored with color c_2

4. Continue the process of Coloring using New colors $c_3, c_4, \dots, c_k$ until we finish coloring all the vertices in the list.

Summit	Degree	Color
A	5	C1
C	5	C2
B	4	C2
D	4	C3
E	4	C1
G	4	C3
F	3	C3
H	3	C4

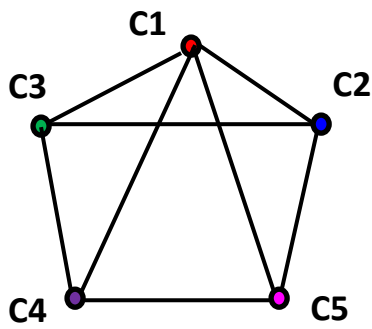
The coloring result of the previous example is as follows:

- The color c_1 is assigned to the vertices {A, E}
- The color c_2 is assigned to the vertices {C, B}
- The color c_3 is assigned to the vertices {D, F, G}
- The color c_4 is assigned to the vertex {H}

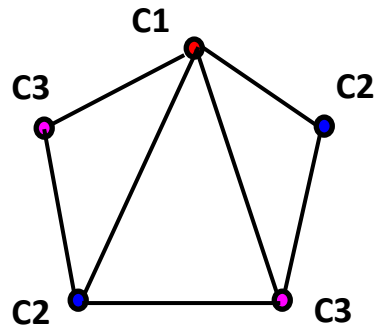
We say that G is 4-colorable

OBS : a bipartite graph is a 2-colorable graph

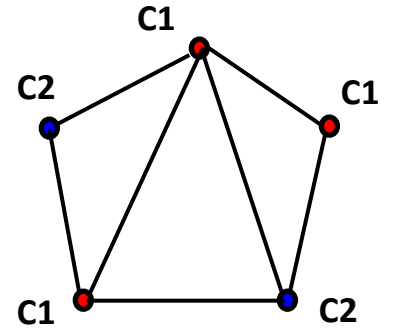
Coloring examples



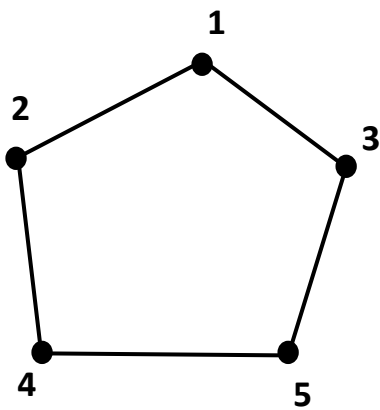
5 colors



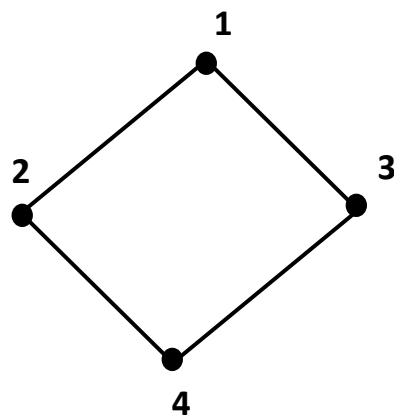
3 colors (sufficient)
 $\gamma(G) = 3$



Incorrect
coloring



Nb. vertices = 5 (odd)
 $\Rightarrow \gamma(G) = 3$
(3 needed colors)



Nb. vertices = 4 (even)
 $\Rightarrow \gamma(G) = 2$
(2 needed colors)

Note: Any planar graph is colorable in 04 colors.

Edges' Coloring

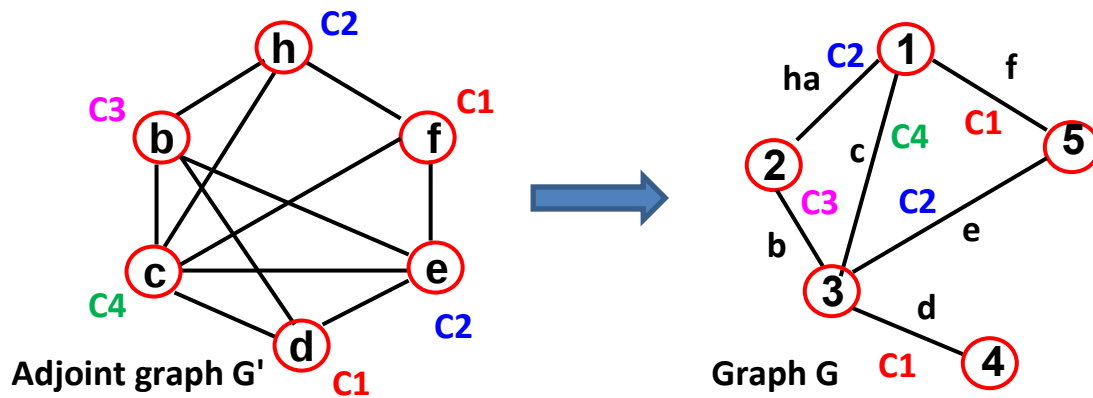
- Similar to vertices' coloring, edges' coloring of a graph involves assigning a color to any edge of the graph so that no two adjacent edges have the same color.
- We define the chromatic index of the graph G (similar to the chromatic number for vertices' coloring) as being the smallest k for which there exists a coloring of the edges, it is noted $\chi(G)$.
- Edges' Coloring of a graph can be reduced to vertices' coloring by working not on the graph itself, but on the adjoint graph G' which is defined as follows:
 1. To each edge of $G(X, U)$, we associate a vertex of $G'(X', U')$.
 2. Two vertices of G' are connected by an edge, if the two corresponding edges of G are adjacent.

Example :



Subsequently, we apply the WELSH POWELL algorithm on the graph G' to color its vertices. Once this coloring is

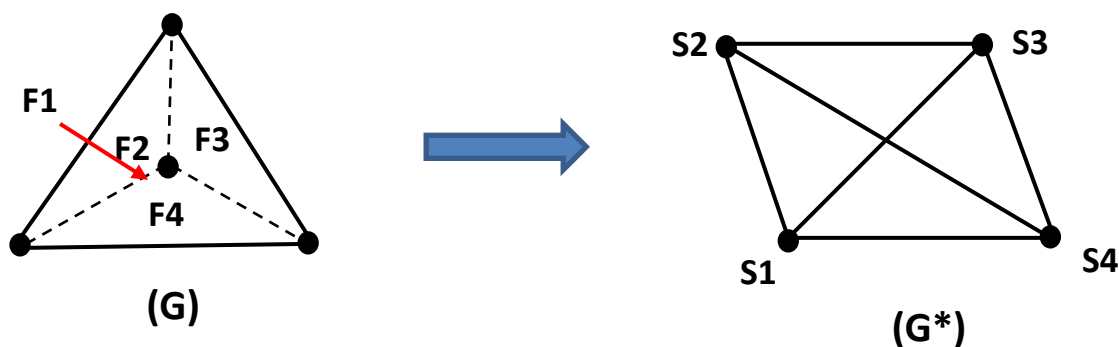
done, we will color the edges of G with the same colors of the corresponding vertices.



Coloring faces

Definition of dual of a graph

- Let $G(X, U)$ be a planar multigraph, the dual G^* of G is the graph defined as follows:
- The faces of G are represented by vertices in G^*
- For each edge of G belonging to the boundary of two faces $F1, F2$, there corresponds an edge in G^* connecting the vertices $S1, S2$ corresponding to the faces $F1, F2$.
- The Dual of a planar graph G is also a planar graph.
 $(G^*)^* = G$



And so, the problem of coloring the faces of a planar graph G comes down to coloring the vertices of its dual graph G^* and vice versa.