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Academic Year 2025/2026
Level: L2
Duration: 2 H 00 min

Final Exam of Numerical Methods (Normal Session)

Exercise 1 : (10 points)

1. **A**. Write an algorithm to solve an $N \times N$ system of linear equations using Gaussian elimination.

   B. Calculate the number of elementary arithmetic operations that this algorithm must perform (complexity analysis).

2. **A**. Write an algorithm to solve a tri-diagonal linear system of order **N**.

   B. Calculate the number of elementary arithmetic operations that this algorithm must perform (complexity analysis).

3. **A**. Write an algorithm to compute an approximate solution of an $N \times N$ system of linear equations using the Gauss-Seidel method after **K** iterations.

   B. Calculate the number of elementary arithmetic operations that this algorithm must perform (complexity analysis).

Exercise 2 : (5 points)

Consider the linear system **(S)**:

$$(S): \begin{cases} x_1 + x_2 + 4x_3 + 2x_4 = -2 \\ 4x_1 + x_2 - x_3 - x_4 = 3 \\ 2x_1 + 4x_2 + 2x_3 + x_4 = -3 \\ x_1 + x_2 - x_3 + 3x_4 = 4 \end{cases}$$

1. Express the system **(S)** in the matrix form **AX=b**.
2. Solve the system **(S)** using Gaussian elimination.
3. Deduce the determinant of matrix **A**.

Exercise 3 : (5 points)

Consider the linear system **(S')**:

$$(S'): \begin{cases} 0.4x_1 - 0.1x_3 - 0.1x_4 = 0.4 \\ 0.2x_1 + 0.4x_2 + 0.1x_4 = -0.1 \\ 0.1x_1 + 0.4x_3 + 0.2x_4 = -0.1 \\ 0.1x_1 - 0.1x_3 + 0.3x_4 = 0.5 \end{cases}$$

1. Verify that the Gauss–Seidel iterative method associated with the linear system **(S')** converges for any initial vector $X^{(0)} \in \mathbb{R}^4$.
2. Compute the first four (4) iterations of the Gauss-Seidel method for the system **(S')** starting from $\boldsymbol{X^{(0)}} \begin{pmatrix} \mathbf{1.5} \\ \mathbf{-1.25} \\ \mathbf{-1.25} \\ \mathbf{1} \end{pmatrix}$, and estimate the error (round to $\mathbf{10^{-4}}$).

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Correction of Final Exam (Normal Session)

Exercise 1 : (10 points)

1. **An algorithm to solve a linear system using Gaussian elimination.**

- Given the matrix A and the vector b.
- The Gaussian elimination algorithm transforms the matrix A into an upper triangular form. It proceeds as follows:

$$\begin{cases} \text{For } k = 1 \text{ to } n-1 \\ \begin{cases} \text{For } i = k+1 \text{ to } n\text{:} \\ \mathrm{P} \leftarrow \dfrac{A_{i,k}^{(k)}}{A_{k,k}^{(k)}} \\ b_i \leftarrow b_i - \mathrm{P}\, b_k \\ \begin{cases} \text{For } j = k \text{ to } n\text{:} \\ A_{i,j} \leftarrow A_{i,j} - \mathrm{P}\, A_{k,j} \end{cases} \end{cases} \end{cases}$$

- Next, the solution of the system is obtained using the backward substitution algorithm:

$$\begin{cases} x_n = \dfrac{b_n}{A_{n,n}} \\ \text{For } i = n-1 \text{ to } 1\text{:} \\ x_i = \dfrac{1}{A_{i,i}}\left(b_i - \displaystyle\sum_{j=i+1}^{n} A_{i,j} x_j\right) \end{cases}$$

A. Algorithm ***(2 points)***

....

B. Complexity analysis ***(2 points)***

....

2. **An algorithm to solve a tri-diagonal linear system of order N.**

A tridiagonal system has the following structure:

$$\underbrace{\begin{pmatrix} a_1 & v_1 & 0 & 0 \\ c_2 & a_2 & \dots & 0 \\ 0 & \dots & \dots & v_{N-1} \\ 0 & 0 & c_N & a_N \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_N \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_N \end{pmatrix}}_{b}$$

- Given the vectors a, v, c *and* b.
- ,Step 1: $\begin{cases} u_1 = a_1 \\ l_i = \frac{c_i}{u_{i-1}}, \quad for\ i = 2\ to\ N \\ u_i = a_i - l_i \times v_{i-1}, \quad for\ i = 2\ to\ N \end{cases}$
- Step 2: $\begin{cases} y_1 = b_1 \\ y_i = b_i - l_i \times y_{i-1}, \quad for\ i = 2\ to\ N \end{cases}$

Step 3: $\begin{cases} x_N = \frac{y_N}{u_N} \\ x_i = \frac{(y_i - v_i \times x_{i+1})}{u_i}, \quad for\ i = 1\ to\ N-1 \end{cases}$

A. Algorithm *(2 points)*

....

B. Complexity analysis *(1 points)*

....

3. **An algorithm to compute an approximate solution of a linear system of order N using the Gauss-Seidel method after K iterations.**

- Given $A, b, X^{(0)}\left(x_1^{(0)}; x_2^{(0)}; \ldots; x_n^{(0)}\right) \in \mathbb{R}^n$, **Tol** as the precision, and/or **Nb_iter** as the number of iterations,
- for k=1 to Nb_iter, or Error > Tol, we compute:

$$x_i^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} A_{i,j} x_j^{(k)} - \sum_{j=i+1}^{n} A_{i,j} x_j^{(k+1)}\right) / A_{i,i};\ \text{for } i = 1 \text{ to } n$$

$$Error = \left\| X^{(k+1)} - X^{(k)} \right\|$$

A. Algorithm *(2 points)*

....

B. Complexity analysis *(1 points)*

....

Exercise 2 : (5 points)

1. **The matrix form AX=b.** (0.5 *points*)

$$\underbrace{\begin{pmatrix} 1 & 1 & 4 & 2 \\ 4 & 1 & -1 & -1 \\ 2 & 4 & 2 & 1 \\ 1 & 1 & -1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} -2 \\ 3 \\ -3 \\ 4 \end{pmatrix}}_{b}$$

2. **The resolution of (S) using Gaussian elimination.** (4 *points*)

Step 1 : $\left(\begin{array}{cccc|c} 1 & 1 & 4 & 2 & -2 \\ 4 & 1 & -1 & -1 & 3 \\ 2 & 4 & 2 & 1 & -3 \\ 1 & 1 & -1 & 3 & 4 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 - 4L_1 \\ L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array}$; Step 2 : $\left(\begin{array}{cccc|c} 1 & 1 & 4 & 2 & -2 \\ 0 & -3 & -17 & -9 & 11 \\ 0 & 2 & -6 & -3 & 1 \\ 0 & 0 & -5 & 1 & 6 \end{array}\right) L_3 \leftarrow L_3 + \frac{2}{3} L_2$

Step 3 : $\left(\begin{array}{cccc|c} 1 & 1 & 4 & 2 & -2 \\ 0 & -3 & -17 & -9 & 11 \\ 0 & 0 & -\frac{52}{3} & -9 & \frac{25}{3} \\ 0 & 0 & -5 & 1 & 6 \end{array}\right) L_4 \leftarrow L_4 - \frac{15}{52} L_3$; we obtain : $\left(\begin{array}{cccc|c} 1 & 1 & 4 & 2 & -2 \\ 0 & -3 & -17 & -9 & 11 \\ 0 & 0 & -\frac{52}{3} & -9 & \frac{25}{3} \\ 0 & 0 & 0 & \frac{187}{52} & \frac{187}{52} \end{array}\right)$ ...... (3 points)

$$(S) \Leftrightarrow \begin{cases} x_1 + x_2 + 4x_3 + 2x_4 = -2 \\ -3x_2 - 17x_3 - 9x_4 = 11 \\ -\frac{52}{3} x_3 - 9x_4 = \frac{25}{3} \\ \frac{187}{52} x_4 = \frac{187}{52} \end{cases} \quad \text{The solution of (S) is : } \begin{cases} x_1 = 1 \\ x_2 = -1 \\ x_3 = -1 \\ x_4 = 1 \end{cases}$$ .... (1 points)

3. **The determinant of matrix A.** *(0.5 points)*

$$\det(A) = (1)(-3)\left(-\frac{52}{3}\right)\left(\frac{187}{52}\right) = 187.$$

Exercise 3 : (5 points)

1. **The convergence of the Gauss-Seidel method** (0.5 points)

The matrix form AX=b.

$$\underbrace{\begin{pmatrix} 0.4 & 0 & -0.1 & -0.1 \\ 0.2 & 0.4 & 0 & 0.1 \\ 0.1 & 0 & 0.4 & 0.2 \\ 0.1 & 0 & -0.1 & 0.3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0.4 \\ -0.1 \\ -0.1 \\ 0.5 \end{pmatrix}}_{b}$$

We have : $\begin{cases} |0.4| > |-0.1| + |-0.1| \\ |0.4| > |0.2| + |0.1| \\ |0.4| > |0.1| + |0.2| \\ |0.3| > |0.1| + |-0.1| \end{cases}$ .The matrix A is strictly diagonally dominant, then the Gauss-Seidel iterative method associated with the system (S′) converges for any initial vector $X^{(0)} \in \mathbb{R}^4$.

2. **The first four (4) iterations of the Gauss-Seidel method for the system (S′)** (4.5 points)

The iterative Gauss-Seidel processes: (0.5 points)

$$\begin{cases} x_1^{(k+1)} = \dfrac{\left(0.4 + 0.1\, x_3^{(k)} + 0.1\, x_4^{(k)}\right)}{0.4} \\ x_2^{(k+1)} = \dfrac{\left(-0.1 - 0.2\, x_1^{(k+1)} - 0.1\, x_4^{(k)}\right)}{0.4} \\ x_3^{(k+1)} = \dfrac{\left(-0.1 - 0.1\, x_1^{(k+1)} - 0.2\, x_4^{(k+1)}\right)}{0.4} \\ x_4^{(k+1)} = \dfrac{\left(0.5 - 0.1\, x_1^{(k+1)} + 0.1\, x_3^{(k)}\right)}{0.3} \end{cases}$$

The first four (4) iterations of the Gauss-Seidel method: (4 points)

K	*0*	*1*	*2*	*3*	*4*
x_1	**1.5**	**0.9375**	**1.0104**	**0.9939**	**1.0025**
x_2	**-1.25**	**-0.9688**	**-1.0117**	**-0.9948**	**-1.0022**
X_3	**-1.25**	**-0.9844**	**-1.0156**	**-0.9941**	**-1.0026**
X_4	**1**	**1.0260**	**0.9913**	**1.0040**	**0.9983**
Error		**0.5625**	**0.0729**	**0.0215**	**0.0085**