Predicate Logic - Semantics

Introduction:

Well-formed formula + Interpretation + Valuation = Truth Value

- An interpretation I consists of:
  - A non-empty set D called the domain of individuals.
  - A relation for each predicate symbol P. $I_P : D^n \longrightarrow \{0,1\}$
  - One operation for each function symbol f. $I_f : D^n \longrightarrow D$
  - One element for each constant symbol a_i. $I_a : d_i \in D$

Example: Let the formula $F = P\ (f(x, y), y)$

- We can find two possible interpretations for this formula:

• D=$\mathbb{Z}$ • $f(x, y)$=,$x - y$ • $P\ (x, y)$= «$x\ >\ y$ »	• D= $\mathbb{R}$ • $f(x, y)$=,$(x^2 - y)/2$ • $P\ (x, y)$= «$x\ =\ y$»

- A valuation is a function that assigns to each variable an element of D.
- The valuation of a formula depends on the valuation of the formula's components: atomic formulas, logical connectives, and qualifiers.

True formula: A formula A is true for an interpretation I if v(A)= 1 (true) in I.

False formula: A formula A is false for an interpretation I if $v(A)$= 0 (false) in I.

Example: Formula: $F = P\ (f(x, y), y)$

Interpretation: D=$\mathbb{N}$, $f(x, y)$= $x - y$, $P\ (x, y)$ = «$x\ >\ y$ »

Valuation:

- V(x) = 8 and V(y)= 2 then the truth value of F is: True
- V(x) = 3 and V(y)= 2 then the truth value of F is: false

Important note:

- ∀x(P) is true iff P(x) is true for any interpretation$I$.

- To prove that a formula as $\exists x\, P(x)$ is true, it suffices to check the proposition for one element.
- To prove that a formula as $\forall x\, P(x)$ is false, it is sufficient to verify it with a counterexample

Examples:

- Let the formula $\exists x\, P(x)$ and the interpretation be, D= $\mathbb{N}$ and $P(x) =$ « $x > 5$»
  - If we assume x= 7 then $\exists x\, P(x)$ is true. Then the formula is true.
- Let the formula $\forall x\, P(x)$ and the interpretation be, D = $\mathbb{N}$ and $P(x) =$ « $x > 5$»
  - We have a counterexample x=2, then the formula is false,

Satisfiable formula:

- A formula A is said to be satisfied if there exists an interpretation I and a valuation v (with assignment of the free variables) such that A is true in $I$and v.
- We note $I \vDash A_v$

Example: Let the formula F= $P\ (f(x,y),y)$

- Interpretation: D=$\mathbb{N}$, $f(x,y)$= $x - y$, $P\ (x,y)$= «$x > y$»
- For $V(x)$= 5 and $V(y)$= 2 then $v(F) = P(f(5,2),2)$. $v(F)$ is true. This valuation satisfies F for $I(I \vDash Av)$
- For $V(x)$= 5 and $V(y)$= 4 then $v(F) = P(f(5,4),4)$. $v(F)$is false. This valuation does not satisfy F for I ($I \nvDash Av$).
- The formula is satisfiable.F

Unsatisfiable formula:

- A formula A is said to be unsatisfied if its truth value is false for all interpretations and all valuations.

Example:

$$F = \exists x\ (P(x,y) \land \neg P(x,y))$$

F is an unsatisfiable formula.

Valid formula:

- Formula A is valid if its truth value is true for all interpretations and all valuations.

- We note ⊨A means A is true in all I and whatever the valuation.
- There are several techniques to check whether a formula is valid or not:
  - Analyzes all possible cases,
  - by the absurd,
  - find a counterexample.

Example:

Show that $F = \forall x \, \forall y \, ((x \neq y) \lor P(x, y) \lor \neg P(y, x))$ is valid.

- There are two possible cases, whatever the interpretation I considered:
  - x = y: $P(x, y) \lor \neg P(y, x) = P(x, x) \lor \neg P(x, x)$ is always true.
  - x ≠ y: $(x \neq y)$ is true because the first predicate is true.
- So is valid (⊨F).

PRENEX NORMAL FORM

Definition: A formula F is said to be in prenex form iff it is of the form:

$$Q1x1 \ \ldots \ Qnxn \ A',$$

- Qi : a quantifier ∀Or ∃
- A' : formula does not contain quantifiers.

Theorem: For every formula A there exists a formula A' in prenex form such as: $A \equiv A'$

Algorithm:

1. Rename variables to avoid conflicts.
2. Elimination of implications and equivalences.
3. Put quantifiers at the beginning of the formula.

Rename variables to avoid conflicts.

Variables are renamed in the following cases:

- Bound variables of different scopes.
- Free and bound variable conflict.

- Variable dependent on several scopes.

Examples:

- $\forall x\ (\exists y\ P(x,y) \wedge \exists x\ Q(x,A)\ )$ After: $\forall x\ (\exists y\ P(x,y) \wedge \exists z\ Q(z,A)\ )$
- $\forall x\ (\exists y\ P(x,y) \wedge M(x,y)\ )$ After: $\forall x\ (\exists y\ P(x,y) \wedge M(x,z)\ )$
- $\forall x \forall z\ (\exists y\ P(x,y) \wedge \exists y\ Q(z,y)\ )$ After: $\forall x \forall z\ (\exists y\ P(x,y) \wedge \exists w\ Q(z,w)\ )$

Eliminate implications and equivalences

As in propositional logic:

- $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- $P \Rightarrow Q \equiv \neg P \vee Q$

Put quantifiers at the beginning of the formula

For this we use the following equivalences:

- $(\forall x\ A) \vee B \equiv \forall x\ (A \vee B)$ iff x is not bound in B
- $(\exists x\ A) \vee B \equiv \exists x\ (A \vee B)$ iff x is not bound in B
- $(\forall x\ A) \wedge B \equiv \forall x\ (A \wedge B)$ iff x is not bound in B
- $(\exists x\ A) \wedge B \equiv \exists x\ (A \wedge B)$ iff x is not bound in B

Example 1: $\forall x(P(x) \Rightarrow \exists x Q(x))$

1. Rename variables to avoid conflicts:

► $\forall x(P(x) \Rightarrow \exists y Q(y))$

2. Eliminate implications and equivalences:

► $\forall x(\neg P(x) \vee \exists y Q(y))$

3. Put quantifiers at the beginning of the formula:

► $\forall x \exists y\ (\neg P(x) \vee Q(y))$ PNF

SKOLEM NORMAL FORM (SNF)

Skolemization is about replacing existentially quantified variables:

• by a constant if the variable is within the scope of no other universal quantification.

• by a function $f(x1, \dots, xn)$ where the xi are the universally quantified variables preceding the existentially quantified variable. (f is called Skolem's function).

Examples:

$\exists x \forall y\ A(x,y)$ becomes $\forall y\ A(B,y)$ (replace x by the constant B).

$\forall y \exists x A(x,y)$ becomes $\forall y\ A(f(y),y)$ (replace x by the function f(y)).

$\forall y \forall z \exists x A(x,y)$ becomes $\forall y\ A(f(y,z),y)$ (replace x by the function f(y,z)).

CONJUNCTIVE NORMAL FORM (CNF)

- The formula is written in SNF
- Distribution of $\vee$ on $\wedge$.

As for the logic of propositions:

- $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$

Example:

- $(A(f(x)) \wedge \neg M(x,f(x))) \vee M(g(x),x)$
- The formula is in CNF:
- $(A(f(x)) \vee M(g(x),x)) \wedge (\neg M(x,f(x)) \vee M(g(x),x))$

Exercise :

Put the following formula in conjective normal form

$\forall x (\forall y\ (P(y) \Rightarrow Q(x,y))) \Rightarrow (\exists y\ Q(y,x))$

1. Rename variables to avoid conflicts.

$\forall x (\forall y\ (P(y) \Rightarrow \neg Q(x,y))) \Rightarrow (\exists y\ Q(y,x))$

$\forall x (\forall y\ (P(y) \Rightarrow \neg Q(x,y))) \Rightarrow (\exists z\ Q(z,x))$

2. Elimination of implications and equivalences

$\forall x\,(\forall y\,(P(y)\Rightarrow Q(x,y)))\Rightarrow(\exists z Q(z,x))$

$\forall\ x\,(\forall y(\neg P(y)\vee Q(x,y)))\Rightarrow(\exists z Q(z,x))$

$\forall\ x\,(\neg(\forall y(\neg P(y)\vee Q(x,y)))\vee(\exists z\ Q(z,x))$

$\forall x\,(\exists y\neg(\neg P(y)\vee Q(x,y)))\vee(\exists z\ Q(z,x))$

$\forall x\,(\exists y\,(P(y)\wedge \neg Q(x,y)))\vee(\exists z\ Q(z,x))$

3. **Put the quantifiers at the beginning of the formula, respecting their order.**

$\forall x(\exists y\,(P\,(y)\wedge \neg Q(x,y)))\vee(\exists z\ Q(z,x))$

$\forall x\exists y\exists z\,((P(y)\wedge \neg Q(x,y))\vee Q(z,x))$

The prenex normal form of the formula $\boldsymbol{\forall x(\forall y\,(P(y)\Rightarrow Q(x,y)))\Rightarrow(\exists y\ Q(y,x))}$

is $\boldsymbol{\forall x\exists y\exists z\,((P(y)\wedge \neg Q(x,y))\vee Q(z,x))}$

4. **Skolemization**

$\forall x\exists y\exists z\,((P(y)\wedge \neg Q(x,y))\vee Q(z,x))$

$\forall x\exists z\,((P(f(x))\wedge \neg Q(x,\ f(x)))\vee Q(z,x))$

$\forall x\exists z\,((P(f(x))\wedge \neg Q(x,\ f(x)))\vee Q(z,x))$

$\forall x\,((P(f(x))\wedge \neg Q(x,\ f(x)))\vee Q(g(x),x))$ SNF

5. Removal of universal quantifiers

$(P(f(x))\wedge \neg Q(x,\ f(x))\vee Q(g(x),x)$

Since all variables are now universally quantified we can remove the $\forall$ (they become implicit).

6. Distribution of $\vee$ on $\wedge$:

$(P(f(x))\wedge \neg Q(x,\ f(x))\vee Q(g(x),x)$

The formula is in CNF:

$(P(f(x))\vee Q(g(x),x))\wedge(\neg Q(x,f(x))\vee Q(g(x),x))$