Predicate Logic – Syntax – Part 2

Atomic formula

An atomic formula (atom) of LP1 is defined by:

- If P_i is a predicate and $t_1, t_2, \dots, t_n$ are terms then $P_i\ (t_1, t_2, \dots, t_n)$ is an atomic formula.
- Example: Brother (Salah, Rachid), Human(s)

Well-formed formula (WFF) :

1. Every atomic formula is a well-formed formula.
2. If A, B are well-formed formulas of LP1 then $\neg A$, $A \wedge B, A \vee B,\ A \Rightarrow B, A \Leftrightarrow B$ are well-formed formulas.
3. If A is a formula and x a variable, then .$\forall x\ (A), \exists x\ (B)$ are well-formed formula.

- The precedence of connectors and quantifiers is presented in the following order:

$$\neg, \wedge, \vee, \forall, \exists, \Rightarrow, \Leftrightarrow$$

Example:

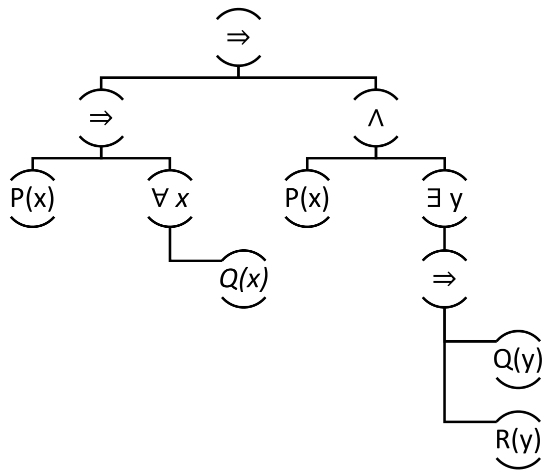
- **$\forall$x A(x) $\vee$ B(x,y) $\Rightarrow$ A(x) $\Leftrightarrow$ B(x,y)**
- Reads: **(($\forall$x(A(x) $\vee$ B(x,y))$\Rightarrow$ A(x)) $\Rightarrow$ B(x,y))**

Parsing tree

As in propositional logic, we can associate any well-formed formula with its parsing tree.

Example:

- $(P(x)\ \Rightarrow\ \forall\, x\, Q(x)\,)\ \Rightarrow\ (P(x)\ \wedge\ \exists\, y\, (Q(y)\ \Rightarrow\ R(y)))$



All sub-formulas of a formula A of LP1 is the smallest set such that:

- If A is an atomic formula, then A is a sub formula.
- If $A(x)$ is a sub-formula of A then $\neg A(x)$ is a sub-formula of A.
- If $(B \wedge C), (B \vee C),\ (B \Rightarrow C), (B \Leftrightarrow C)$ are sub-formulas of A then B and C are sub-formulas of A.
- If $\forall\, x\, B\ \exists\, x\, B$ are subformulas of A then B is a sub-formula of A.

Example: The set of subformulas of F:$(P(x)\ \Rightarrow\ \forall\, x\, Q(x)\,)\ \Rightarrow\ (P(x)\ \wedge\ \exists\, y\,(Q(x)\ \Rightarrow\ R(x)))$

SF(S):$\{P(x), Q(x), R, \forall x\, Q(x), Q(x) \Rightarrow R(x), P(x)\ \Rightarrow\ \forall x\, Q(x), \exists y\, Q(y) \Rightarrow R(y), P(x)\ \wedge$
$(\exists y\, Q(y) \Rightarrow R(y)), F\}$

Free variables and bound variables

- In the formula $\forall x\ (A)$, (resp. $\exists x\ (A)$, we say that the formula A is in the scope of the quantifier ∀(resp.∃).
    - Example: ∃x (P(x)∧Q(s))

      ∃x P(x) ∧ ∃x Q(x)
- The occurrence of x means the position occupied by x in formula A.
    - Example: in this formula $\forall\, x\, P(x)\ \Rightarrow\ (Q(x) \wedge\ \exists\, y\, Q(y))$, the variable x has two occurrences.
- A variable x is said to be a **<u>bound variable</u>** of a formula A if and only if it has an occurrence in the scope of a quantifier of the formula A.
- If an occurrence of a variable is not bound, then it is free.

Examples:

- F1: P(A,f(x)) Atomic formula containing an occurrence of *x*, *x* is therefore free.
- F2: $P(B,B) \Rightarrow \forall x\, (Q(x) \vee \exists x\, P(x,y))$, y is free.
  - $\forall x$ can only bind one free variable i.e. x in Q(x).
  - The occurrence of x in P(x,y) is related by $\exists x$.
  - So *x* is bound variable (but differently).

Closed / open formula

- Formulas with <u>no free variables</u> are called **closed formulas**. Those containing a free variable are called **open formulas**.
- Examples:
- In . $\forall x\ (P(x) \wedge P(y) \wedge Q(A))$, x is bound, y is free, so the formula is open.
- $\exists y \urcorner (P(Hakim, y))$, y is bound, then the formula is closed.
- $\forall x\, \forall y\ (Q(x,y) \Rightarrow \exists z\, P(z))$, x and y are bound by $\forall$ and z is bound by $\exists$, then this formula is closed.