Propositional Logic – Semantics

Interpretation:

- In our course, we denote F (False) by 0 and T (True) by 1.
- Interpretation is an assignment of meaning to the symbols of a formal language.
- For a propositional formula P composed of different atomic variables: $p_1, \ldots, p_n$, an interpretation I of P is an assignment of truth values to $p_1, \ldots, p_n$.
    - **Example:** Let P: $\Rightarrow (q \land r)$. $\{I(p) = 1, I(q) = 0, I(r) = 1\}$ be an interpretation of the formula P.
- A formula composed of n atomic variables admits 2^n interpretations.
    - **Example:** The formula P: $(p \Rightarrow (q \land r)$ composed of three atomic formulas$\{p, q, r\}$, then it admits 2^3=8 interpretations.
- **Model of a formula:** An interpretation I is a model for the formula P if it is true, i.e. $I(P) = 1$.

Truth Tables

- The truth table represents the different possible interpretations of the formula based on the interpretations of the atomic variables and the logical connectives that connect them.
- The columns define the truth values of different atomic variables appearing in the formula and the initial formula.
- Each line will correspond to a possible situation.
- In practice, we often add columns for the truth values of the different possible subformulas.
- The truth value of the formula depends on the truth values of the subformulas, implicitly the atomic variables.
- If the formula consists of n atomic variables, we thus obtain 2^n ways of combining their truth value, described by 2^n lines of the table.
- The truth table presented below allows us to define the logical connectives in propositional logic.

p	q	¬p	p ∧ q	p ∨ q	p ⇒ q	p ⇔ q
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

Remarks:

- We say that p ∨ q is true if and only if either p or q are true.
- We say that p∧q is true if and only if p and q are true.
- We say that p⇒q is false if and only if p is true and q is false.
- We say that p⇔q is false if and only if p and q are different.

Example:

- Construct the truth table of the formula F=((¬q)∧(p∨q)) ⇔p

p	q	¬q	p ∨ q	(¬q) ∧ (p ∨ q)	F
0	0	1	0	0	**1**
0	1	0	1	0	**1**
1	0	1	1	1	**1**
1	1	0	1	0	**0**

- The set of models of Formula F is:
- {I(p)=0 and I(q)=0, I(p)0= and I(q)=1, I(p)=1 and I(q)=0}

p	q
0	0
0	1
1	0

Validity and satifisability

Tautology

- A propositional formula is said to be a **tautology (valid)** if and only if its truth value is always true, regardless of the truth values of the atomic variables that compose it.
- P is tautology if for any interpretation.$I(F) = 1$

Example: Here is the truth table of the formula F: ((¬q) ∧ (p∨q)) ⇒ p

- Is P valid?

P1: ((p⇒q)∧p)⇒q

So, P1 is a valid formula

p	q	p ⇒ q	((p ⇒ q) ∧ p)	**((p ⇒ q) ∧ p) ⇒ q**
0	0	1	0	**1**
0	1	1	0	**1**
1	0	0	0	**1**
1	1	1	1	**1**

Insatisfiable

- A propositional formula is said to be antilogy (Antilogy, contradictory) if its truth value is always false, whatever the truth values of the propositional variables that compose it.
- F is antilogy if for any interpretation $I(F) = 0$

Example: The truth table of the formula F: (¬p) ∧ p

- We see that the formula P is always false, for all truth values of p and q. We say F is an insatisfiable,

Invalidity

- A formula P is said to be invalid if its truth value is false in at least one interpretation.
- Is P2 invalid?

P2: ¬p∨¬q

So, P2 is an invalid formula

p	q	¬p ∨ ¬q
0	0	**1**
0	1	**1**
1	0	**1**
1	1	**0**

Satisfiability:

- A formula is said to be satisfiable (or semantically consistent) if and only if its truth value is true of at least one interpretation (I(F)=1).
- A formula is said to be unsatisfiable (or inconsistent, antilogy) if and only if its truth value is always false in any interpretation.

Logical equivalence

- In order to verify a equivalence, we have two techniques,
  - Use the truth table,
  - Use equivalence chains based on equivalence rules.
- Truth table:F1: p $\Rightarrow q$ F2: $\neg p \vee q$

p	q	$p \Rightarrow q$
0	0	**1**
0	1	**1**
1	0	**0**
1	1	**1**

p	q	¬p	$\neg p \vee q$
0	0	1	**1**
0	1	1	**1**
1	0	0	**0**
1	1	0	**1**

- F1 and F2 admit the same models, So $F1 \equiv F2$

Exercise : Use a truth tables to show that $P\ and\ Q$ are equivalent,

- $P: \neg(p \Rightarrow \neg q) \vee r$
- $Q: (p \wedge q) \vee r$

Equivalent transformations

- Neutral element
  - $p \wedge V \ \equiv p$
  - $p \vee \text{ F } \equiv p$
- Idempotent laws
  - $p \wedge \text{p} \ \equiv p$
  - $p \vee p \equiv p$
- Double negation laws
  - $\neg(\neg p) \equiv p$
- Absorbent element
  - $p \wedge F \equiv F$
  - $p \vee V \equiv V$

- Commutative laws
  - $P \wedge Q \equiv Q \wedge P$
  - $P \vee Q \equiv Q \vee P$
- Associative laws
  - $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
  - $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
- Distributive laws
  - $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
  - $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
- De Morgan's Law
  - $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
  - $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- Material implication
  - $P \Rightarrow Q$(elimination of implication)$\equiv (\neg P) \vee Q$
- Material equivalence
  - $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Example : Say whether the following equivalence is correct, .$P \equiv Q$

$$Q: \neg(p \Rightarrow \neg q) \vee r \equiv ?\ P: (p \wedge q) \vee r$$

Answer:

$\neg(p \Rightarrow \neg q) \vee r$(Material implication)$\equiv \neg(\neg\, p \vee \neg q) \vee r$

(De Morgan's Law)$\neg(\neg\, p \vee \neg q) \vee r \equiv (\neg\neg p \wedge \neg\neg q) \vee r$

$(\neg\neg p \wedge \neg\neg q) \vee r$(Law of double negative)$\equiv (p \wedge q) \vee r$

Exercise:

- Prove that the following formulas are tautologies?
  - $(\neg q \wedge (p \Rightarrow q)) \Rightarrow \neg q \vee p$
  - $((p \vee q) \wedge \neg q) \Rightarrow p$
- Check whether the following equivalence is correct, $P \equiv Q$.
  - P: $(\neg q \wedge (p \vee r)) \Rightarrow p$
  - Q: $\neg (\neg p \wedge (\neg q \wedge r))$