People's Democratic Republic of Algeria
Ministry of higher education and scientific research
University of Mohamed Boudiaf - M'sila

Faculty of Mathematics and Computer Science
Department of Computer Science

2nd Year License (2L)

Data Structures and Algorithms 3 (DSA3)

Semester : 03
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- ➢ We are at your disposal to help you

CHAPTER 4

Trees

Plan

I. Introduction

- A data structure is said to be **linear** if its elements form a **sequence or a linear list**.
- Previous linear data structures that we have studied like an **array, stacks, queues and linked lists** organize data in **linear order.**
- A data structure is said to be **non-linear** if its elements form a **hierarchical classification** where data items appear at **various levels.**
- **Trees and Graphs are widely used non-linear data structures.**

I. Introduction

- In computer science, a **tree is a very general and powerful** data structure that resembles a real tree.
- It consists of an **ordered set of linked nodes in a connected graph**, in which each node has at most **one parent node,** and **zero or more child nodes** with a specific order.
- In this chapter, in particular, we will explain one of the non-linear data structures known as a tree, which is easy to maintain on the computer.

II. Non-Linear Data Structures

- Non-linear data structures are those where **data items are not arranged sequentially**, unlike linear data structures.
- In these data structures, **elements are stored in a hierarchical or a network-based structure** that does not follow a sequential order.
- These data structures allow **efficient searching, insertion, and deletion of elements** from the structure.
- Examples of Non-Linear Data Structures:
  - Trees
  - Graphs, etc.

II. Non-Linear Data Structures

Properties of Non-Linear Data Structures

The Non-linear data structures have the following properties.

- Non-linear data structures **do not follow a sequential order.**
- Elements are **stored in a hierarchical or a network-based** structure.
- These data structures allow **efficient searching, insertion, and deletion of elements**.
- Non-linear data structures are **used to solve complex problems** where data cannot be arranged in a linear manner.

III. Tree Data Structure

- The **tree is a Non-Linear data structure in which data are stored in a hierarchical structure** and consists of **nodes** connected by **edges**.
- It is also defined as a **hierarchical collection of nodes.** The collection can be **empty**. Each node has **a parent node and zero or more child nodes.**
- The node at the top of the hierarchy is called the **root node**.
- Trees are widely used in computer science and are an essential part of many algorithms and data structures.

III. Tree Data Structure

- Trees represent a special case of more general structures known as graphs. Figure 4.1 shows a tree and a non-tree.

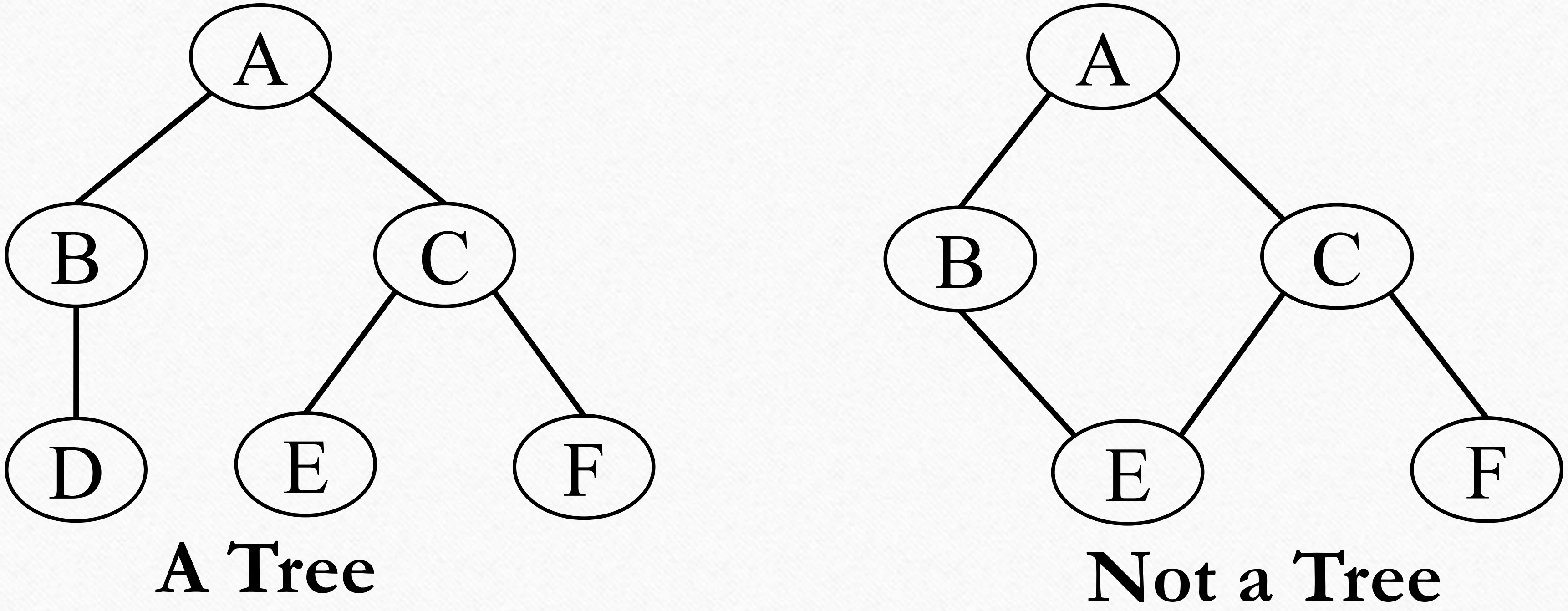
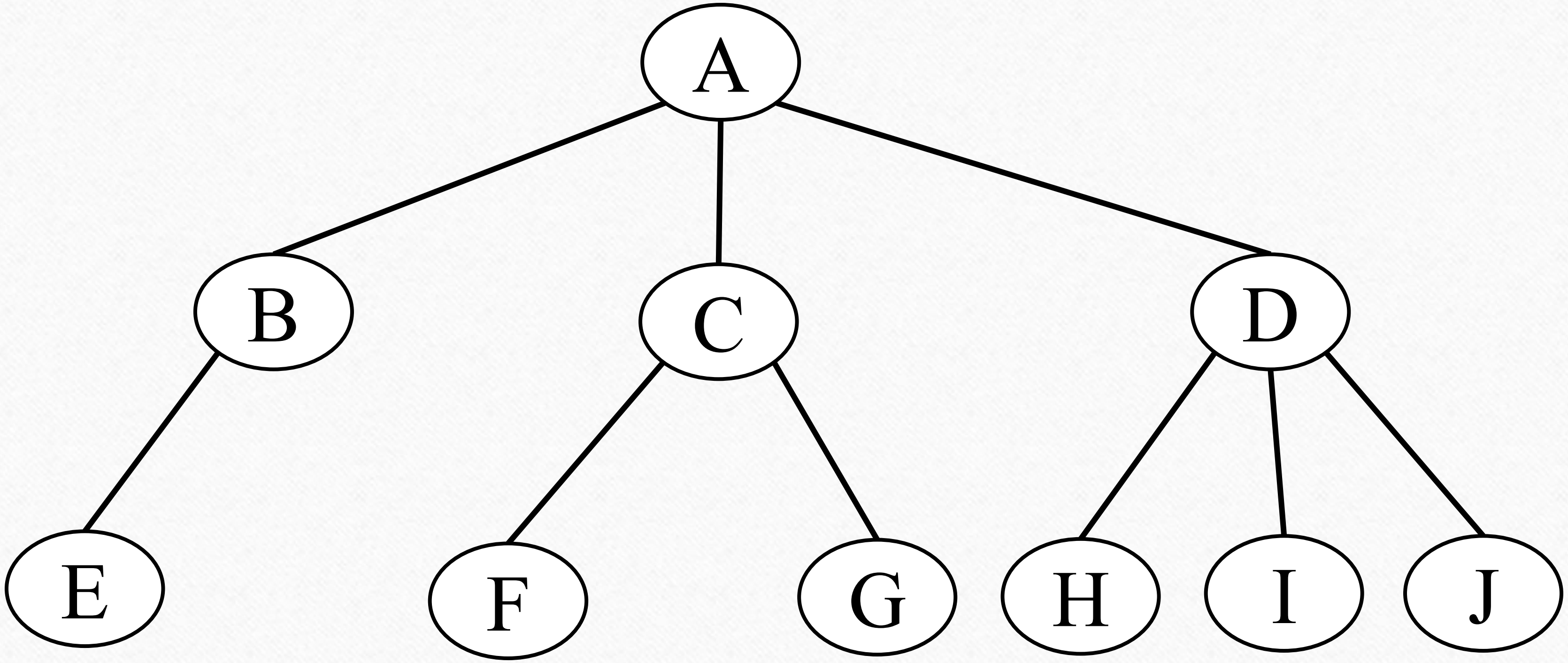


Figure 4.1 : A Tree and not a Tree.

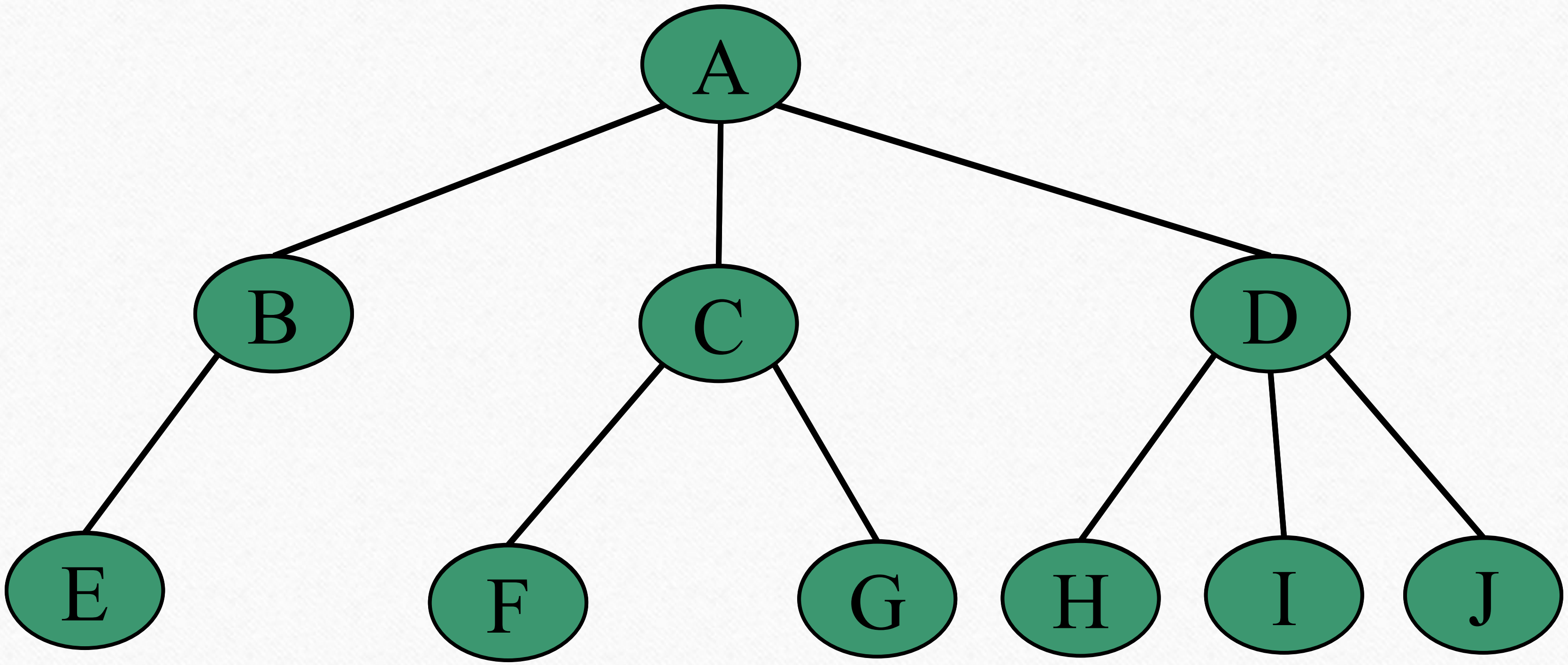
IV. Basic Tree Terminology

- Below lists some general tree terminology (many of these terms are derived from terms used for family relationships):



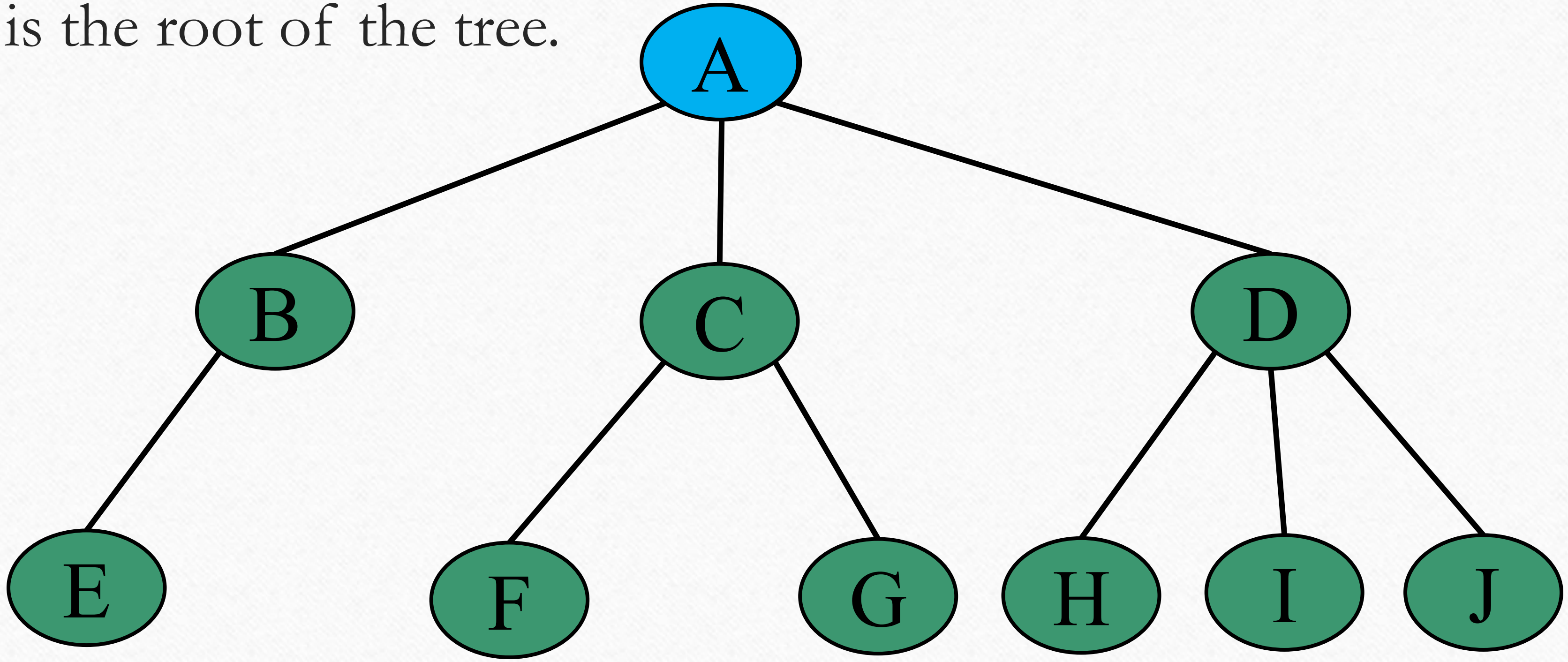
IV. Basic Tree Terminology

- **Node:** Each element of a tree is called as node. In the previous figure there are **10** nodes.



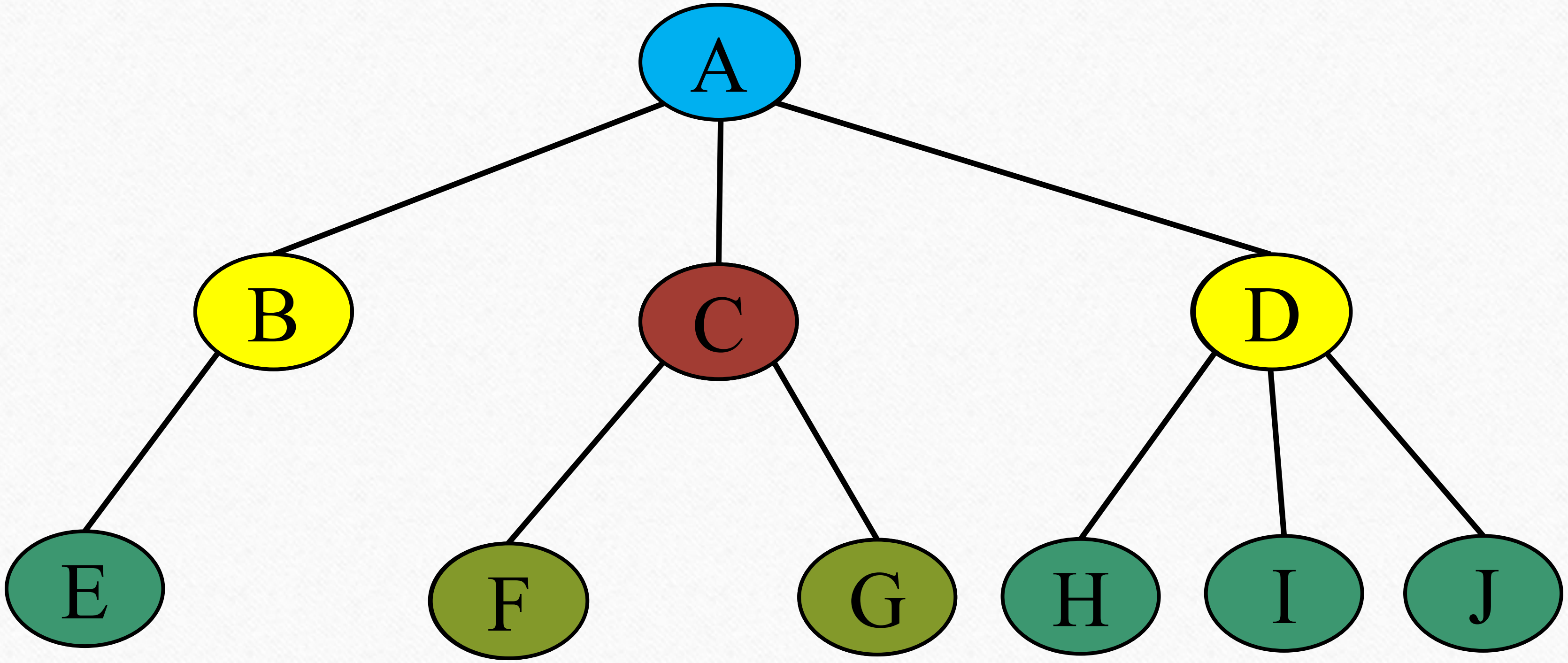
IV. Basic Tree Terminology

- **Root:** The root of a tree is the "top-most" vertex in the tree, from which all other nodes are directed away from. In the tree above, node **A** is the root of the tree.



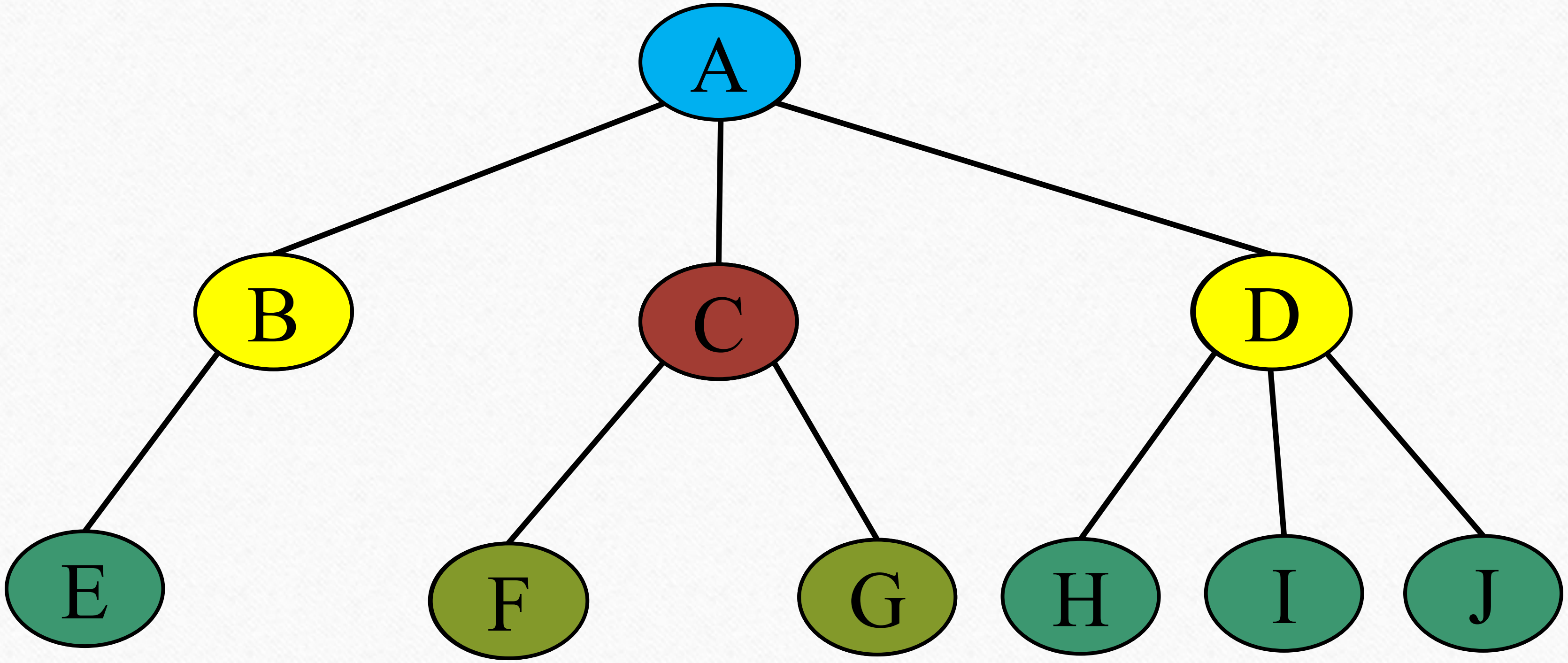
IV. Basic Tree Terminology

- **Parent:** Parent of a node is the immediate predecessor of a node. In the tree above, **C** is the parent of **F** and **G**.



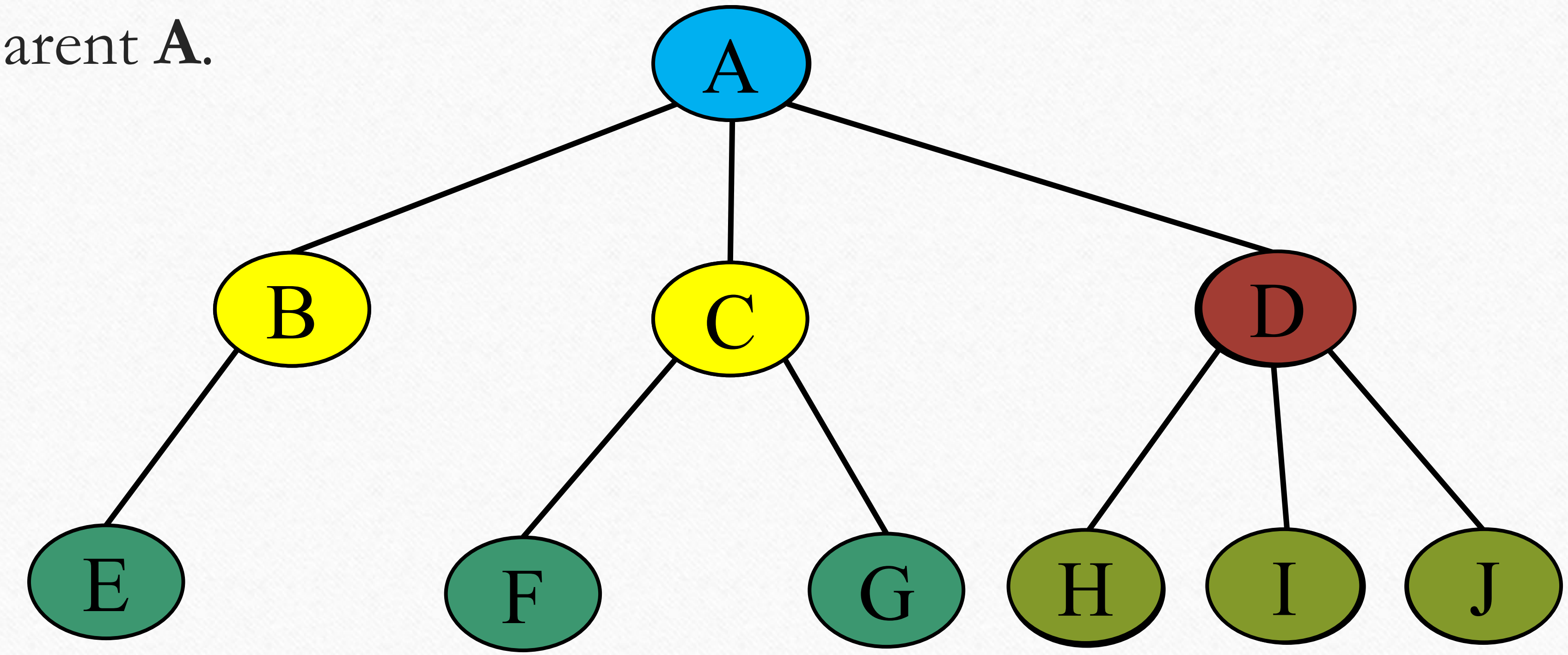
IV. Basic Tree Terminology

- **Child:** Each immediate successor of a node is known as child. In the tree above, **B**, **C**, **D** are children of **A**.



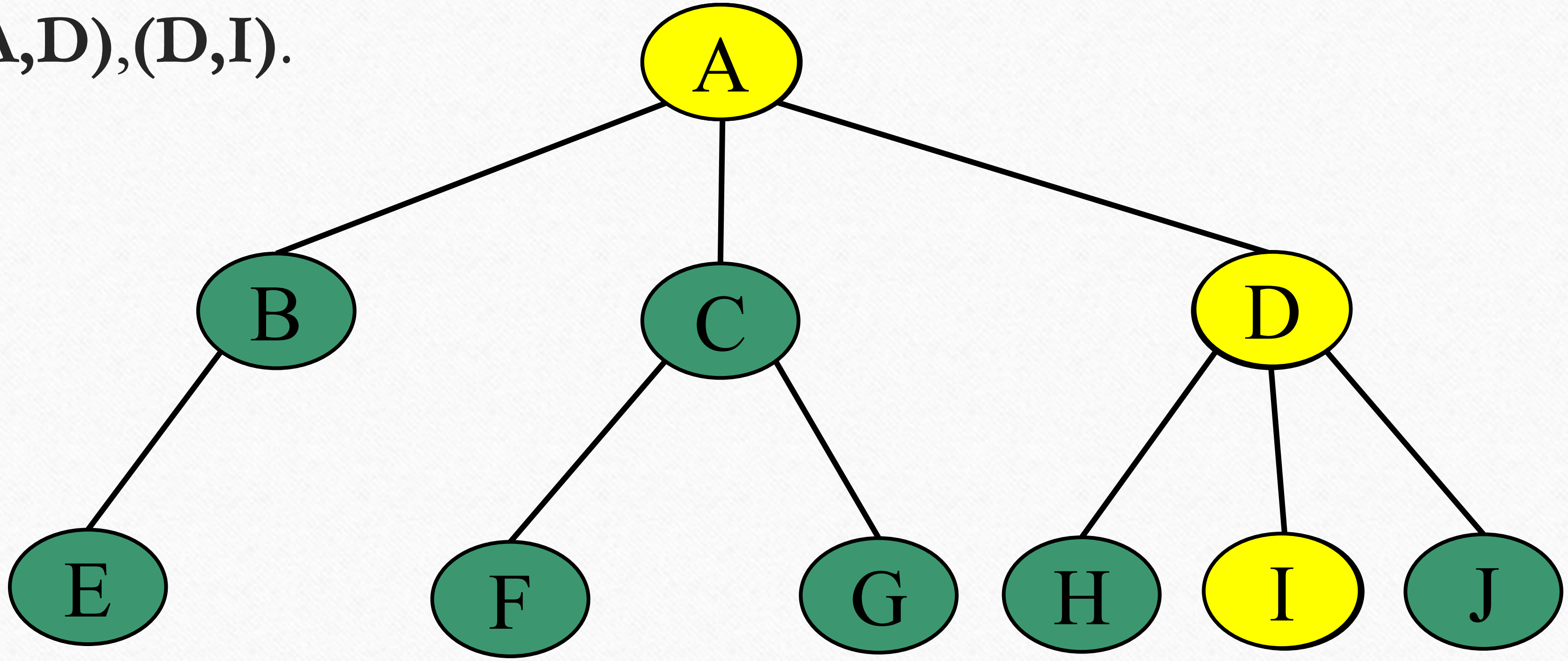
IV. Basic Tree Terminology

- **Sibling:** Two nodes are siblings if they share the same parent. In the tree above, **B**, **C**, and **D** are siblings, since they share the parent **A**.



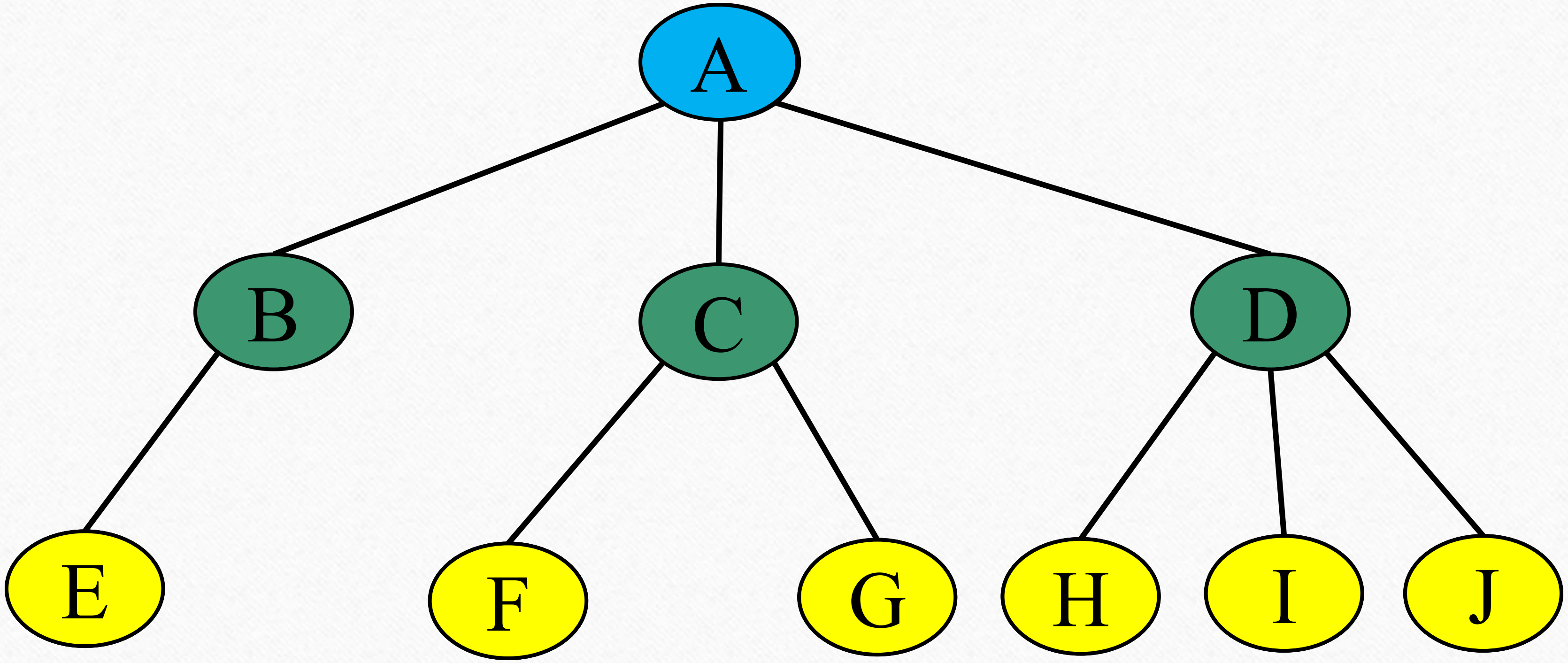
IV. Basic Tree Terminology

- **Path:** Path is the sequence of consecutive **edges** from the source node to the destination node path between **A** and **I** is **(A,D)**,**(D,I)**.



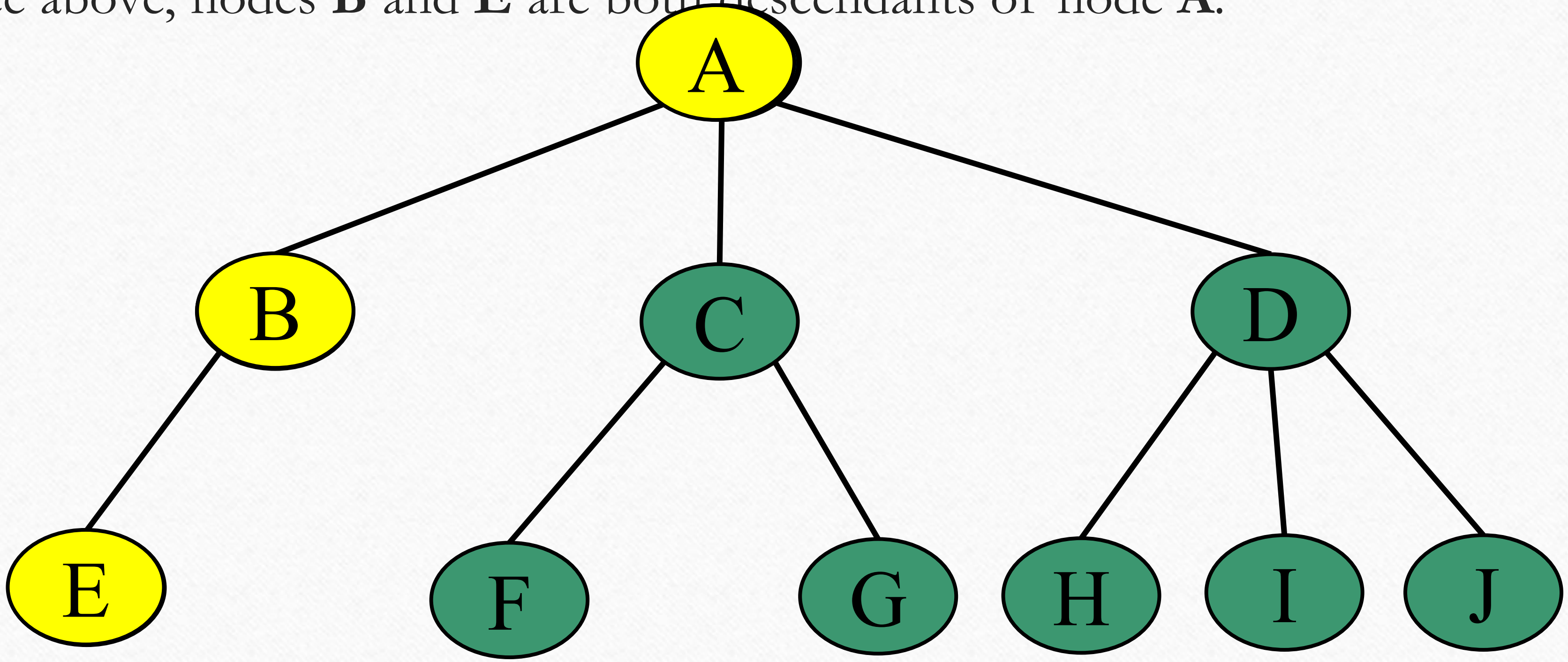
IV. Basic Tree Terminology

- **Leaf** : is a node that has no child node. In the tree above, **E**, **F**, **G**, **H**, **I,** and **J** are leaves.



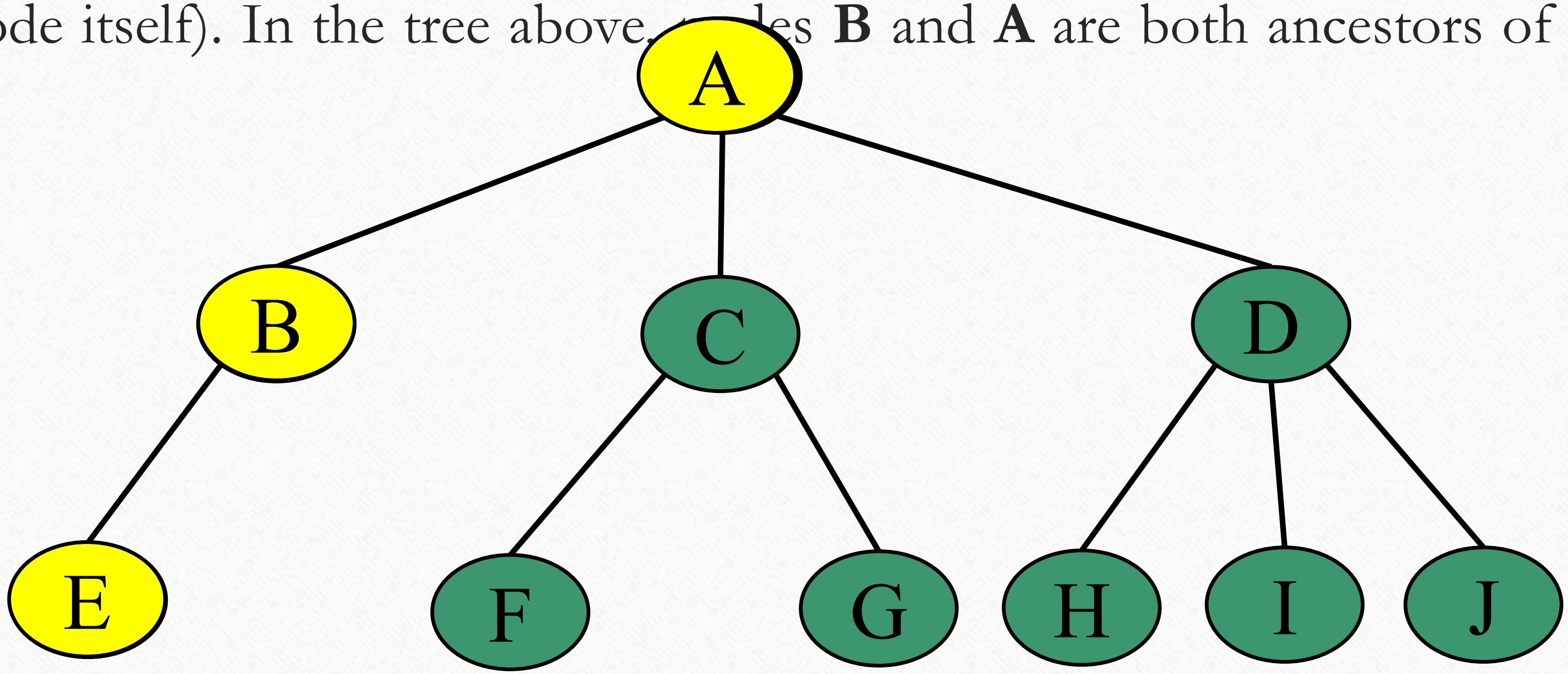
IV. Basic Tree Terminology

- **Descendant:** The descendants of a given node consist of any node that exists along a path from the given node to a leaf node (including the leaf node). In the tree above, nodes **B** and **E** are both descendants of node **A**.



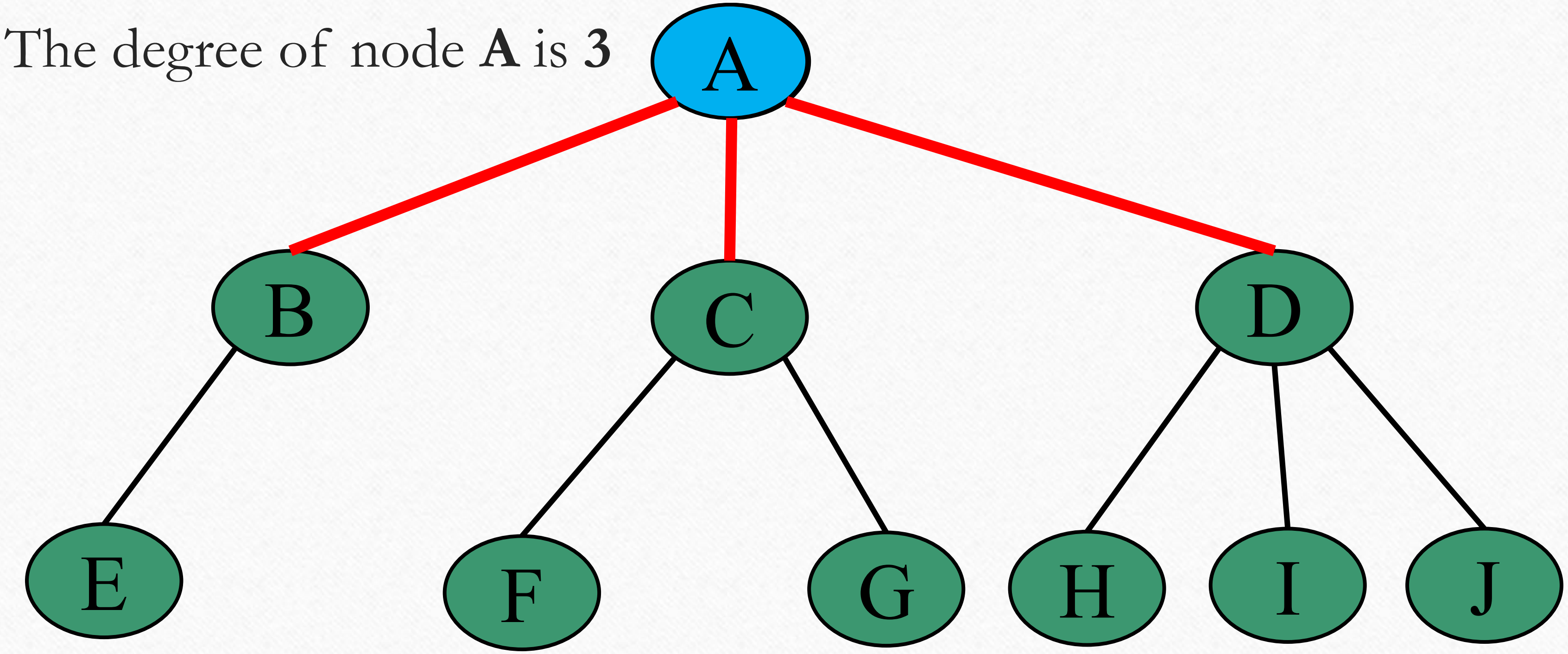
IV. Basic Tree Terminology

- **Ancestor:** The ancestors of a given node consist of any node that exists along the path from the given node to the root node (including the root node itself). In the tree above, nodes **B** and **A** are both ancestors of node **E**.



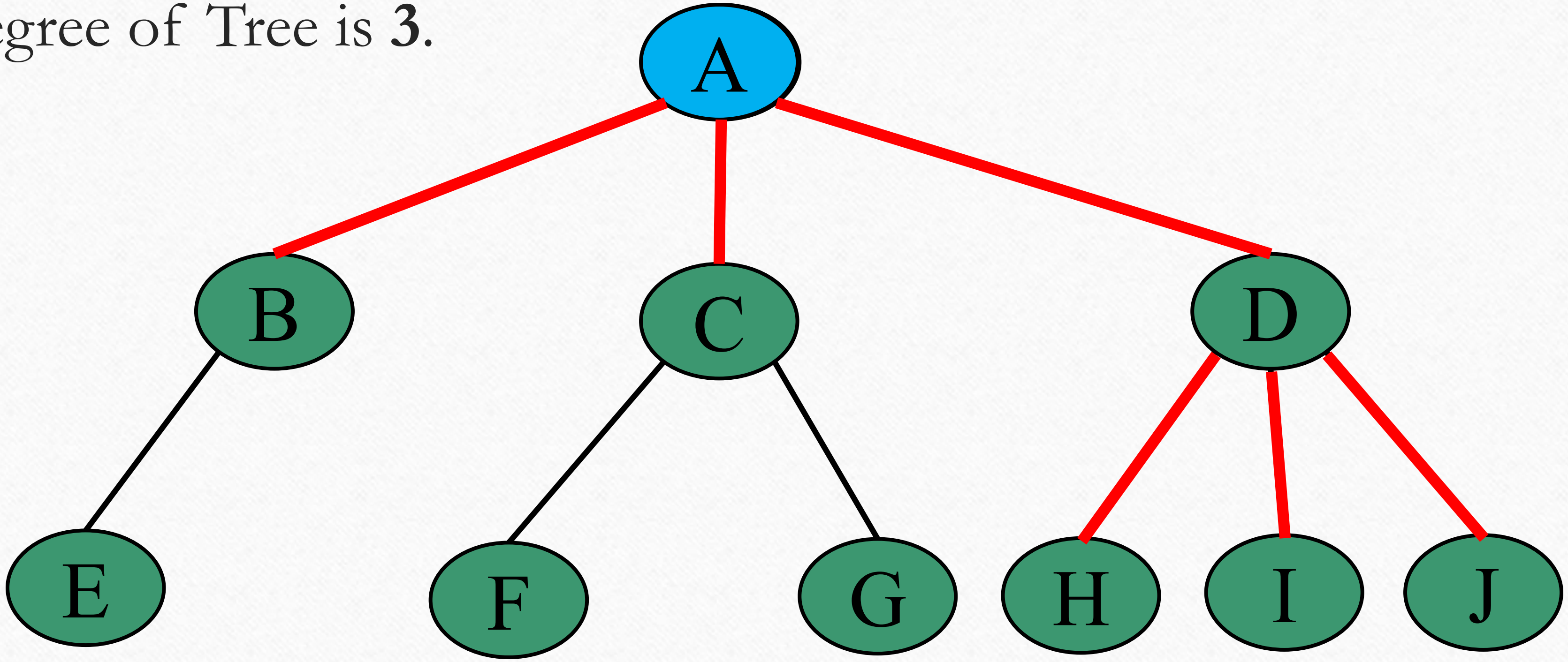
IV. Basic Tree Terminology

- **Degree of a Node:** The number of sub-trees of a node in a given tree. In the tree above,
  - The degree of node **A** is **3**



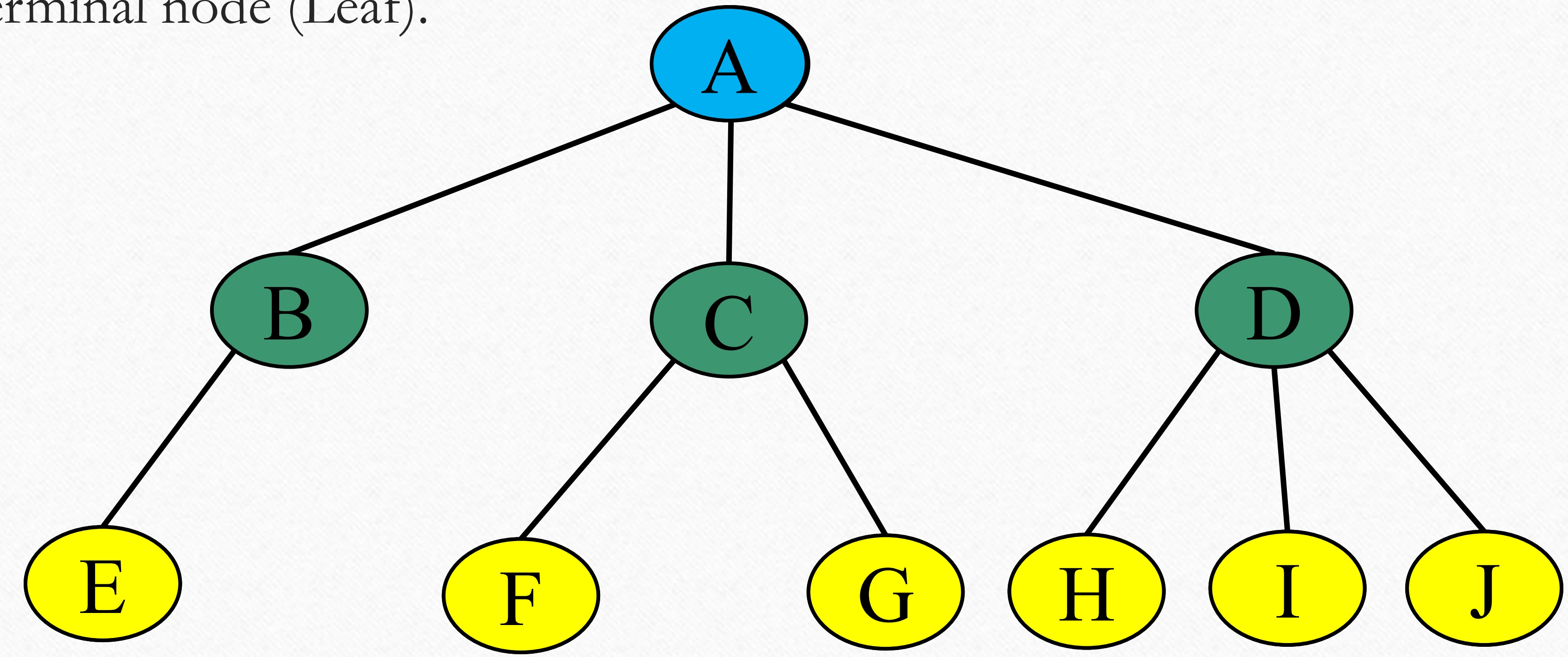
IV. Basic Tree Terminology

- **Degree of Tree:** The maximum degree of nodes in a given tree. In the tree above, the maximum degree of nodes **A** and **D** is **3**. So, the degree of Tree is **3**.



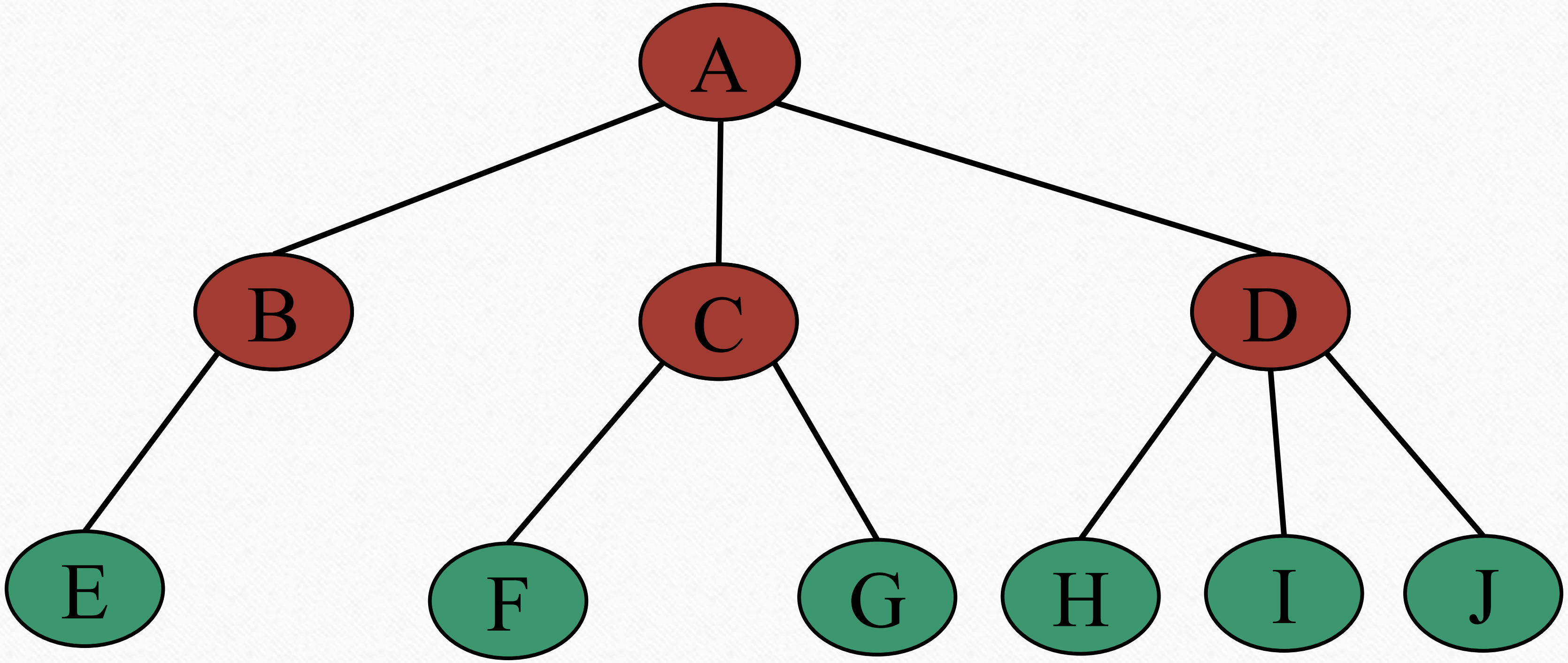
IV. Basic Tree Terminology

- **Terminal Node (Leaf):** A terminal node (or leaf node) is a node that has no children or has a degree zero. In the tree above, nodes **E** to **J** are examples of terminal node (Leaf).



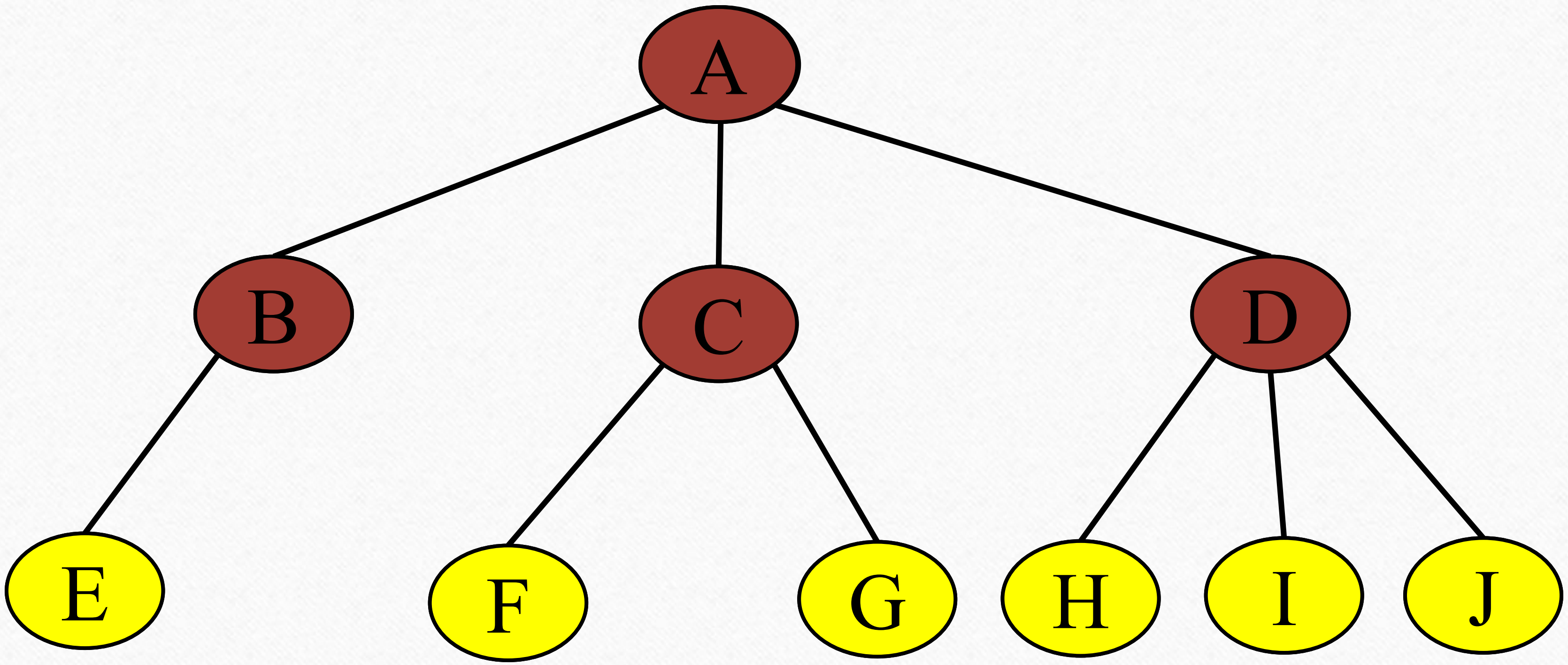
IV. Basic Tree Terminology

- **Internal Node:** An internal node is a node with children. In the tree above, nodes **A** to **D** are examples of internal nodes.



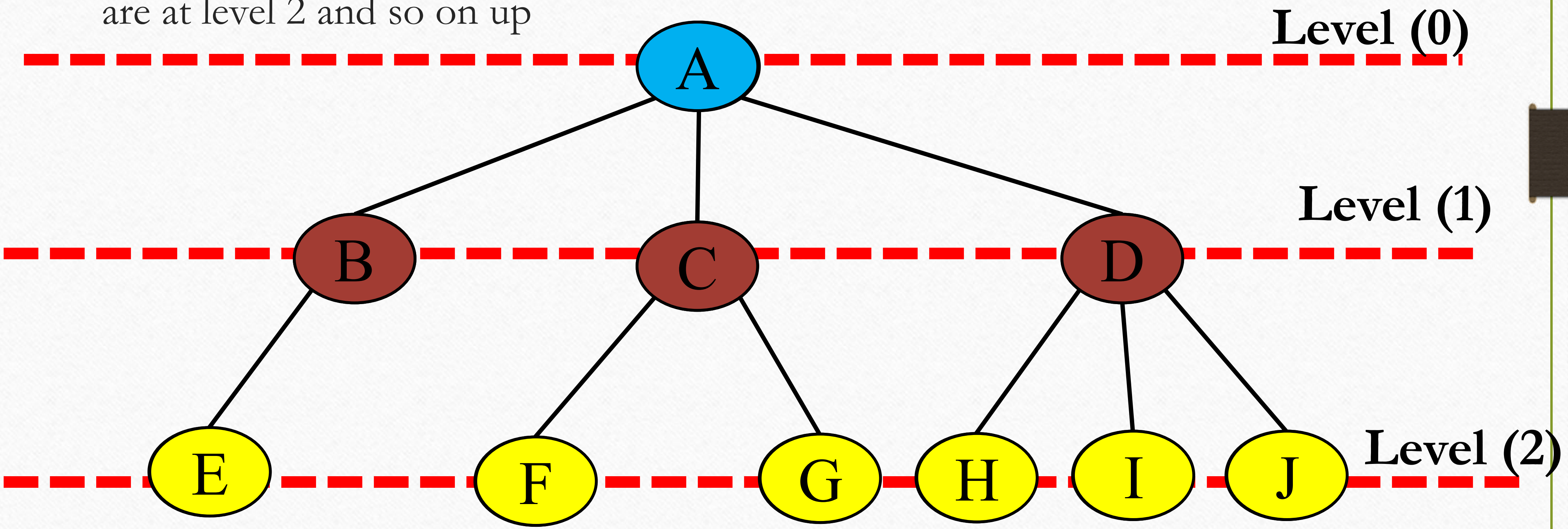
IV. Basic Tree Terminology

➢ Internal Node Vs Terminal Node (Leaf)



IV. Basic Tree Terminology

- **Level:** The entire tree structured is leveled in such a way that the root is always at the level **0**, then its immediate children are at level 1, and their immediate children are at level 2 and so on up

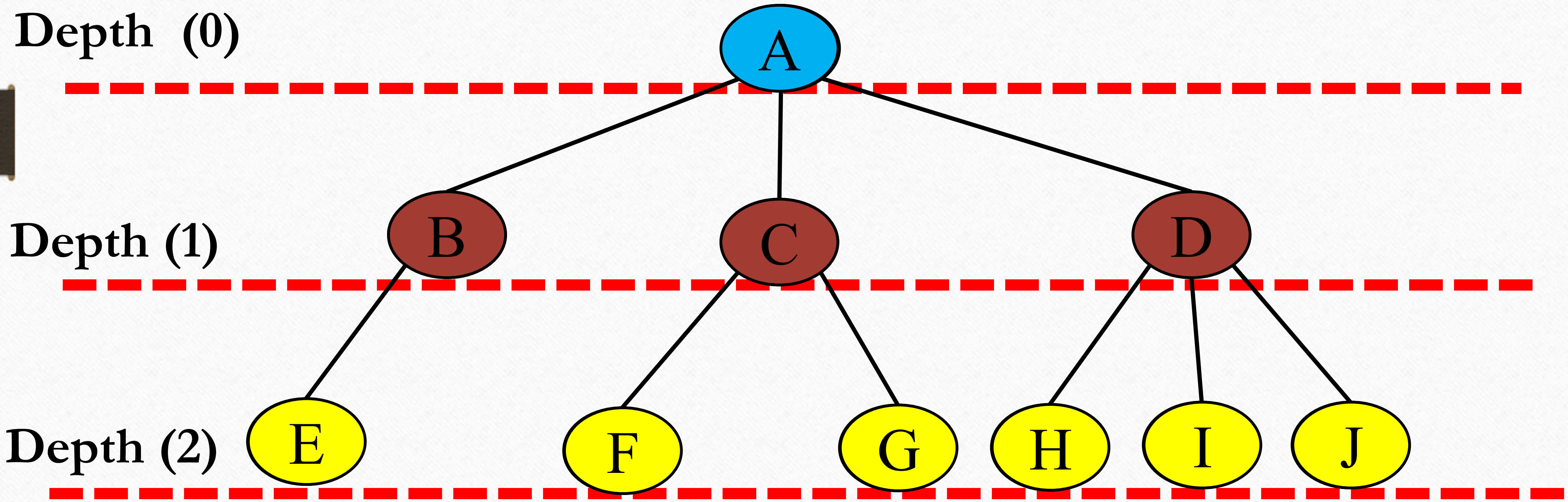


IV. Basic Tree Terminology

- **Depth:** The depth of a node represents **how far it is from the top of the tree.**
- The **root** has a depth of **0**.
- If a node is one edge away from the root, it has a depth of **1**. If a node is two edges away from the root, it has a depth of **2**.
- We can define the depth of a node recursively:
  - **Depth(root) = 0**
  - **Depth(node) = Depth(parent) + 1**

IV. Basic Tree Terminology

- **Depth:** The depth of a node represents how far it is from the top of the tree.

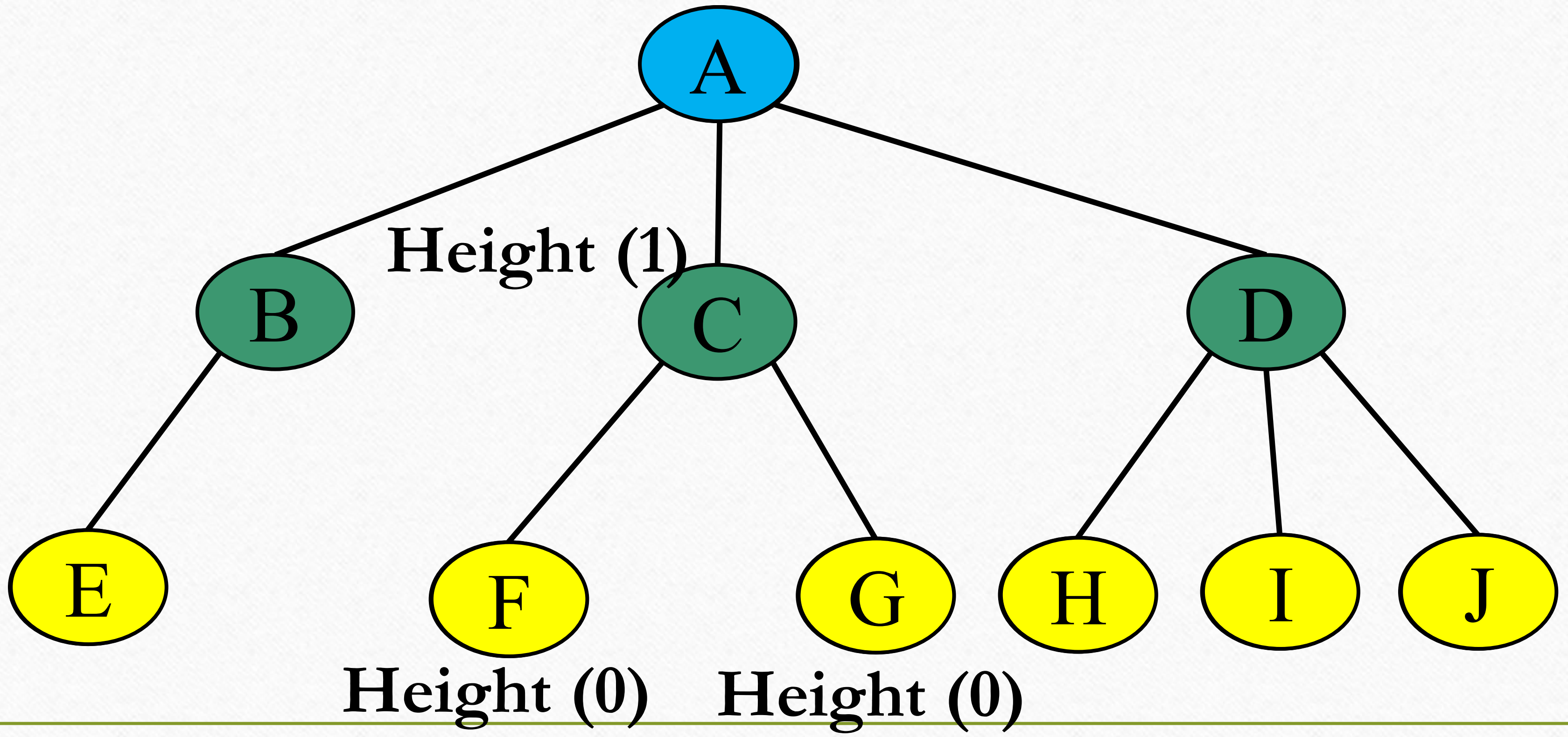


IV. Basic Tree Terminology

- **Height:** The height of a node represents **how far it is from the bottom of the tree**.
- **Leaf** nodes have a height of **0**.
- If a node has more than one child, its height is equal to one plus the largest height of any of its children. We can define the height of a node recursively:
  - **Height(leaf) = 0**
  - **Height(node) = max(Height(children)) + 1**

IV. Basic Tree Terminology

- **Height:** The height of a node represents **how far it is from the bottom of the tree.**



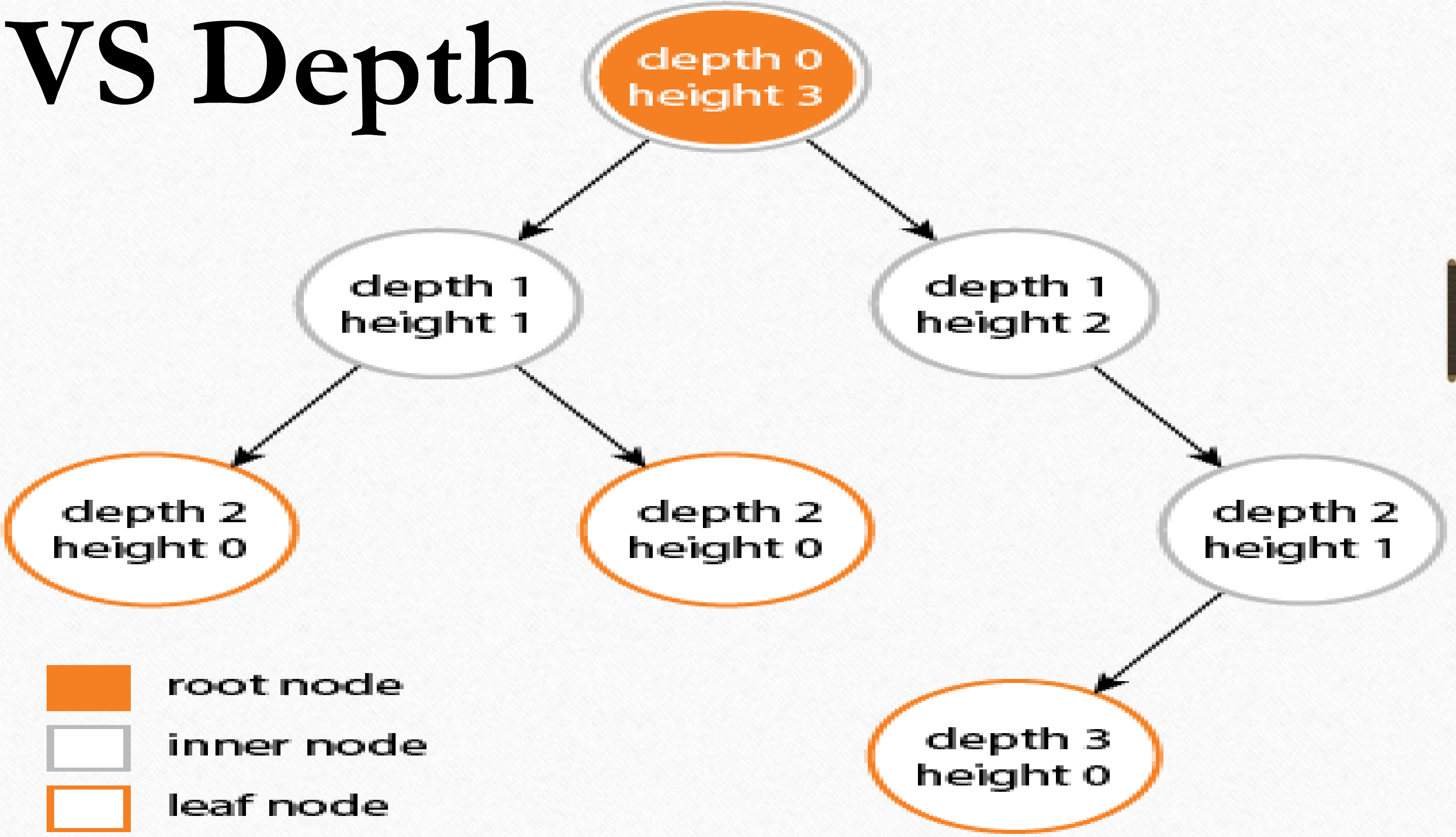
IV. Basic Tree Terminology

Remark:

In a tree, the largest height of any node and the largest depth of any node should be the same value.

IV. Basic Tree Terminology

Height VS Depth

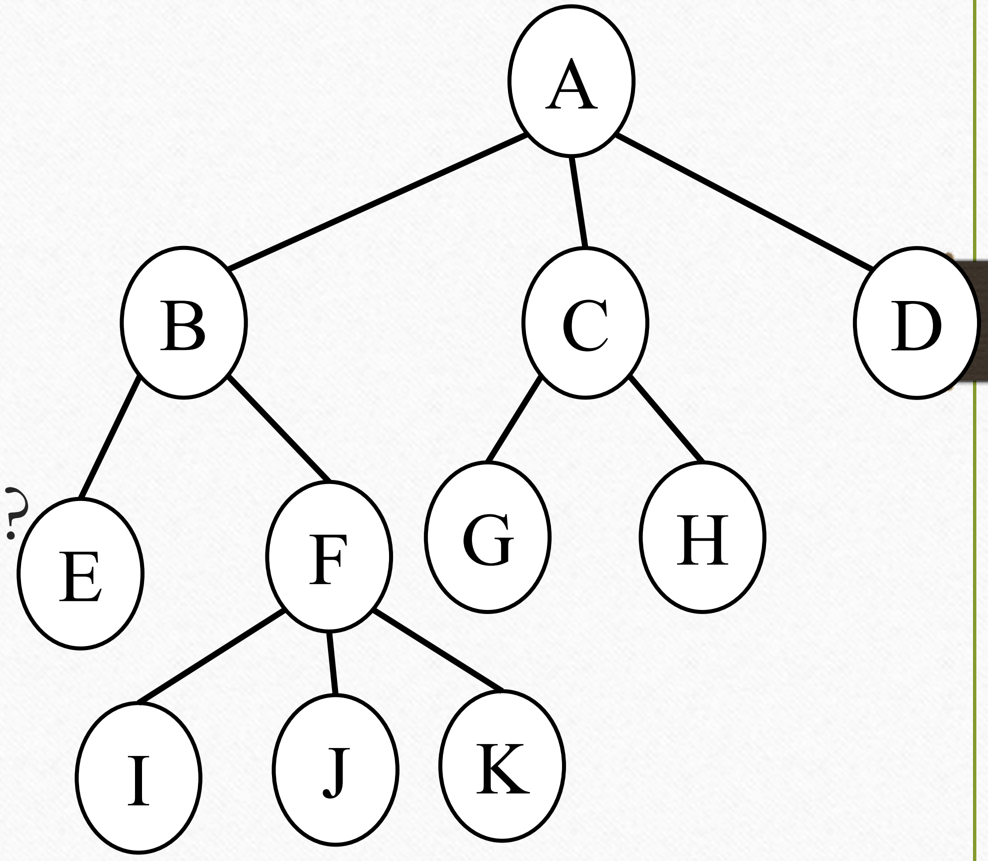


IV. Basic Tree Terminology

Example :

Consider the following tree.

- Which node is the **root**?
- Which nodes are **leaf nodes**?
- Which nodes are **internal nodes**?
- What is the **maximum depth** of the tree?
- What is the **height** of **node B**?



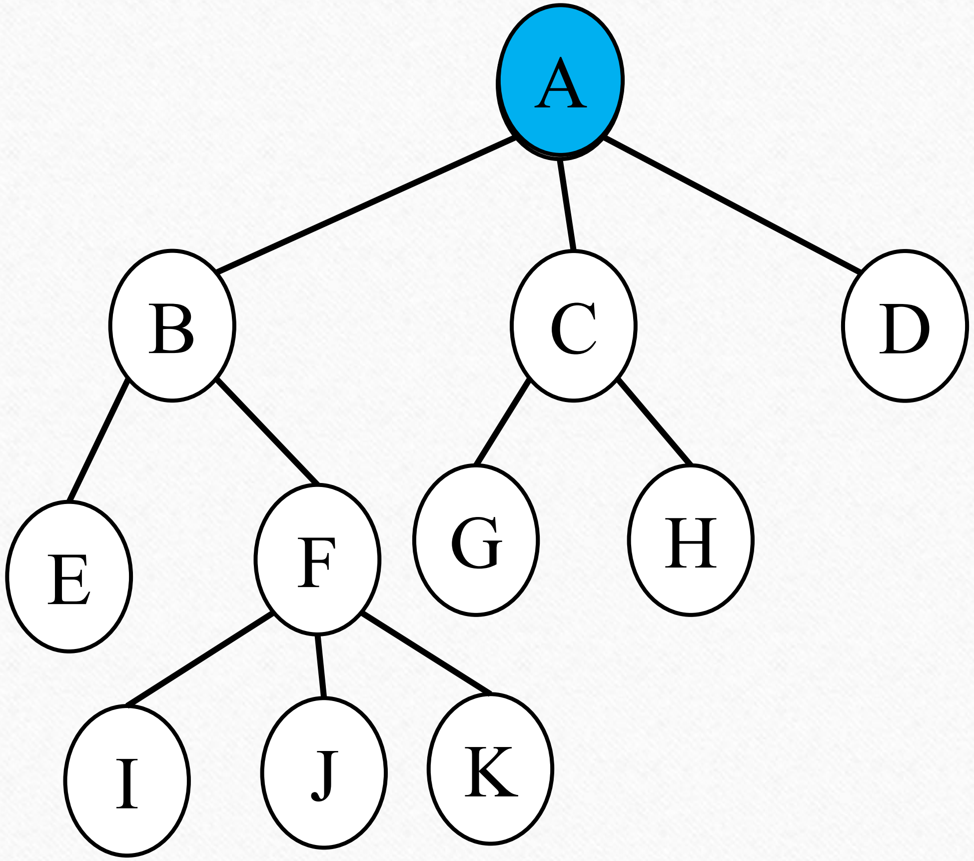
IV. Basic Tree Terminology

Example :

Consider the following tree.

- Which node is the **root**?

In this tree,
node **A** is the **root**.



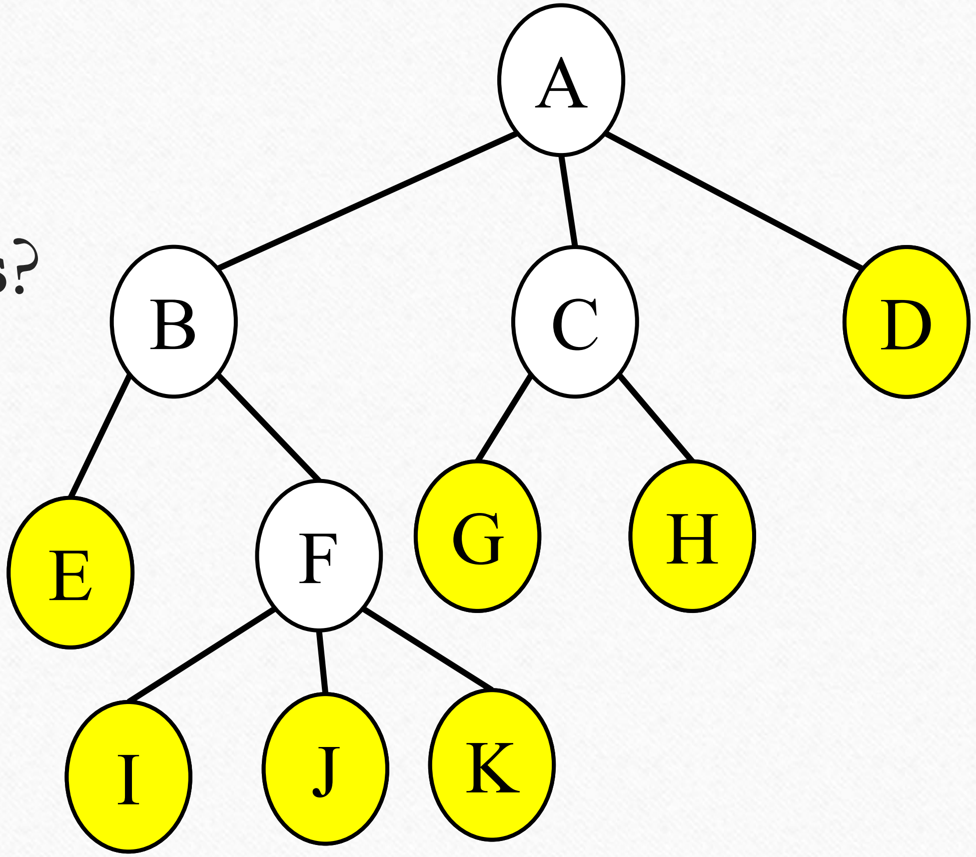
IV. Basic Tree Terminology

Example :

Consider the following tree.

- ➢ Which nodes are **leaf nodes**?

Nodes **D, E, G, H, I, J,** and **K** are **leaves** because they have no children.

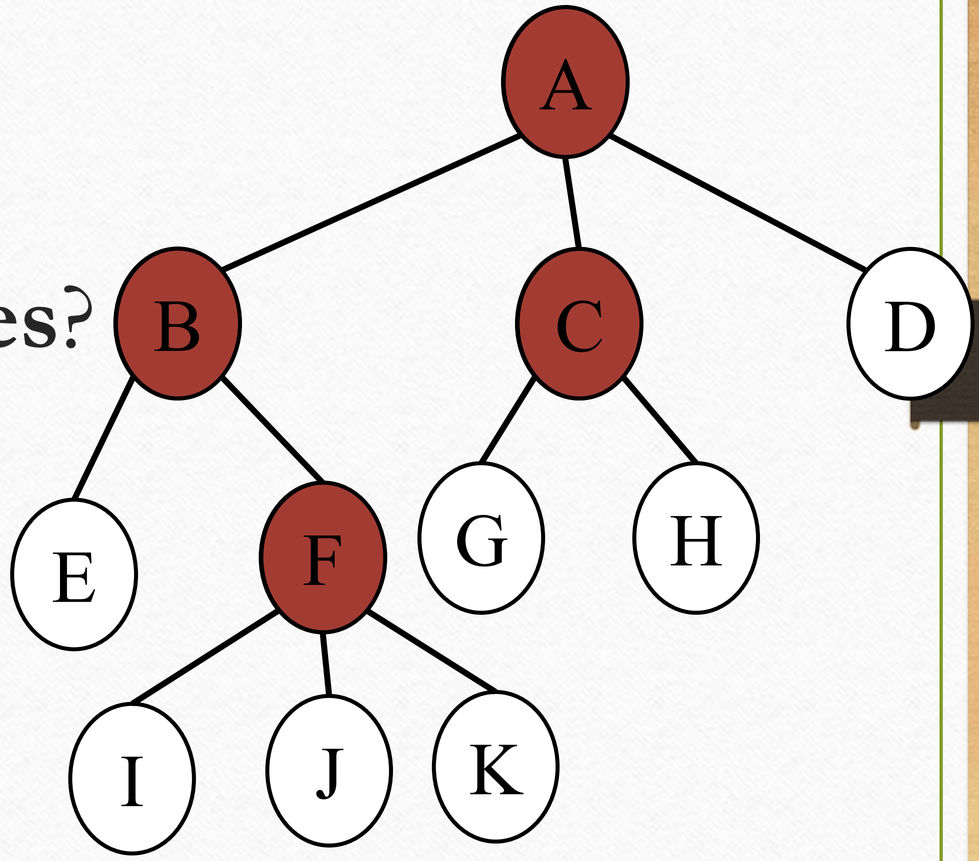


IV. Basic Tree Terminology

Example :

Consider the following tree.

Which nodes are **internal nodes**?



Nodes **A**, **B**, **C**, and **F** are **internal nodes** because they have children.

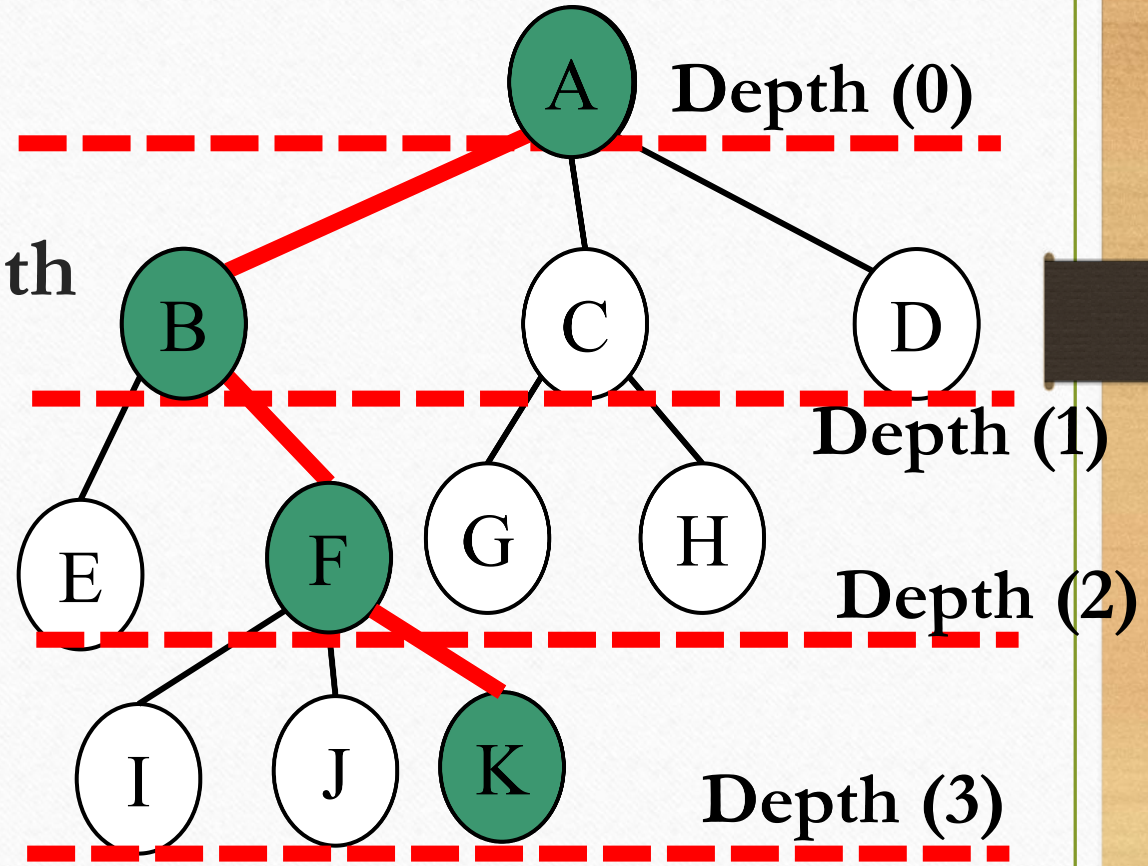
IV. Basic Tree Terminology

Example :

Consider the following tree.

- What is the **maximum depth** of the tree?

The **maximum depth of the tree** is **3.**



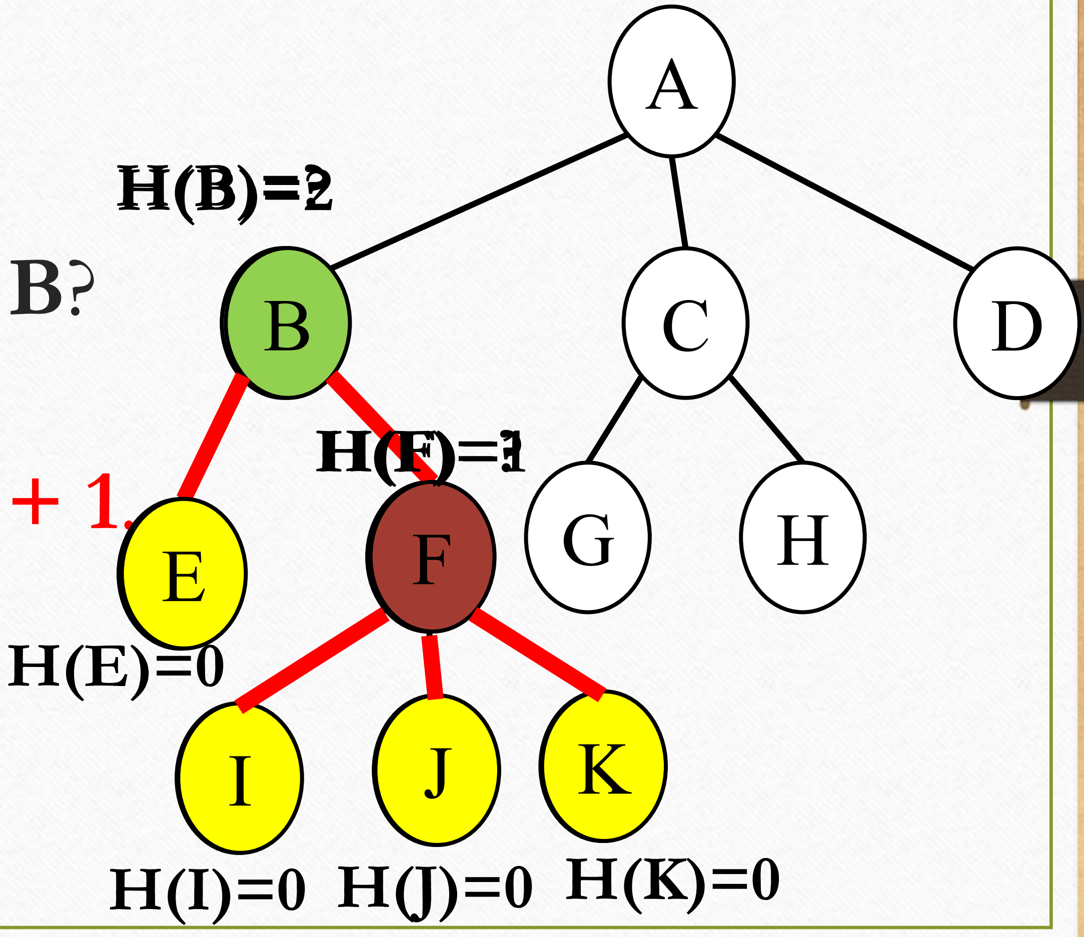
IV. Basic Tree Terminology

Example :

Consider the following tree.

➢ What is the **height** of **node B**?

The **height** of **B** is equal
to **max(height(E), height(F)) + 1**.
Since **F** is the child with
the larger height (height(E) = **0**,
height(F) = **1**),
the **height** of **B** is **1 + 1 = 2**.



V. Types of the Tree Data Structure

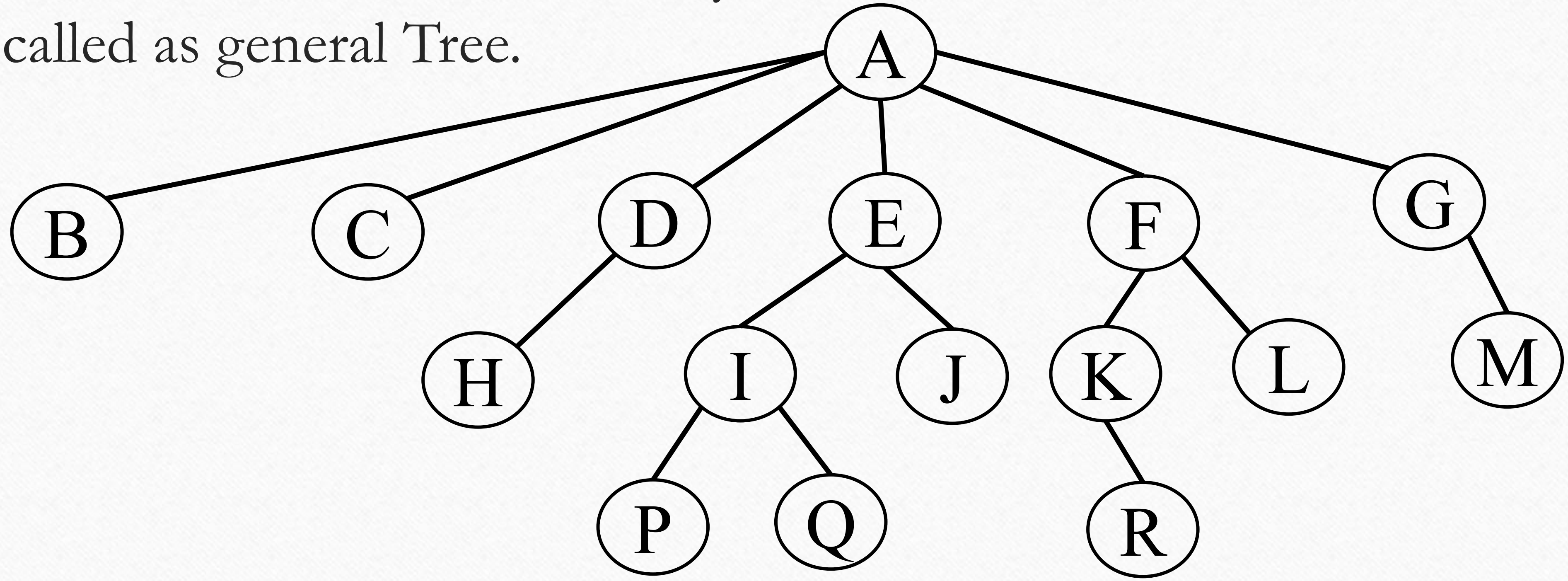
- Based on the **number of children** for each node in the tree, it is classified into two to types :

- ❑ **General (Generic) Tree**
- ❑ **Binary Tree**

V. Types of the Tree Data Structure

General (Generic) Tree

- In a tree, node can have any number of children. Then it is called as general Tree.



V. Types of the Tree Data Structure

General (Generic) Tree

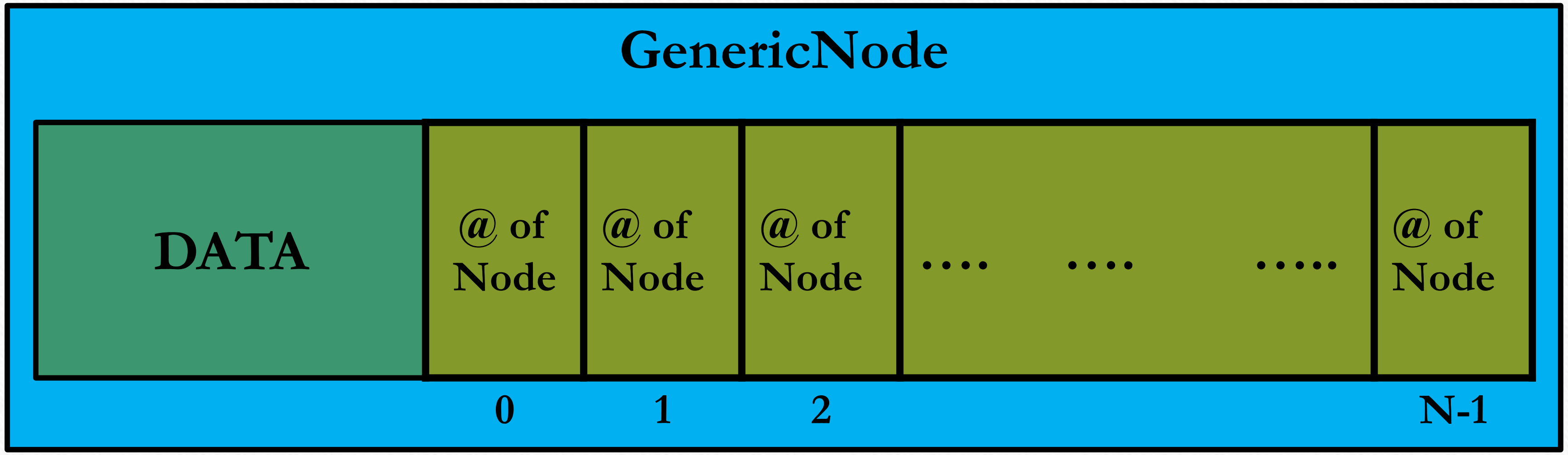
Declaration

```
struct GenericNode {
    int  data;
    GenericNode* children[N];
};
typedef GenericNode *GTREE;
```

V. Types of the Tree Data Structure

General (Generic) Tree

Node in General Tree



V. Types of the Tree Data Structure

Binary Tree

- In general, tree nodes can have any number of children. A **binary tree** is a tree-type **non-linear data structure** with a maximum of two children for each parent (each node can have at most two children).
- A binary tree is either **empty** or consists of a node called the **root** together with two binary trees called **the left subtree and the right subtree.**
- Furthermore, every node in a **binary tree** has a left and right reference along with the data element.

V. Types of the Tree Data Structure

Binary Tree

- Binary trees are **easy to implement** because they have a small, fixed number of child links.
- Because of this characteristic, binary trees are the **most common types of trees** and form the **basis of many importan**t data structures.

V. Types of the Tree Data Structure

Binary Tree

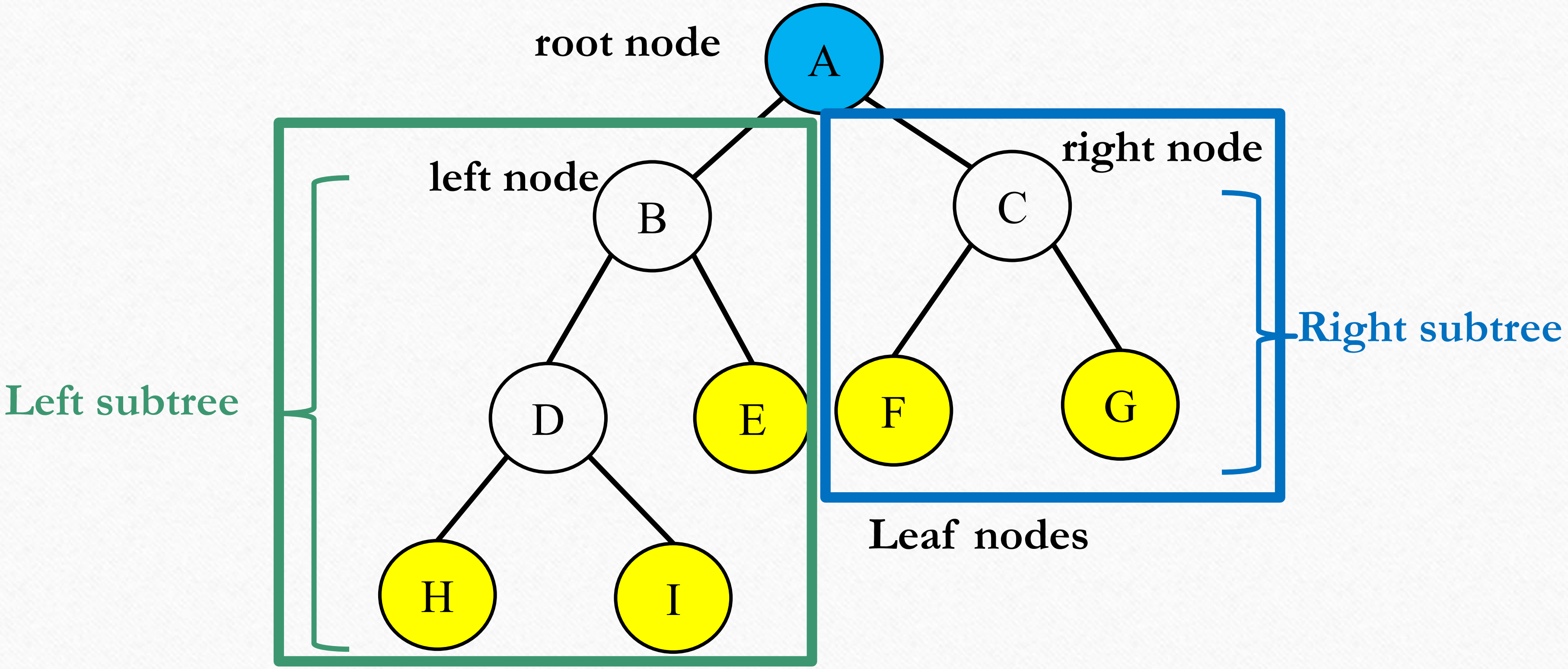


Figure 4.2 : Binary Tree.

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

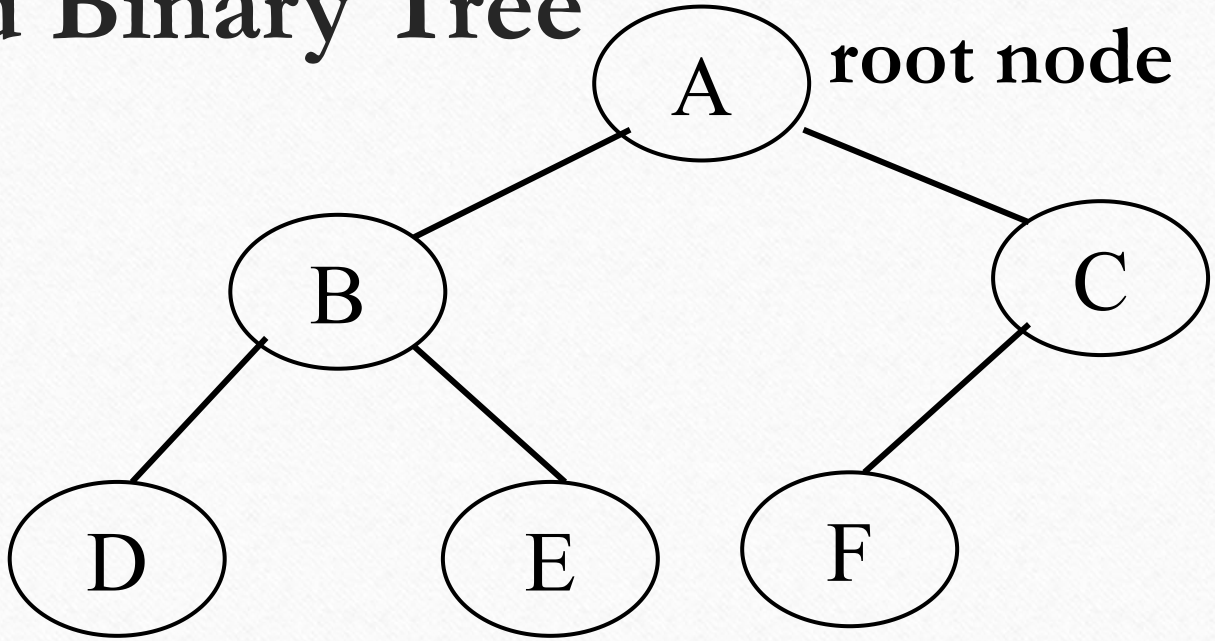
Rooted Binary Tree

- A rooted binary tree is a binary tree that satisfies the following two properties:
  - It has a root node.
  - Each node has at most 2 children.

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

Rooted Binary Tree



Root Binary Tree

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

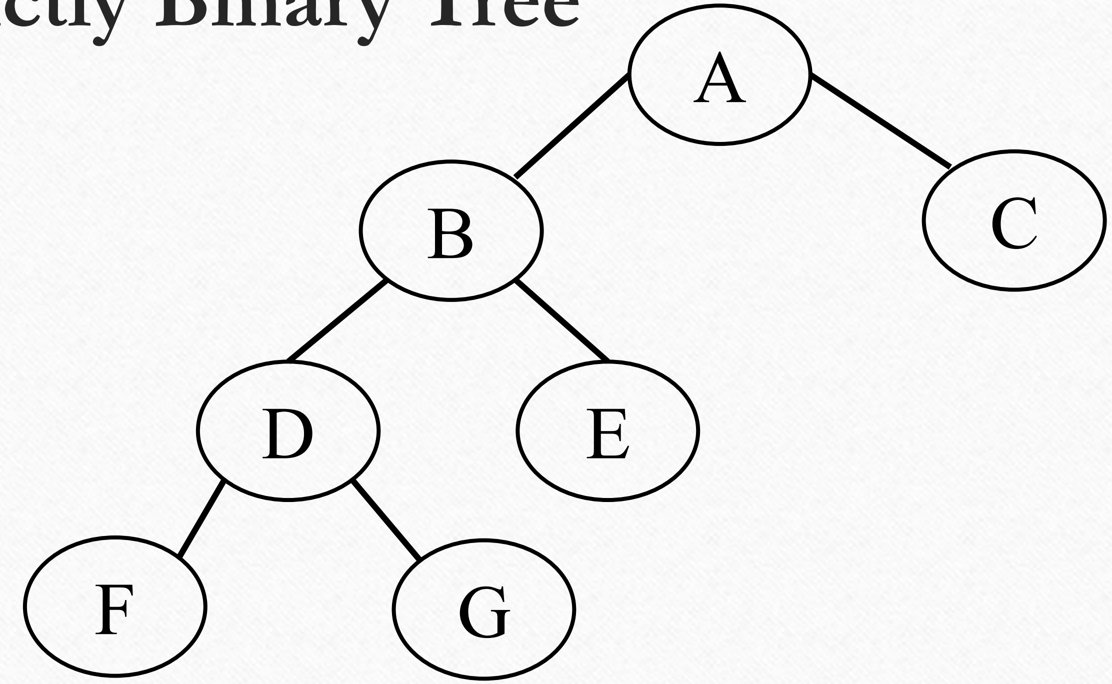
Full / Strictly Binary Tree

- Full binary tree is also called as **Strictly binary tree**.
- It is a special kind of binary tree that has **either zero children or two children**.
- It means that all the nodes in that binary tree should either have two child nodes of its parent node or the parent node is itself the leaf node or the external node.
- Here is the structure of a full (strictly) binary tree.

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

Full / Strictly Binary Tree



Full (Strictly) Binary Tree

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

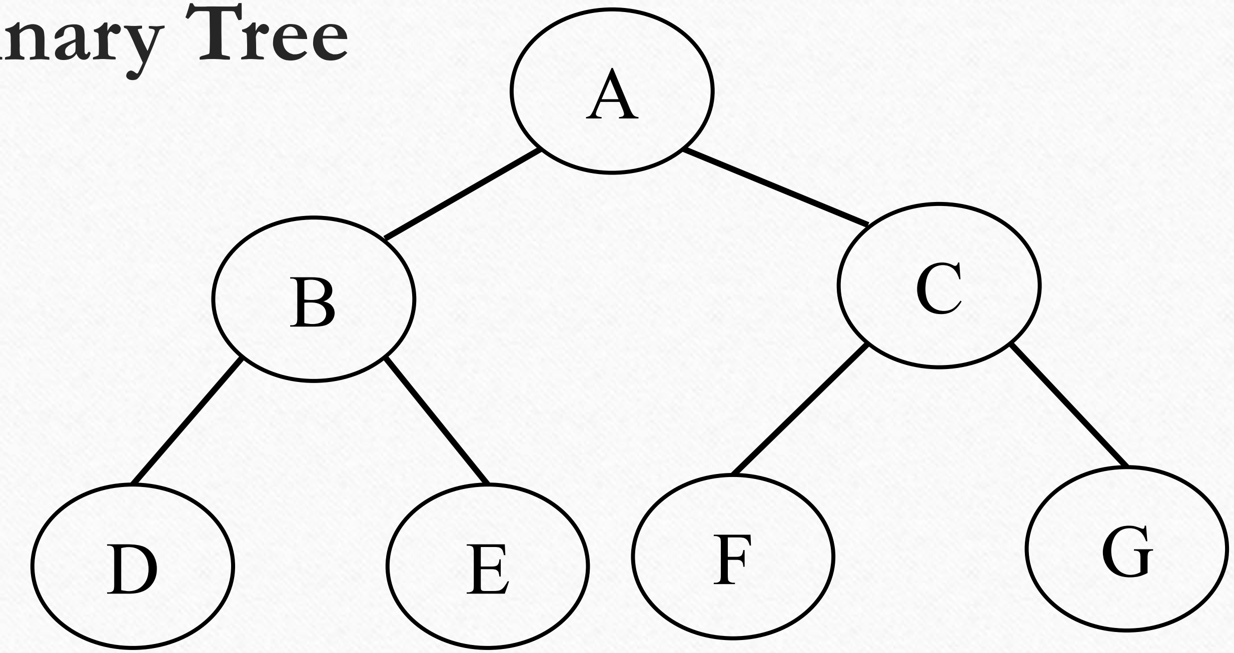
Perfect Binary Tree

- A binary tree is said to be "perfect" if all the internal nodes have strictly two children, and every external or leaf node is at the same level or same depth within a tree.
- A perfect binary tree having height "***H***" has $\mathbf{2^{H+1} - 1}$ node.
- Here is the structure of a perfect binary tree:

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

Perfect Binary Tree



Perfect Binary Tree

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

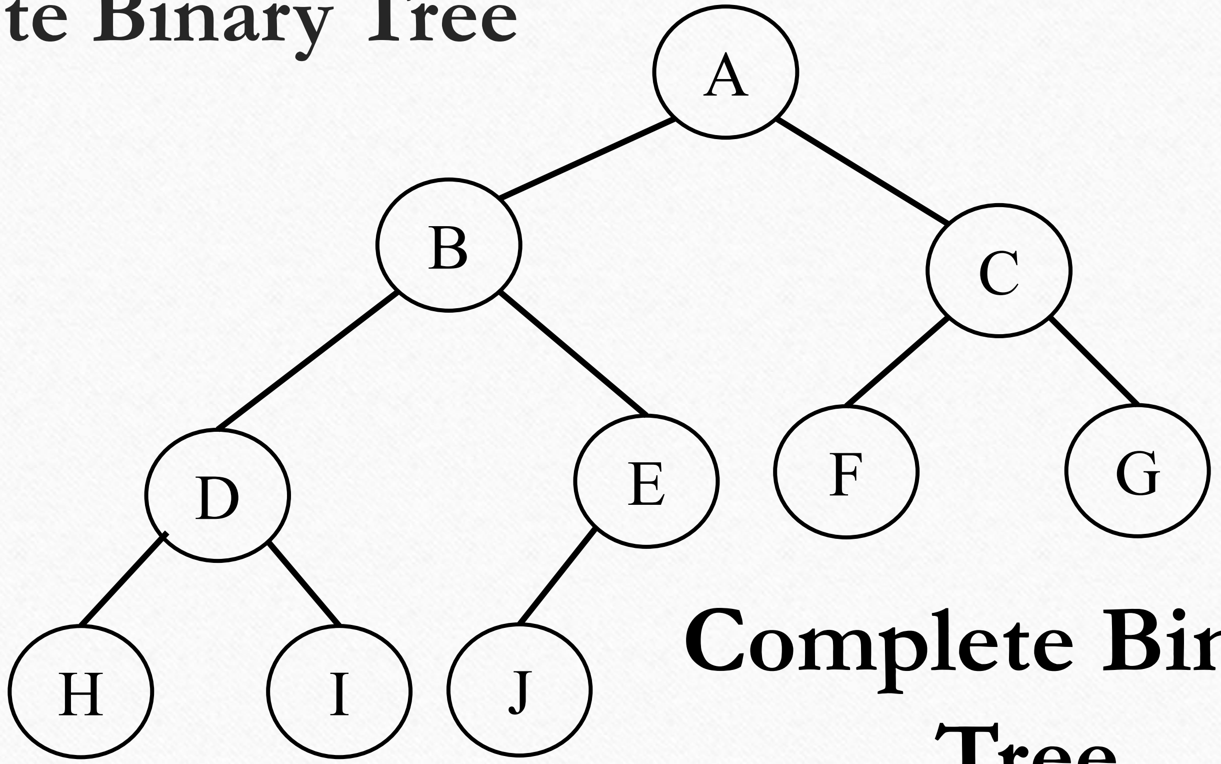
Complete Binary Tree

- An almost complete binary tree is another specific type of binary tree where all the tree levels are filled entirely with nodes, except the lowest level of the tree.
- Also, in the last or the lowest level of this binary tree, every node should possibly reside on the left side.
- Here is the structure of an almost complete binary tree:

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

Complete Binary Tree



Complete Binary Tree

V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

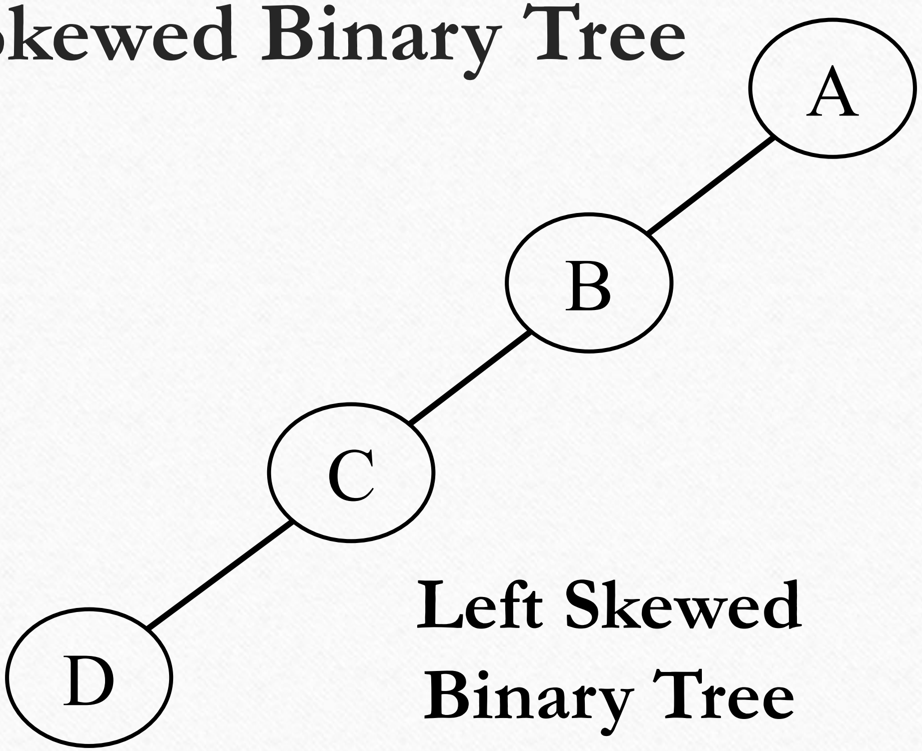
Skewed Binary Tree

- A skewed binary tree is a binary tree that satisfies the following two properties :
  - All the nodes except one node has one and only one child.
  - The remaining node has no child.
- Also, a skewed binary tree is a binary tree of $\boldsymbol{n}$ nodes such that its **depth** is $(\boldsymbol{n-1})$.

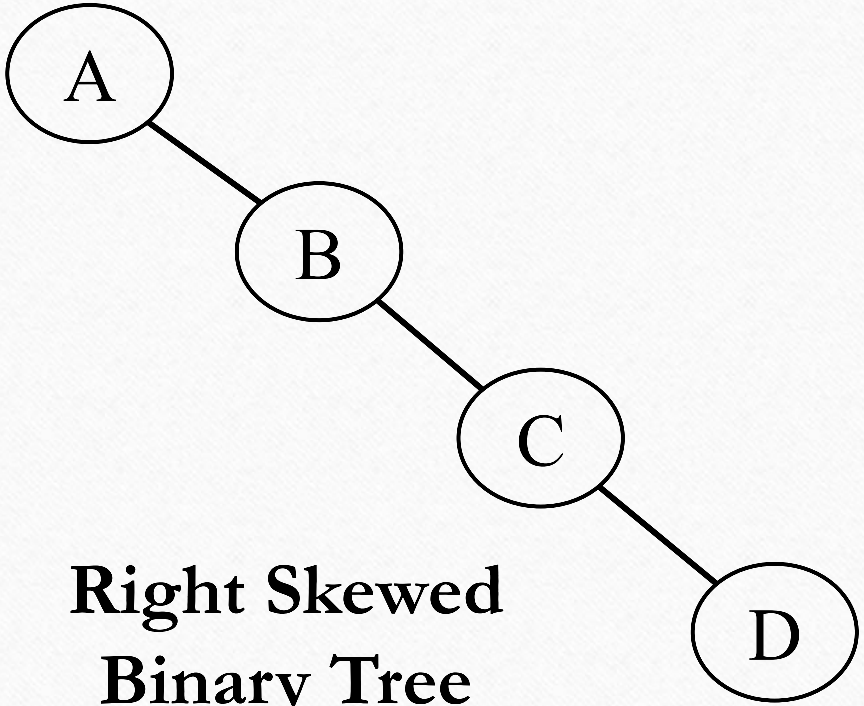
V. Types of the Tree Data Structure (Binary Tree)

Types of Binary Trees

Skewed Binary Tree



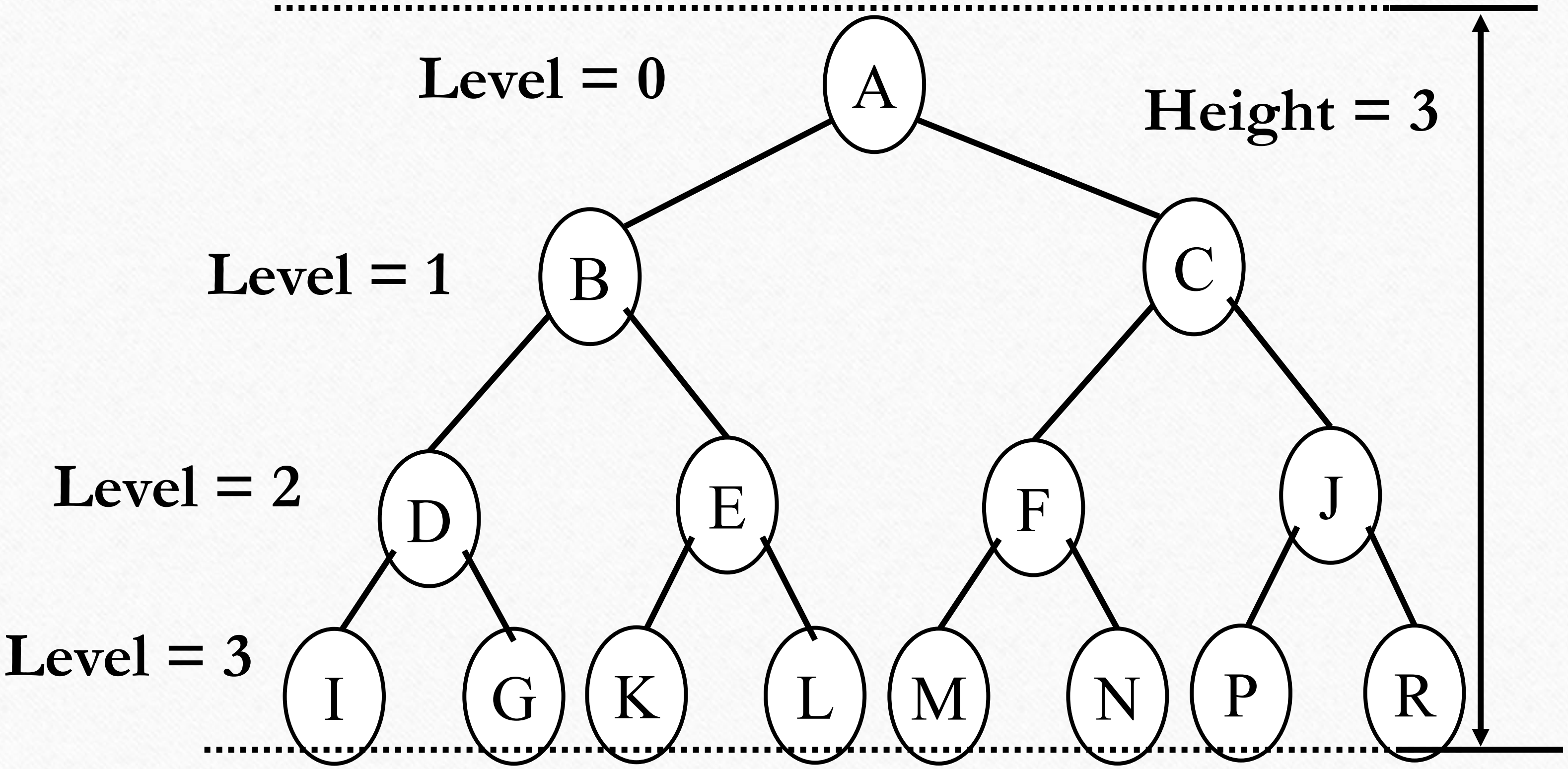
Left Skewed Binary Tree

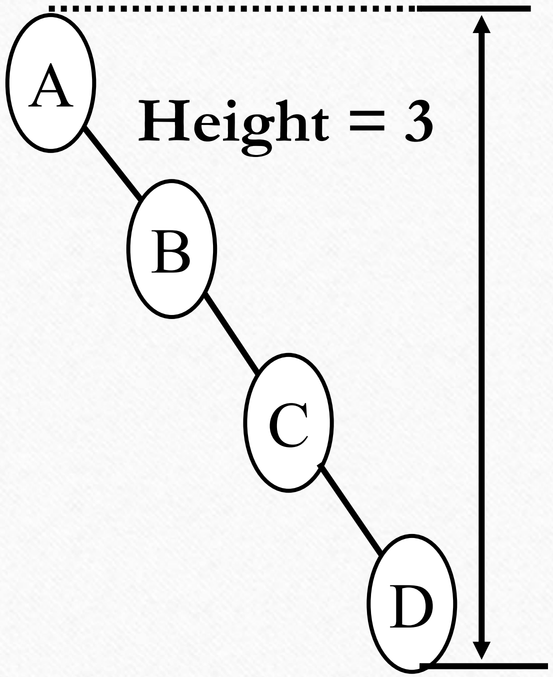


Right Skewed Binary Tree

V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees





V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

Some of the important properties of a binary tree are as follows:

- A tree consisting of only a root node has a height of **0**.

V. Types of the Tree Data Structure (Binary Tree)

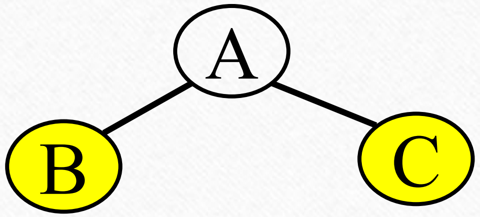
Properties of Binary trees

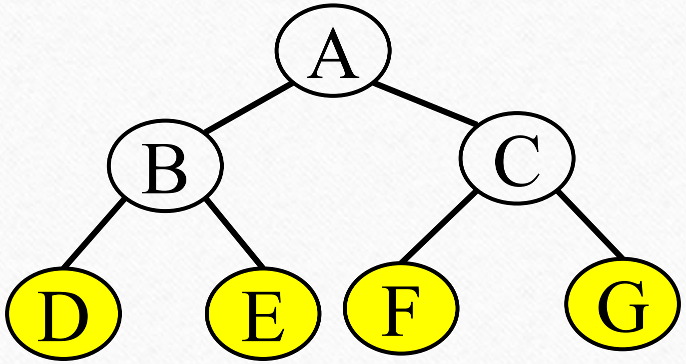
- If $\boldsymbol{H}$ is a height of a binary tree, then
  - Maximum number of leaves $= \mathbf{2}^{\boldsymbol{H}}$
  - Maximum number of nodes $= \mathbf{2}^{\boldsymbol{H}+\mathbf{1}} - \mathbf{1}$
  - Minimum number of nodes $= \boldsymbol{H} + \mathbf{1}$

V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

- If $\boldsymbol{H}$ is a height of a binary tree, then
  - Maximum number of leaves $= \mathbf{2}^{\boldsymbol{H}}$





H = 0
Max nbr of leaves $= \mathbf{2^0 = 1}$

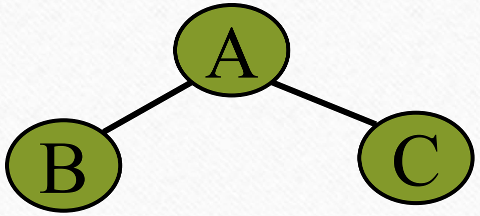
H = 1
Max nbr of leaves $= \mathbf{2^1 = 2}$

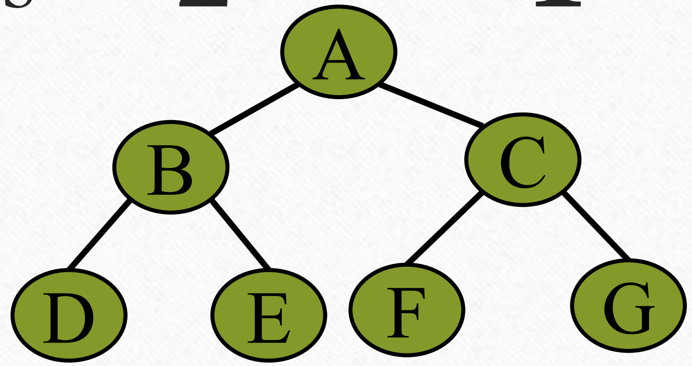
H = 2
Max nbr of leaves $= \mathbf{2^2 = 4}$

V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

- If $\boldsymbol{H}$ is a height of a binary tree, then
  - Maximum number of nodes $= \mathbf{2^{H+1} - 1}$



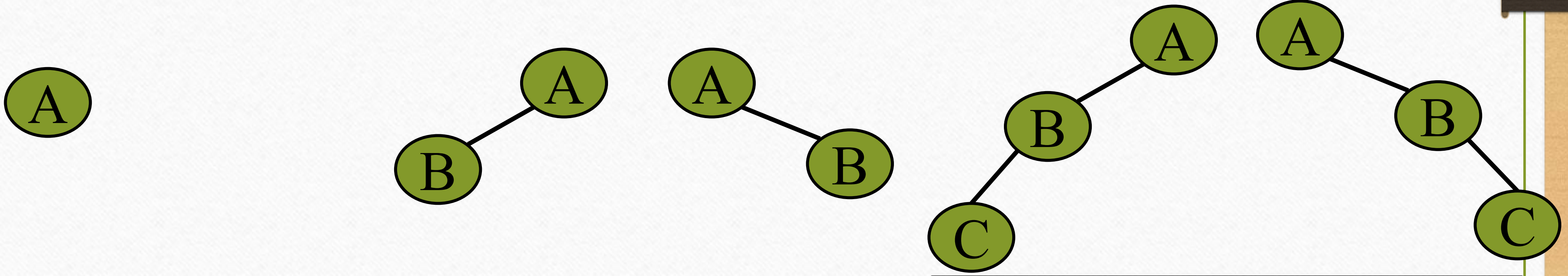


H = 0 Max nbr of nodes = $\mathbf{2^{0+1} - 1 = 1}$	H = 1 Max nbr of nodes =$\mathbf{2^{1+1} - 1 = 3}$	H = 2 Max nbr of nodes =$\mathbf{2^{2+1} - 1} = 7$

V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

- If $\boldsymbol{H}$ is a height of a binary tree, then
  - Minimum number of nodes $= \boldsymbol{H} + \mathbf{1}$

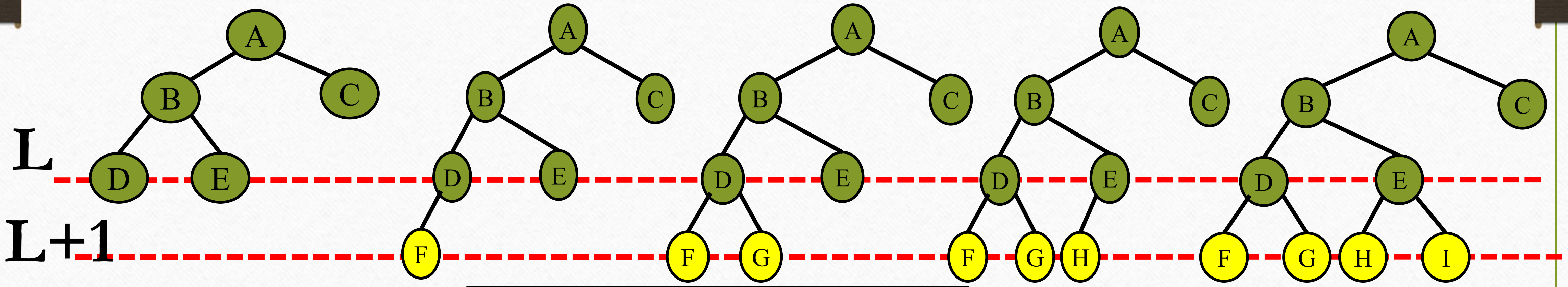


H = 0 Min nbr of nodes = **0** + **1** = **1**	H = 1 Min nbr of nodes =**1** + **1** =2	H = 2 Min nbr of nodes =**2** + **1** = 3

V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

- If a binary tree contains N nodes at level L, it contains at most $2 * N$ nodes at level $L + 1$.

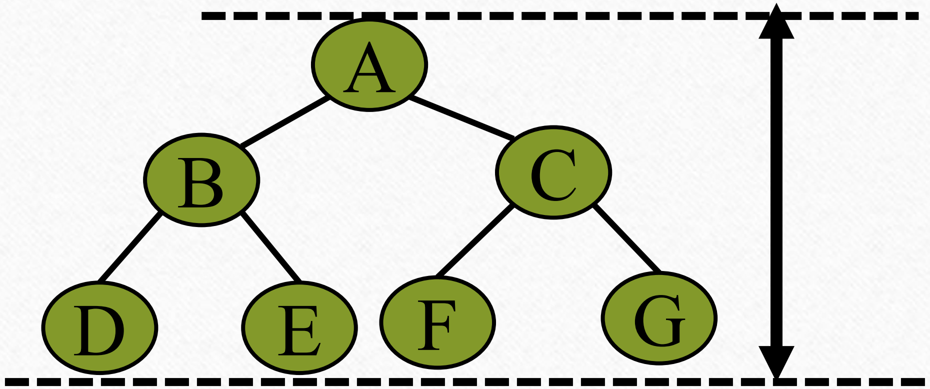


Max nbr in (L)=N = 2
Max nbr in (L +1)=N*2 = 2*2 = 4

V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

- In a Binary Tree with $\boldsymbol{N}$ nodes, the minimum possible height or the minimum number of levels is $\boldsymbol{Log_2(N+1)-1}$
- A Binary Tree with $\boldsymbol{L}$ leaves has at least $|\boldsymbol{Log_2(N)}| + \ \boldsymbol{1}$ levels

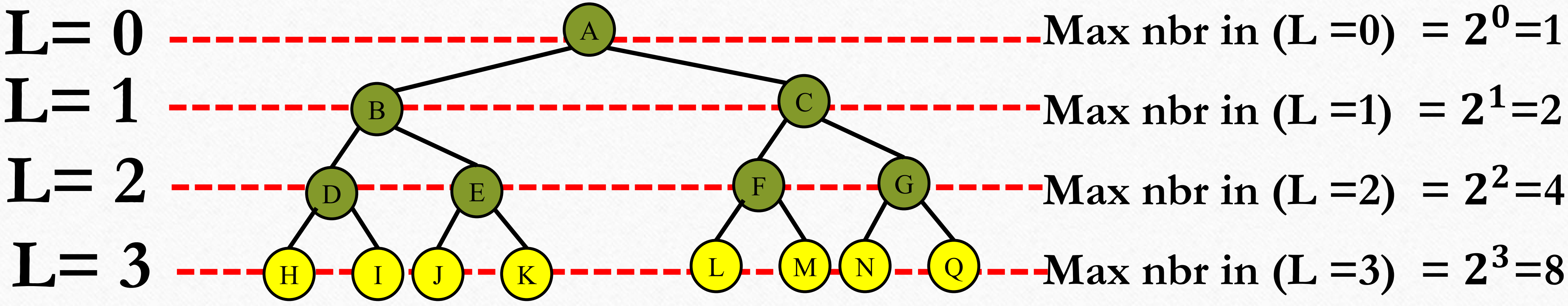


N =7
Height = log(7+1)=log(8)= 2

V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

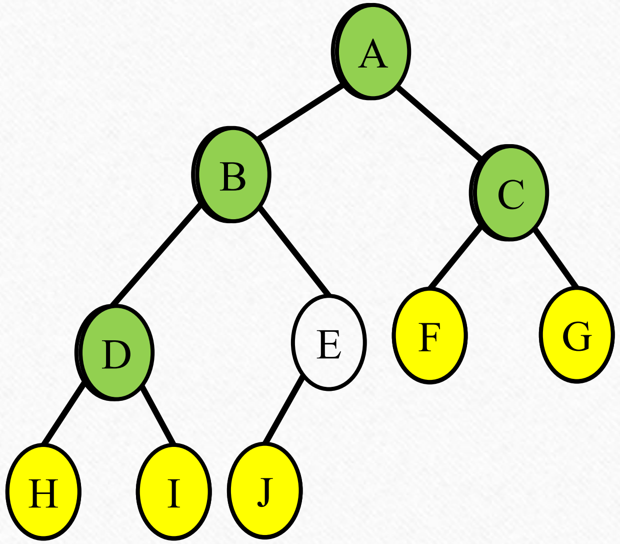
- The maximum number of nodes at any level L in a binary tree $= \mathbf{2^L}$



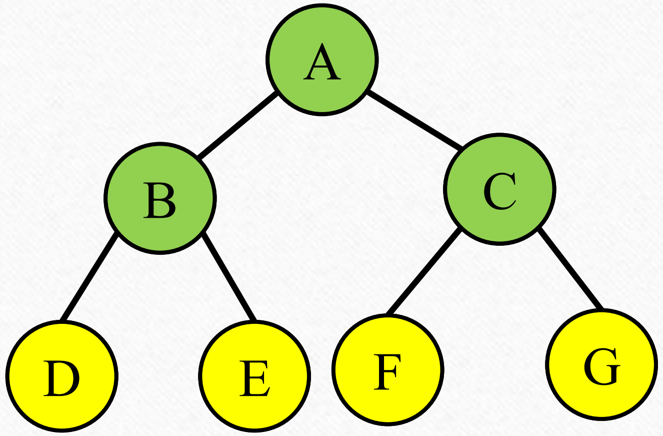
V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

- Total Number of leaf nodes in a Binary Tree is the **Total Number of nodes with 2 children + 1**



Total Nbr of leaf = 5
Nbr of nodes with 2 child + 1 = 4 + 1 = 5



Total Nbr of leaf = 4
Nbr of nodes with 2 child + 1 = 3 + 1 = 4

V. Types of the Tree Data Structure (Binary Tree)

Properties of Binary trees

- The number of nodes $\boldsymbol{N}$ in a **full** binary tree, where $\boldsymbol{H}$ is the height of the tree, is at least $\boldsymbol{N = 2 * H + 1}$ and at most. $\boldsymbol{N = 2^{H+1} - 1}$
- The total number of edges in a binary tree with $\boldsymbol{N}$ node is $\boldsymbol{N - 1}$.
- The number of level nodes in a **perfect** binary tree, is $\boldsymbol{L = (N + 1)/2}$.
- This means that a **perfect** binary tree with $\boldsymbol{L}$ levels has $\boldsymbol{N = 2L - 1}$ nodes.
- The number of **internal nodes** in **an almost complete** binary tree of $\boldsymbol{N}$ nodes is $\boldsymbol{N/2}$.

V. Types of the Tree Data Structure

Convert a Generic Tree to Binary Tree

- If you are given a generic tree $\boldsymbol{T}$, where each node can have more than two children, you can turn this generic tree into a binary tree $\boldsymbol{T'}$ using the following procedure (starting from the root):

1) If a node $\boldsymbol{v}$ has $\boldsymbol{k}$ children $\boldsymbol{v_1, v_2, v_3, \cdots, v_k}$, first set $\boldsymbol{v_1}$ as the left child of $\boldsymbol{v}$ in $\boldsymbol{T'}$.

2) Then, set $\boldsymbol{v_2, v_3, v_4, \cdots, v_k}$ so, that they become a chain of right children of $\boldsymbol{v_1}$ in $\boldsymbol{T'}$.

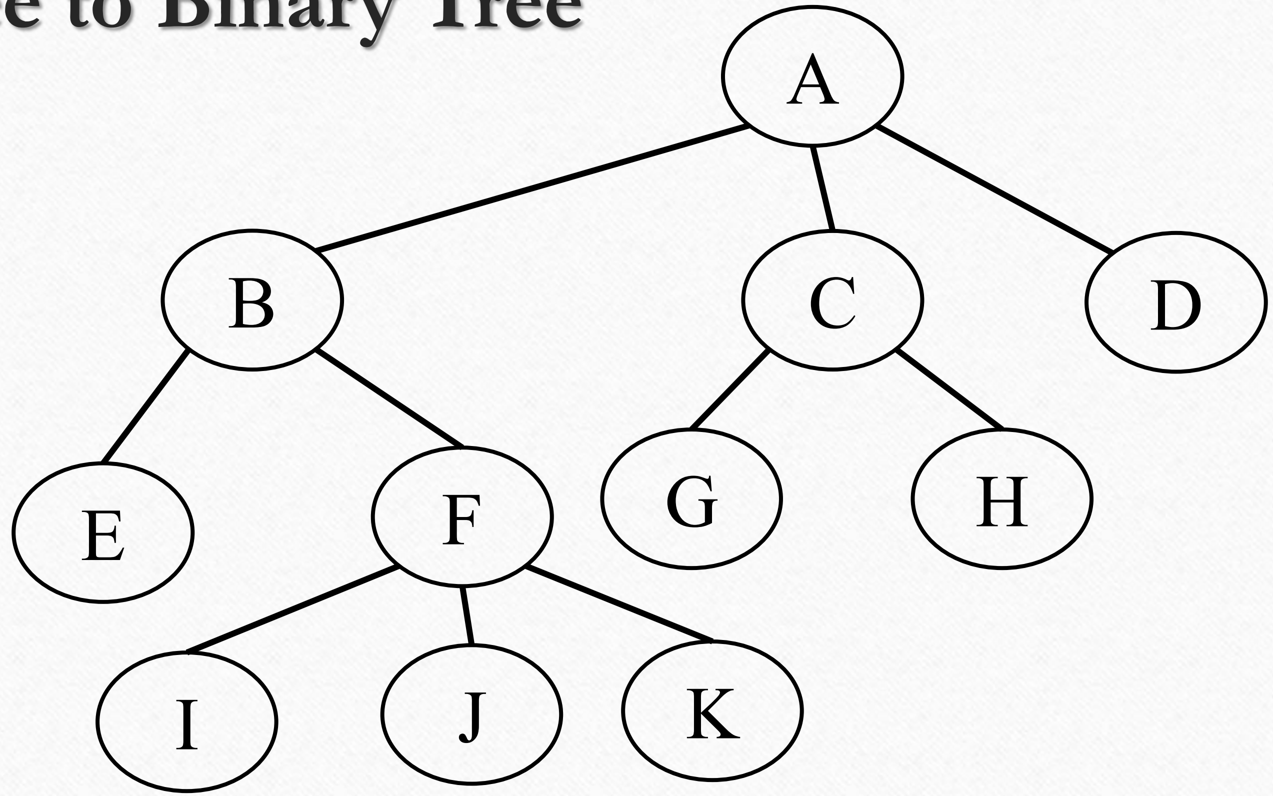
3) Repeat these two steps recursively for the remaining nodes in the tree,

V. Types of the Tree Data Structure

Convert a Generic Tree to Binary Tree

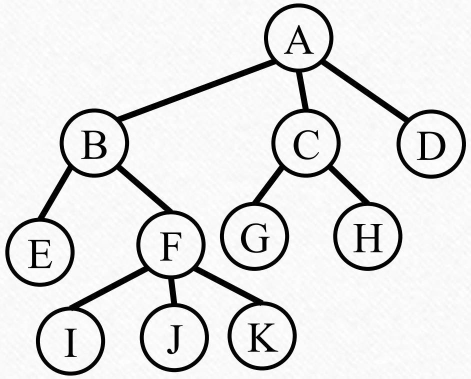
Example :

For instance, suppose we are given the following generic tree, and we want to convert it into a binary tree:



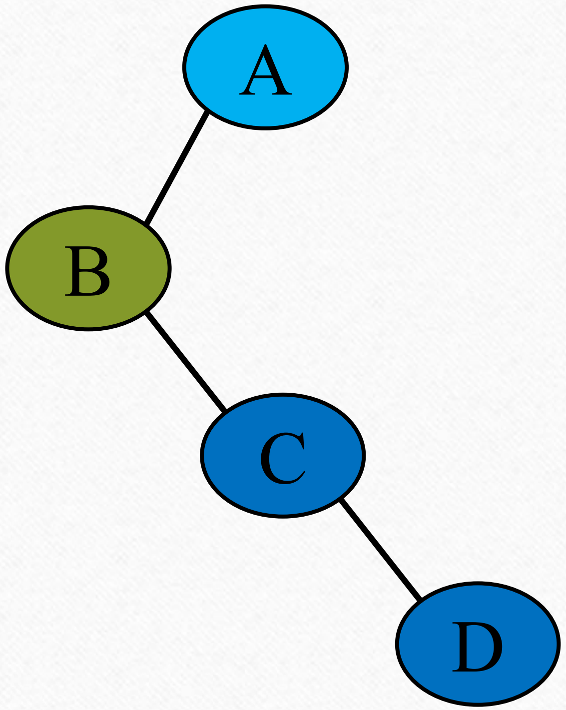
V. Types of the Tree Data Structure

Convert a Generic Tree to Binary Tree



Example :

1) Using the algorithm discussed previously, we first consider the root node $\boldsymbol{A}$. Node $\boldsymbol{A}$ has three children: $\boldsymbol{B}, \boldsymbol{C}$, and $\boldsymbol{D}$, so we set $\boldsymbol{B}$ as the left child of $\boldsymbol{A}$, and then set $\boldsymbol{C}$ and $\boldsymbol{D}$ as a chain of right children of $\boldsymbol{B}$:

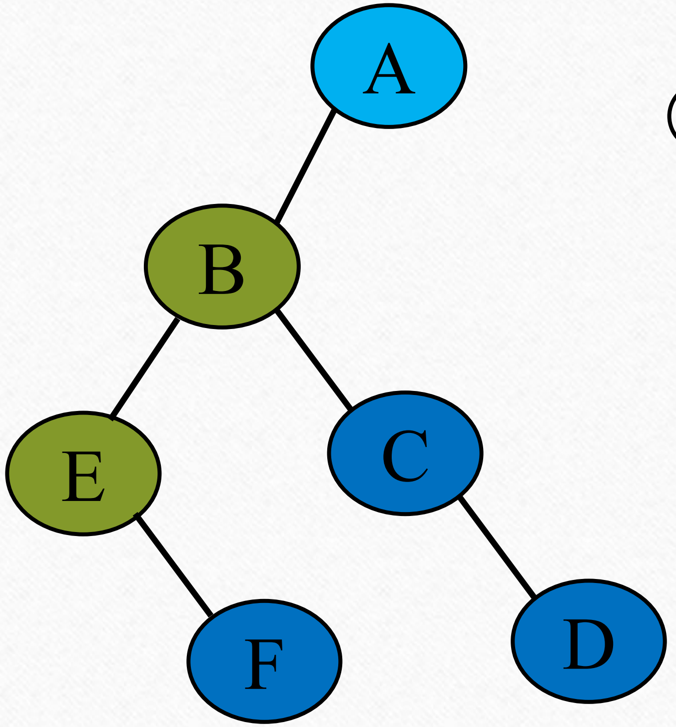


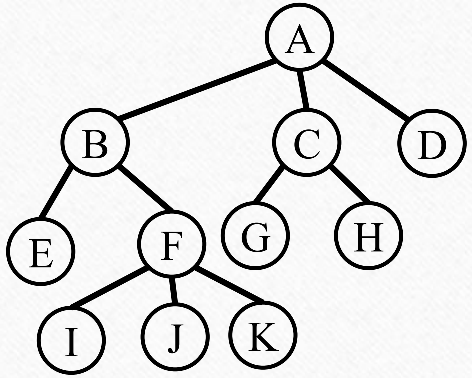
V. Types of the Tree Data Structure

Convert a Generic Tree to Binary Tree

Example :

2) We will repeat these steps recursively. Next, we look at **B**, which has two children: **E** and **F**. We set **E** as the left child of **B**, and set **F** as the right child of E:



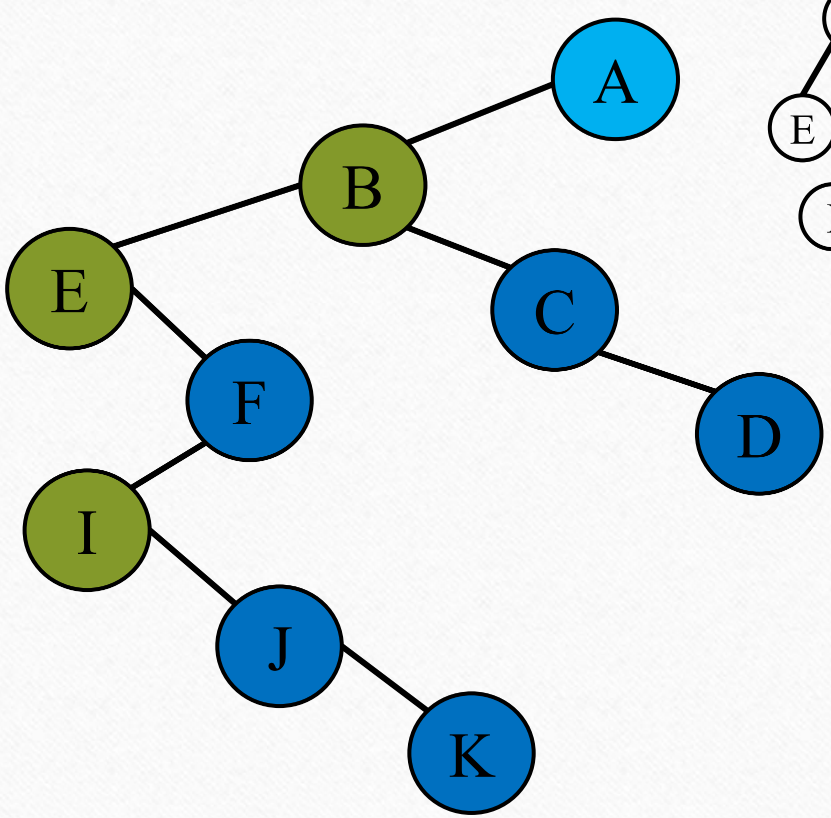


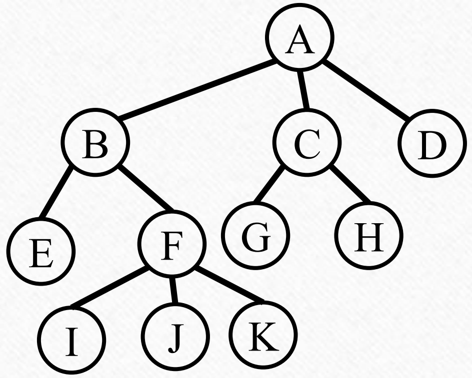
V. Types of the Tree Data Structure

Convert a Generic Tree to Binary Tree

Example :

3) Node **E** has no children, so we do not have to do any additional work for **E**. Node **F**, however, has three children: **I, J**, and **K**. We will set node **I** as the left child of node **F**, and set nodes **J** and **K** as a chain of right children of node **I**.



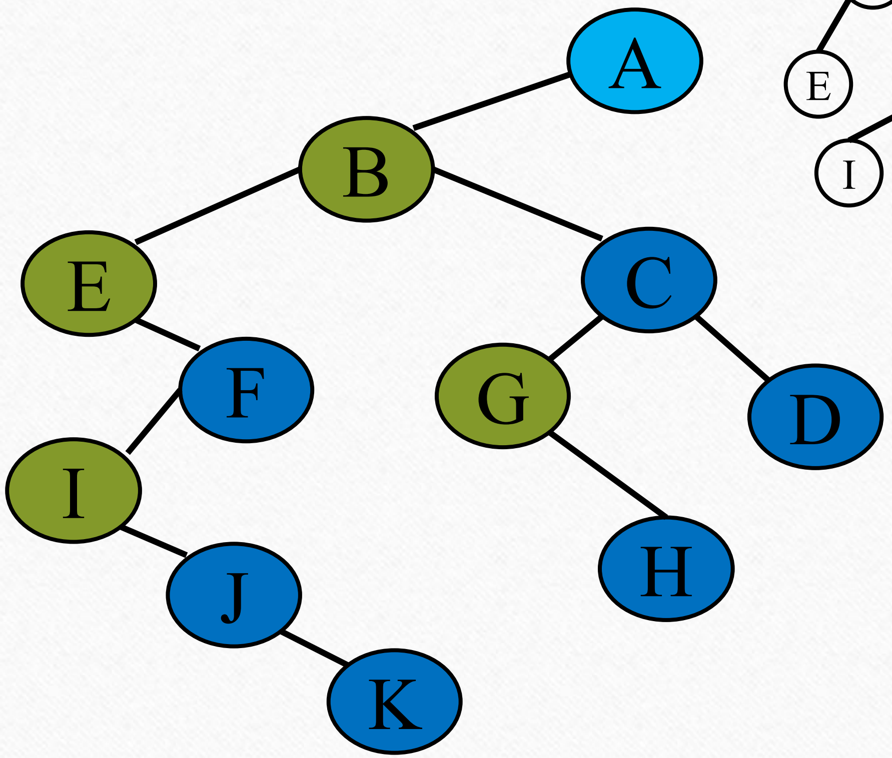


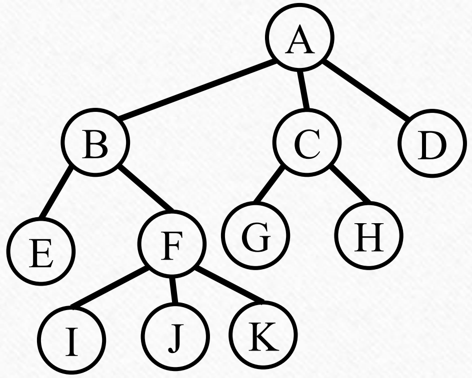
V. Types of the Tree Data Structure

Convert a Generic Tree to Binary Tree

Example :

4) Node C has two children: **G** and **H**. We will set **G** as the left child of **C**, and **H** as the right child of **G**:





V. Types of the Tree Data Structure

Representation of Binary Trees

There are two ways in which a binary tree can be represented. They are:

A. **Array-Based Tree Implementation**

B. **Pointer-Based Tree Implementation**

V. Types of the Tree Data Structure

Array-Based Tree Implementation

- If we implement a binary tree using an array, we essentially flatten out the tree and store its nodes sequentially in memory, level by level.
- To implement a binary tree using an array, we will follow these rules:

V. Types of the Tree Data Structure

Array-Based Tree Implementation

- Elements start being stored from position **1**, with position **0** remaining vacant.
- The root of the binary tree is placed at index **1** of the array.
- The left child of the node at index $\boldsymbol{i}$ should be placed at index $\mathbf{2}\boldsymbol{i}$. If node $\boldsymbol{i}$ has no left child, then index $\mathbf{2}\boldsymbol{i}$ should be empty.

V. Types of the Tree Data Structure

Array-Based Tree Implementation

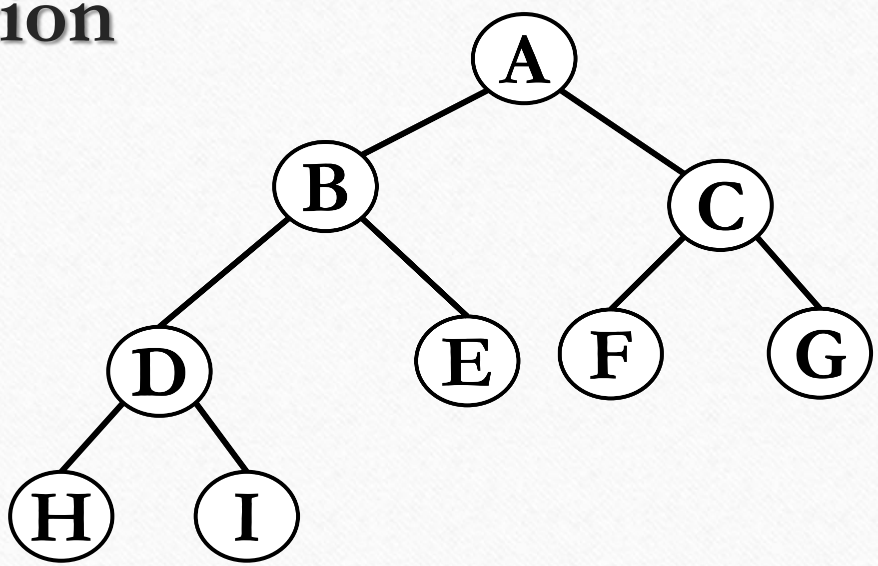
- The right child of the node at index $\boldsymbol{i}$ should be placed at index $\mathbf{2}\boldsymbol{i}+\mathbf{1}$. If node $\boldsymbol{i}$ has no right child, then index $\mathbf{2}\boldsymbol{i}+\mathbf{1}$ should be empty.
- The empty positions in the tree where no node is connected are represented in the array using $\mathbf{-1}$, indicating the absence of a node

V. Types of the Tree Data Structure

Array-Based Tree Implementation

Example 1:

	A	B	C	D	E	F	G	H	I
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]

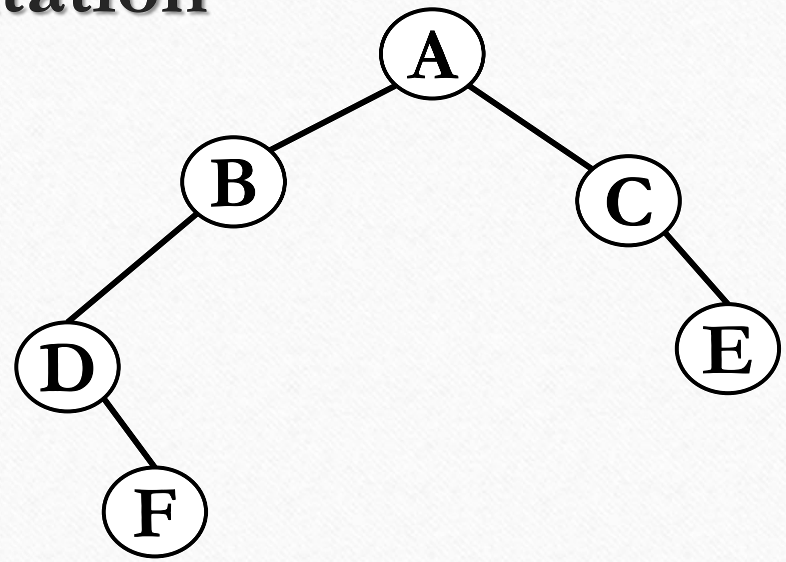


- However, binary trees need not be complete, so we cannot avoid potential gaps in our underlying array.

V. Types of the Tree Data Structure

Array-Based Tree Implementation

Example 2:



	A	B	C	D			E		F
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]

V. Types of the Tree Data Structure

Array-Based Tree Implementation

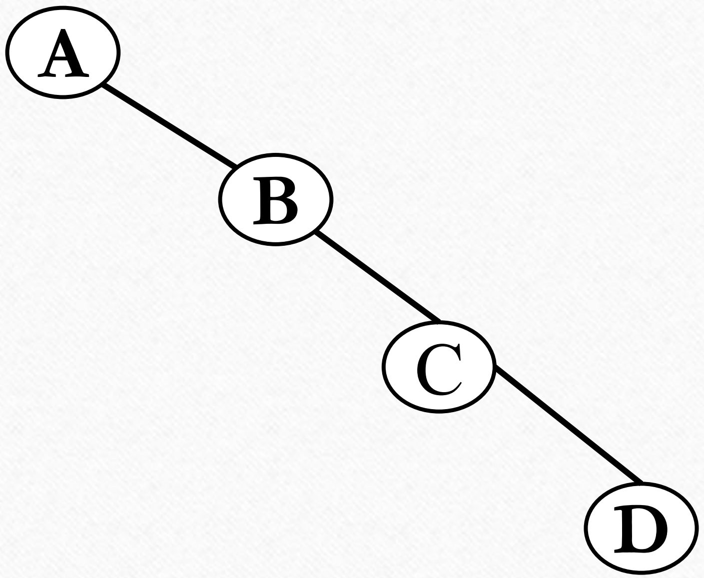
Example 2:

- For example, if we implemented the following binary tree using an array, indices **5**, **6**, and **8** would be empty:
- Because indices in the underlying array may be skipped, an array-based approach can be space-prohibitive for sparse trees (as memory would be wasted by allocating space for nodes that do not exist).

V. Types of the Tree Data Structure

Array-Based Tree Implementation

Example 3:



	A		B				C								D
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]

V. Types of the Tree Data Structure

Space Complexity

- The worst case occurs when we have a **stick**, as shown in the example above.
- When this happens, only a **single node** can exist at each depth of the tree.
- This is because the maximum number of nodes that can exist at any depth $\boldsymbol{d}$ is $\mathbf{2^{d-1}}$, assuming the root has a depth of $\mathbf{1}$.
- Thus, the number of array positions needed to support that depth $\boldsymbol{d}$ is $\mathbf{2^{d-1}}$.

V. Types of the Tree Data Structure

Space Complexity

- If we have a stick, the depth of our tree would be $\boldsymbol{n}$, and the array size we need to support a depth of d is:

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-2} + 2^{n-1} = 2^n - 1$$
$$= \mathcal{O}(2^{\mathrm{n}})$$

- The space complexity is $\mathcal{O}(2^{\mathrm{n}})$, which forces us to use $\mathcal{O}(2^{\mathrm{n}})$ memory.

V. Types of the Tree Data Structure

Pointer-Based Tree Implementation

- An alternative to the array-based approach is the **pointer-based approach.**
- In this approach, pointers are used to link the various nodes of the tree.

V. Types of the Tree Data Structure

Pointer-Based Tree Implementation

Binary Tree Components

- There are three **binary tree components**. Every binary tree node has these three components associated with it. It becomes an essential concept for programmers to understand these three binary tree components:
    - Data element
    - Pointer to left subtree
    - Pointer to right subtree

V. Types of the Tree Data Structure

Pointer-Based Tree Implementation

Binary Tree Components

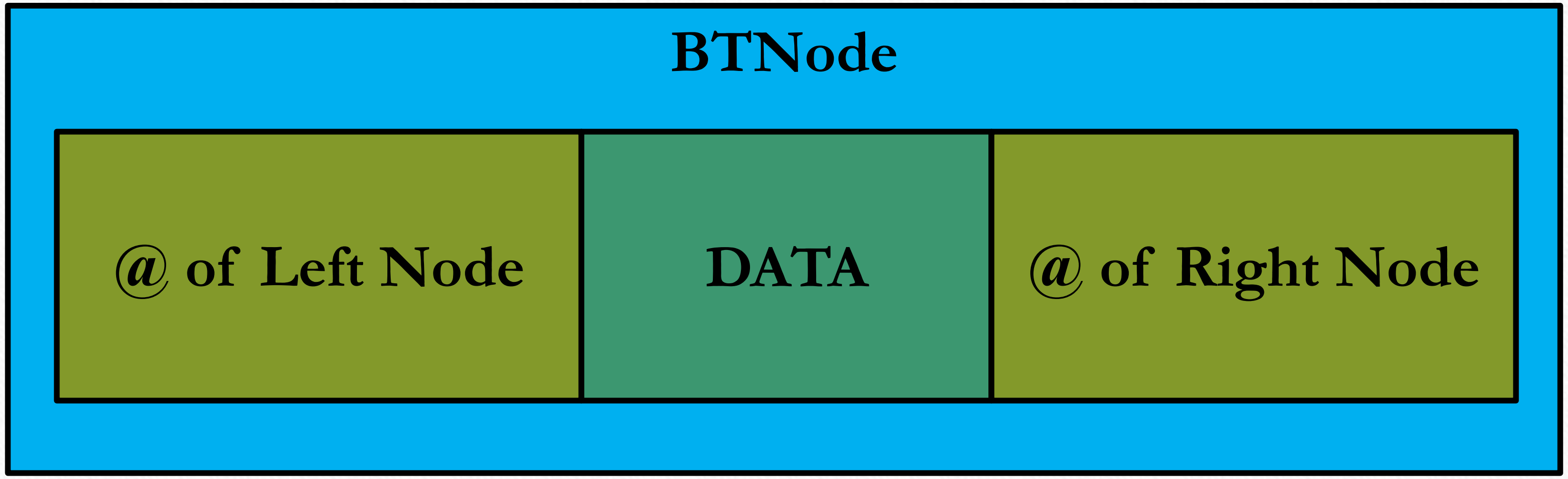
- These three binary tree components represent a node.
- The data resides in the middle.
- The left pointer points to the child node, forming the left sub-tree.
- The right pointer points to the child node at its right, creating the right subtree.

V. Types of the Tree Data Structure

Pointer-Based Tree Implementation

Binary Tree Components

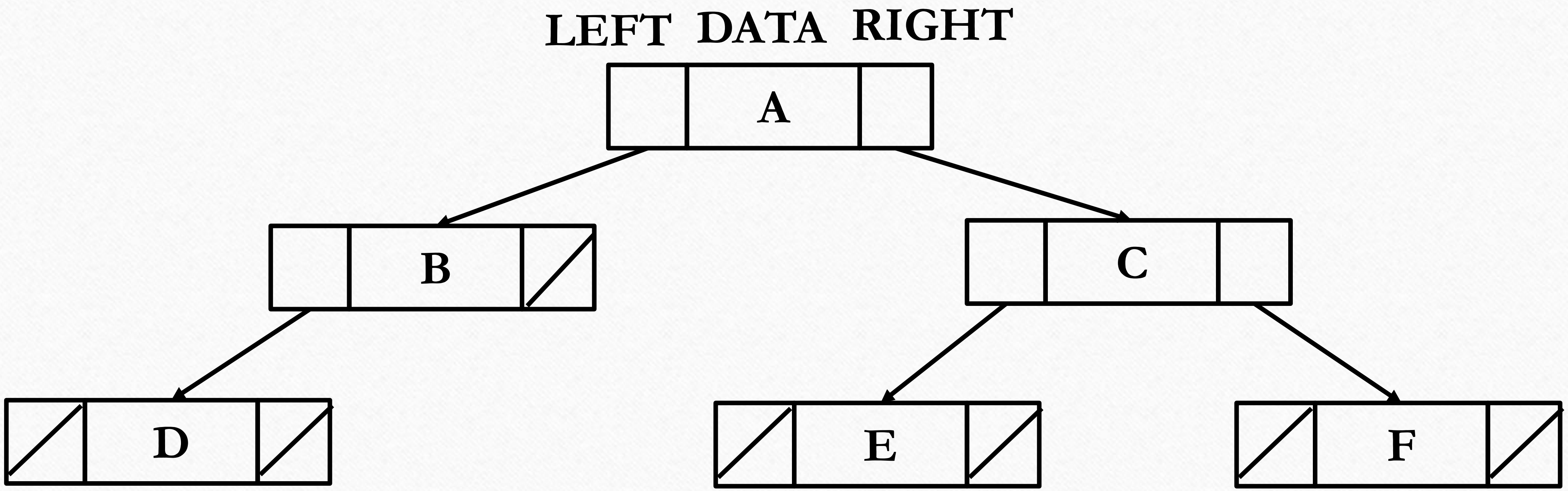
Node in Binary Tree



V. Types of the Tree Data Structure

Pointer-Based Tree Implementation

Binary Tree Components



V. Types of the Tree Data Structure

Binary Tree Node Declarations

```
struct BTNode {
  int data;
  BTNode *left;
  BTNode *right;
};
typedef BTNode *BTREE;
```

Now, we might have a variable of type **BTREE** called BTree:

```
BTREE BTree;
```

V. Types of the Tree Data Structure

Binary Tree Node Declarations

- The pointers storing **NULL** value indicates that there is no node attached to it.
- An alternative would be to store a parent pointer within each node, as shown in the Node definition below.
- **If** the node is the root, the parent pointer is **NULL** (**nullptr**);
- **Otherwise**, it points to the node's parent.

V. Types of the Tree Data Structure

Binary Tree Node Declarations

- Using this approach, the time complexity of finding a node's parent would be $\mathcal{O}(\mathbf{1})$ instead of $\mathcal{O}(\boldsymbol{n})$, since we have direct access to the parent.

```
struct BTNode {
    int data;
    BTNode *parent;
    BTNode *left;
    BTNode *right;
};
```

V. Types of the Tree Data Structure

Binary Tree Node Declarations

- However, this implementation is rarely used because it is memory intensive, and the benefit of a parent isn't often worth the extra memory.
- Usually, **we do not need a parent pointer**, since the behavior of a parent pointer can be emulated using recursion (i.e., when a recursive call unrolls, we automatically move from a child to its parent; there's no need to store an explicit pointer).

V. Types of the Tree Data Structure

Tree Traversals

- If you want to process every node in a tree, you will have to complete a **tree traversal**:
- a systematic method for processing every node in a tree.
- In this section, we will look at four different tree traversal methods: the **preorder**, **inorder**, **postorder**, and **level-order traversals**, each of which visit the nodes of a tree in a different order.
- The first three of these traversals (preorder, inorder, postorder) are recursive, while the level-order traversal is iterative.

V. Types of the Tree Data Structure

Tree Traversals

- The names of these traversals actually provide insight into their behavior:
- the prefix of the traversal name ("pre", "in", and "post") tells us when to process the parent node (i.e., the node you are currently visiting) relative to its children.
- In a **preorder traversal**, the parent node is processed **before** its left and right children;
- in a **postorder traversal**, the parent node is processed **after** its left and right children;
- and in an **inorder traversal**, the parent node is processed **in between** its left and right children.

V. Types of the Tree Data Structure

Preorder Traversal

- To conduct a preorder traversal, you would complete the following steps, in this order:
    1) process the parent node (the node the recursion is currently on)
    2) recursively process the left subtree
    3) recursively process the right subtree

V. Types of the Tree Data Structure

Preorder Traversal

- The code for a preorder traversal is shown below:

```
void preorder(BTREE BTree) {
   if (!BTree) return;
   process(BTree -> data);      // process the current node
   preorder(BTree -> left);     // recurse into left child
   preorder(BTree -> right);    // recurse into right child
}
```

V. Types of the Tree Data Structure

Preorder Traversal

- Note that the **process()** function in the code above is just a placeholder for the work that is done on each node.

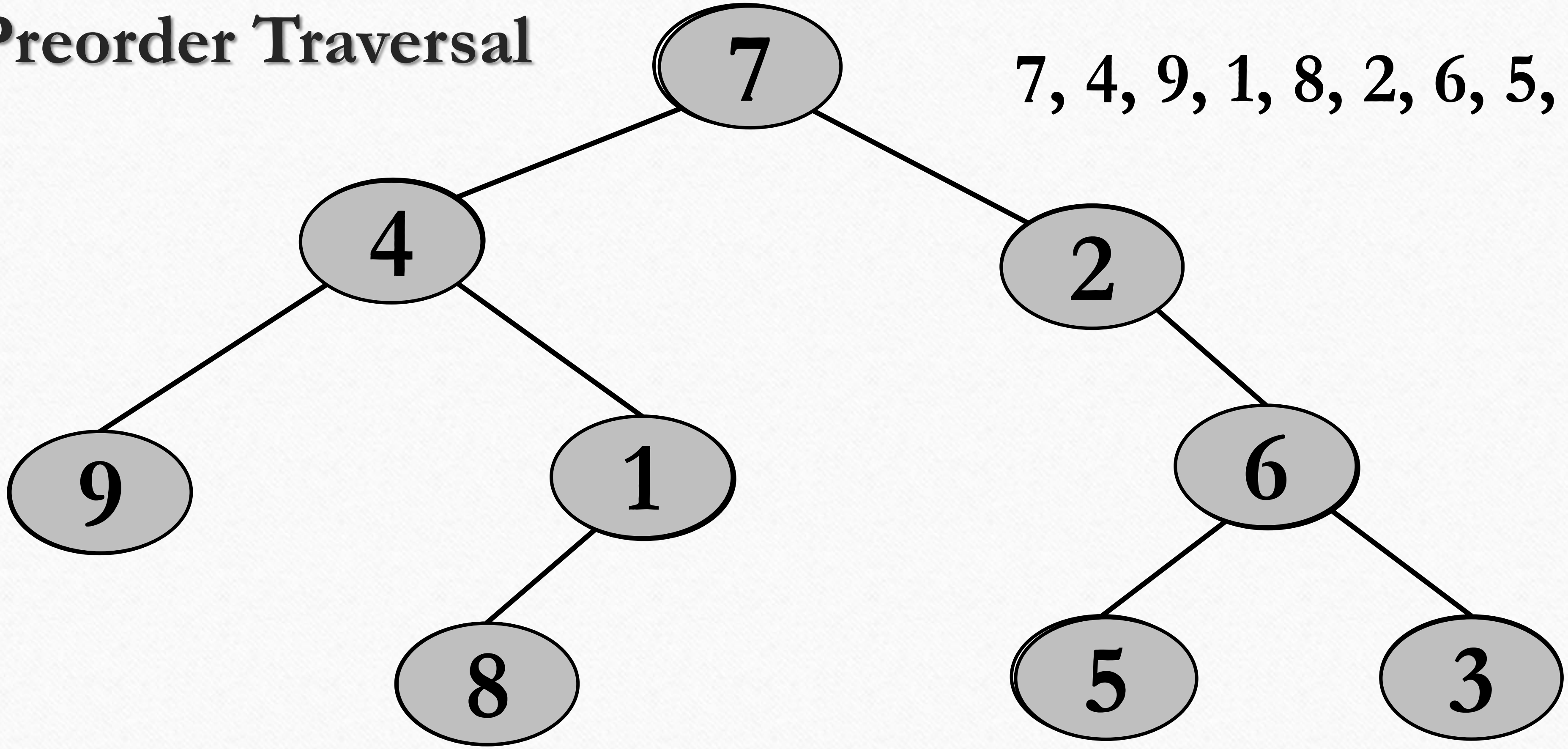
Example:

- For example, to print out the nodes in a preorder traversal, we use a print statement in **process()** or replace **process()** with a print statement:

```
void process(int data) {
    cout << data << endl;
}
```

V. Types of the Tree Data Structure

Preorder Traversal



7, 4, 9, 1, 8, 2, 6, 5, 3

V. Types of the Tree Data Structure

Uses of Postorder Traversal:

- Postorder traversal is used to **delete the tree**.
- Postorder traversal is also useful to get the postfix expression of an expression tree.
- Postorder traversal can help in **garbage collection** algorithms, particularly in systems where manual memory management is used.

V. Types of the Tree Data Structure

Inorder Traversal

- To conduct an inorder traversal, you would complete the following steps, in this order:
  1) recursively process the left subtree
  2) process the parent node (the node the recursion is currently on)
  3) recursively process the right subtree

V. Types of the Tree Data Structure

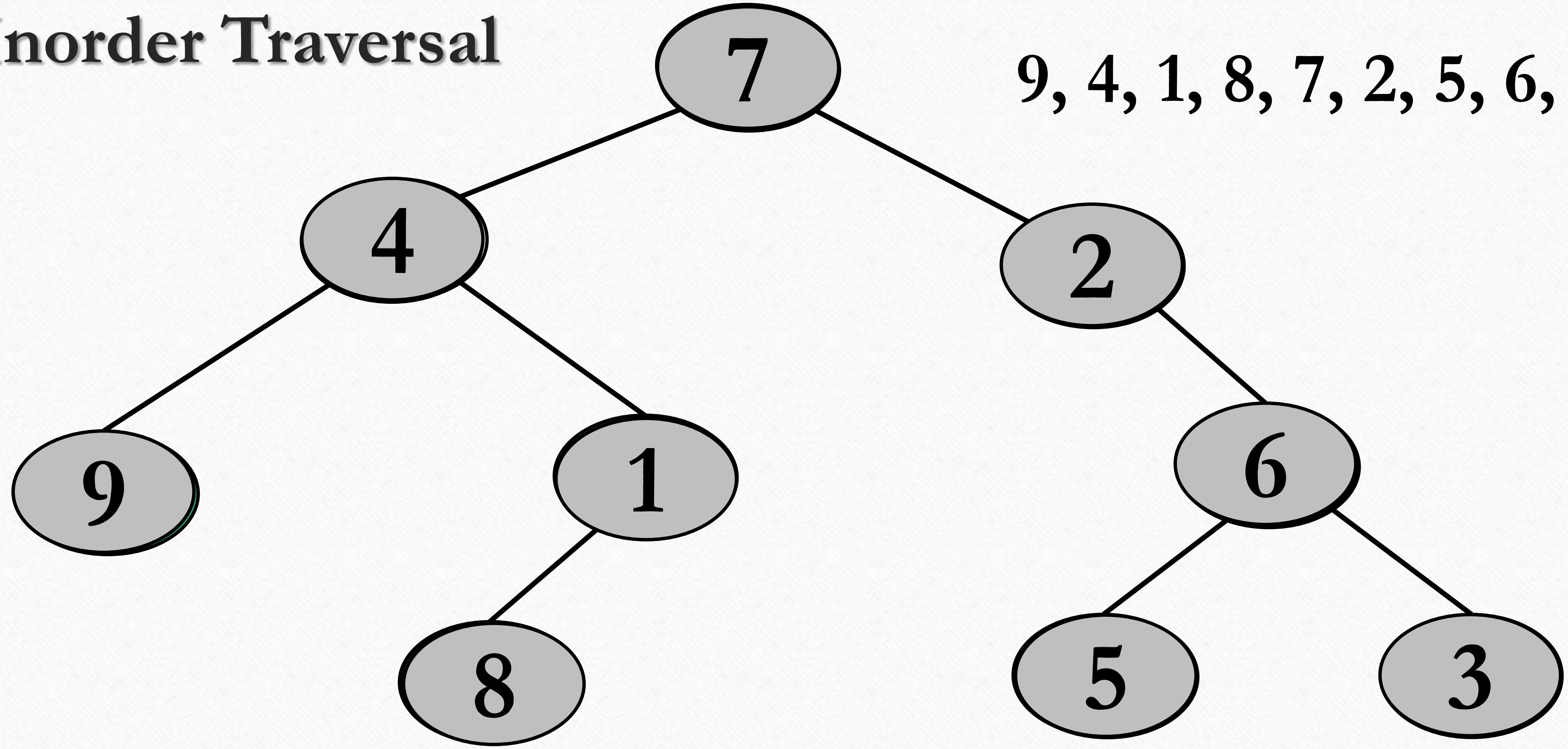
Inorder Traversal

- The code for an inorder traversal is shown below:

```
void inorder(BTREE BTree) {
   if (!BTree) return;
   inorder(BTree -> left);        // recurse into left child
   process(BTree -> data);       // process the current node
   inorder(BTree -> right); // recurse into right child
}
```

V. Types of the Tree Data Structure

Inorder Traversal



9, 4, 1, 8, 7, 2, 5, 6, 3

V. Types of the Tree Data Structure

Uses of Inorder Traversal:

- In the case of binary search trees (BST), Inorder traversal gives nodes in non-decreasing order.
- To get nodes of BST in non-increasing order, a variation of Inorder traversal where Inorder traversal is reversed can be used.
- Inorder traversal can be used **to evaluate arithmetic expressions stored in expression trees.**

V. Types of the Tree Data Structure

Postorder Traversal

- To conduct a postorder traversal, you would complete the following steps, in this order:
  1) recursively process the left subtree
  2) recursively process the right subtree
  3) process the parent node (the node the recursion is currently on)

V. Types of the Tree Data Structure

Postorder Traversal

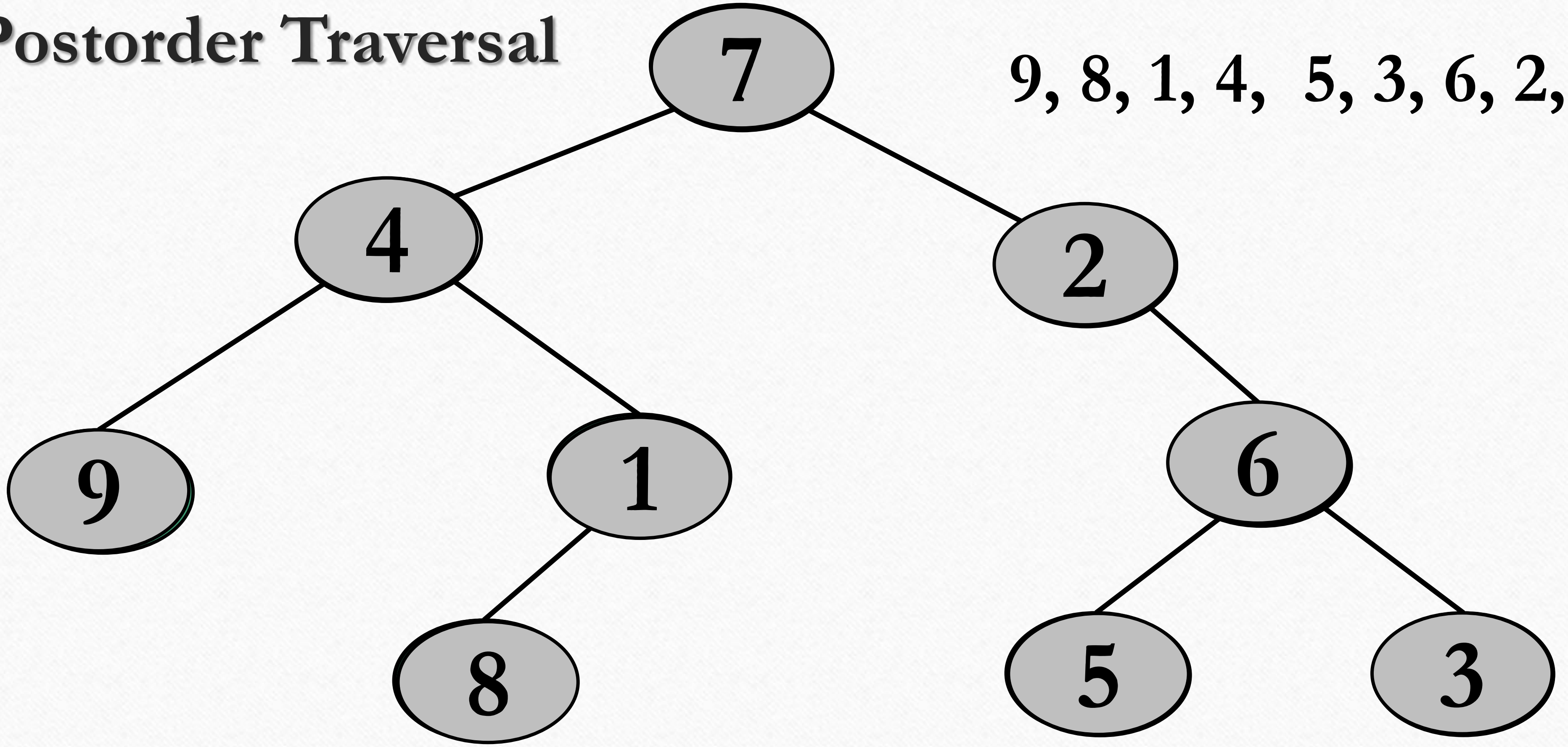
- The code for an postorder traversal is shown below:

```
void postorder(BTREE BTree) {
   if (!BTree) return;
   postorder(BTree -> left); // recurse into left child
   postorder(BTree -> right);// recurse into right child
   process(BTree -> data);      // process
   the current node
}
```

V. Types of the Tree Data Structure

Postorder Traversal

9, 8, 1, 4, 5, 3, 6, 2, 7



V. Types of the Tree Data Structure

Uses of Preorder Traversal:

- Preorder traversal is used to **create a copy** of the tree.
- Preorder traversal is also used to get prefix expressions on an expression tree.

V. Types of the Tree Data Structure

Level-Order Traversal

- The **level-order traversal** is another traversal method that can be used to process the nodes of a tree.
- In a level-order traversal, nodes are processed **in order of increasing depth**, where nodes on the same level (i.e., nodes that have the same depth) are **processed from left to right.**

V. Types of the Tree Data Structure

Level-Order Traversal

- Unlike the other three traversal methods, the **level-order traversal is iterative** rather than recursive.
- To conduct a level-order traversal, a **queue** is used in place of recursive calls.
- The code for a level-order traversal is shown below:

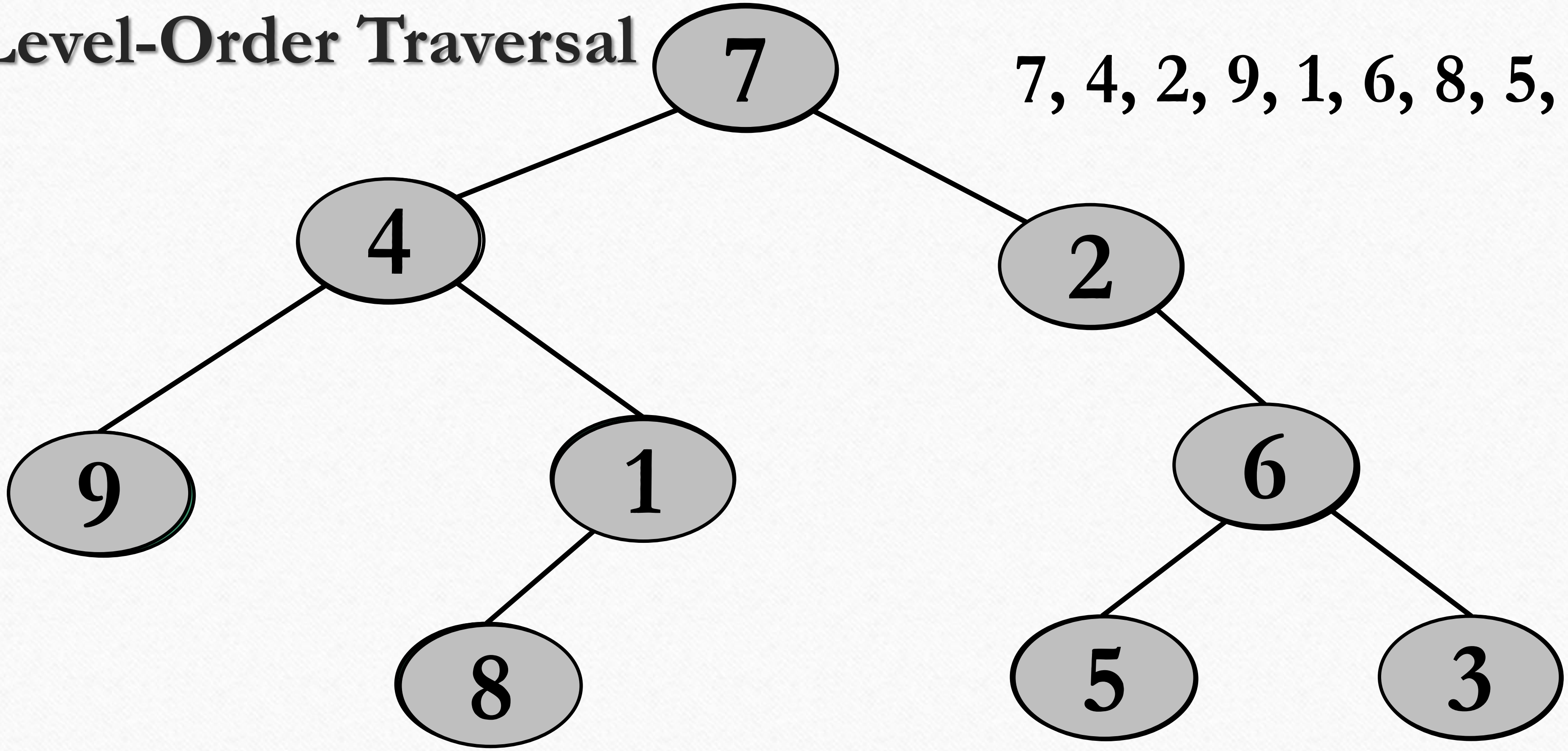
V. Types of the Tree Data Structure

Level-Order Traversal

```
void level_order(BTREE BTree) {
    if (!BTree) return;
    QUEUE bfs = Init_Queue();
    EnQueue (bfs, BTree);                   // push root into queue
    while (!isEmpty(bfs)) {                 // while queue not empty
        BTNode* current = DeQueue(bfs);
        process(current -> data);
        if (current->left)      EnQueue(bfs, current->left);
        if (current->right)     EnQueue(bfs, current-> right);
    } // while
} // level_order()
```

V. Types of the Tree Data Structure

Level-Order Traversal



7, 4, 2, 9, 1, 6, 8, 5, 3

V. Types of the Tree Data Structure

Example :

Traverse the following binary tree in pre, post, inorder and level order:

- **Preorder** traversal yields:

  P, F , B, H, G , S, R, Y, T, W, Z

- **Inorder** traversal yields:

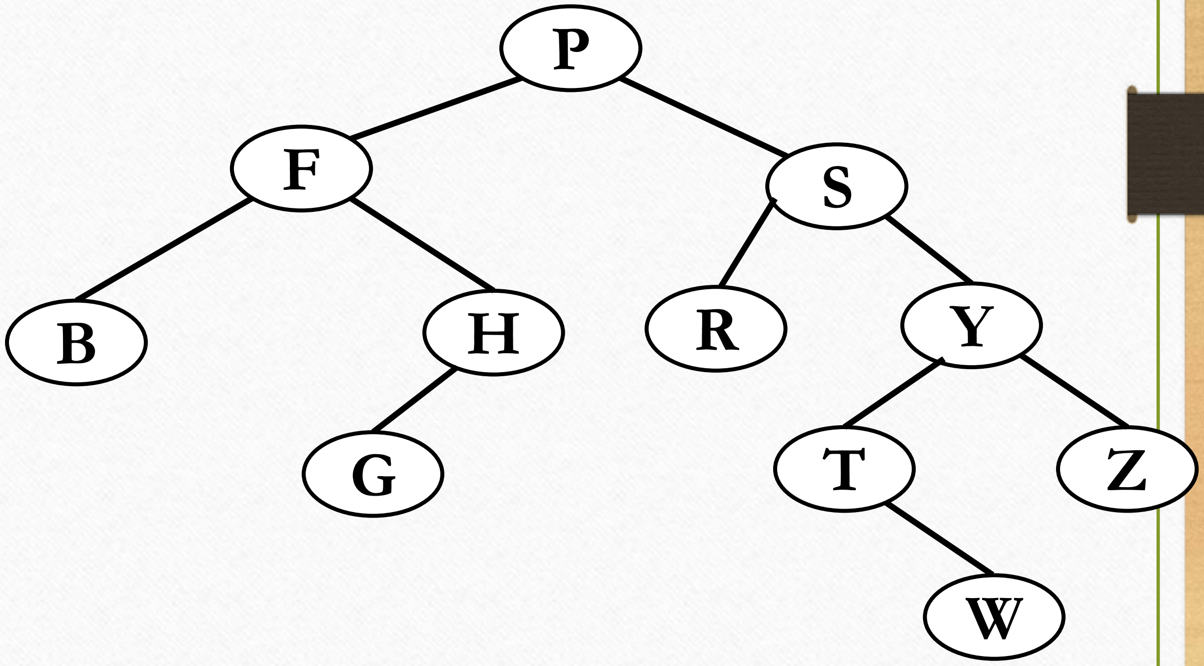
  B, F , G , H, P, R, S, T, W, Y, Z

- **Postorder** traversal yields:

  B, G , H, F , R, W, T, Z, Y, S, P

- **Level order** traversal yields:

  P, F , S, B, H, R, Y, G , T, Z, W



V. Types of the Tree Data Structure

Binary Search Trees (BST)

- One pitfall of the standard binary tree is that **its elements are not ordered in any manner.**
- This can make **searching difficult:**
- if you wanted to find a given element, that element could be anywhere in the tree!
- To address this issue, we will introduce the **binary search tree (BST)**, which is a binary tree whose elements are ordered based on a sorting invariant.

V. Types of the Tree Data Structure

Binary Search Trees (BST)

- A special type of binary trees, called **binary search tree** is a binary tree in **symmetric order**.
- Each node has a key, and for each node in the tree the value of any node in its **left subtree is less** than the value of the node and the value of any node in its **right subtree is greater** than the value of the node.

V. Types of the Tree Data Structure

Binary Search Trees (BST)

A standard non-empty binary search tree exhibits the following properties:

- The key of any node is always **greater** than all of the keys in its left subtree.
- The key of any node is always **less** than or equal to all of the keys in its right subtree.
- Both the left and right subtrees of any node are **also binary search trees themselves.**

V. Types of the Tree Data Structure

Binary Search Trees (BST)

The following given figure is an example of a binary search tree.

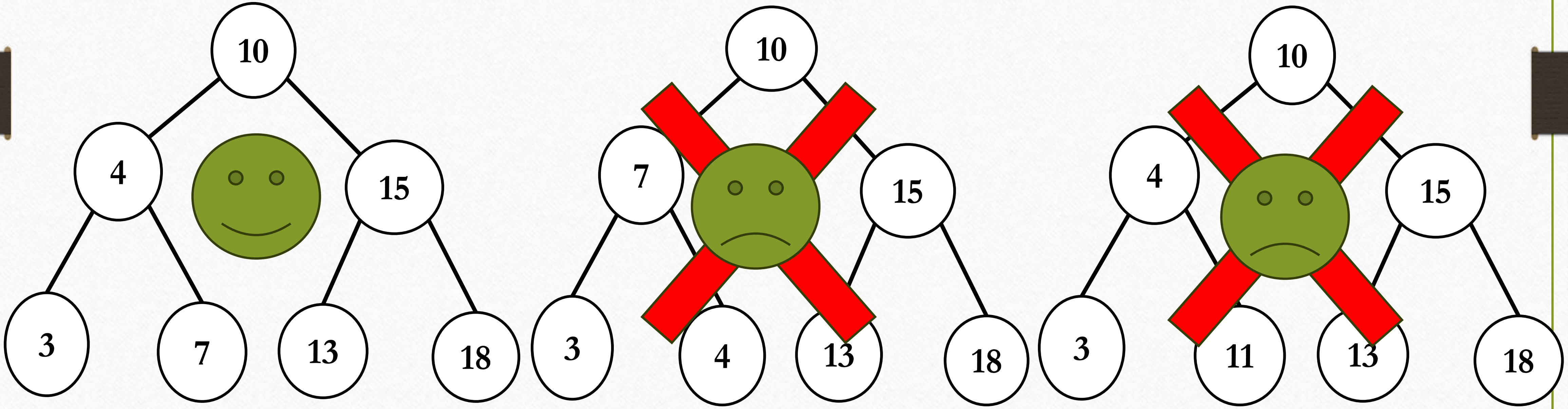
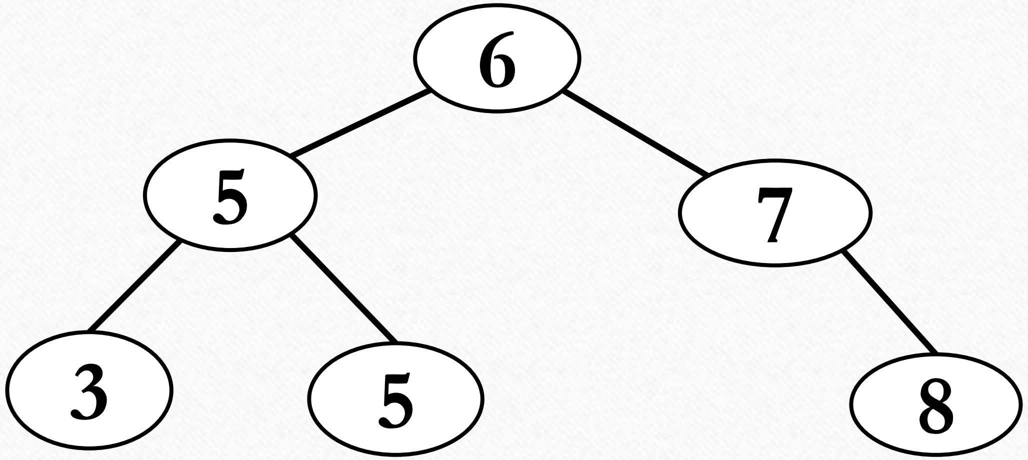


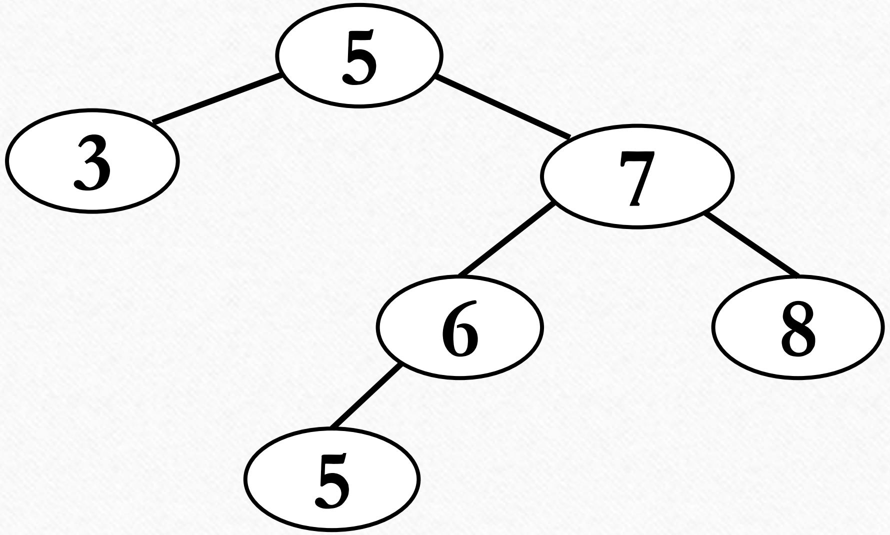
Figure 4.3: Representing a binary search tree

V. Types of the Tree Data Structure

Binary Search Trees (BST)

- For a tree to be a valid binary search tree, only the rules specified above need to be met. As a result, it is possible for two binary search trees to have **different structures**, even if they have the same keys. Two examples are shown below:





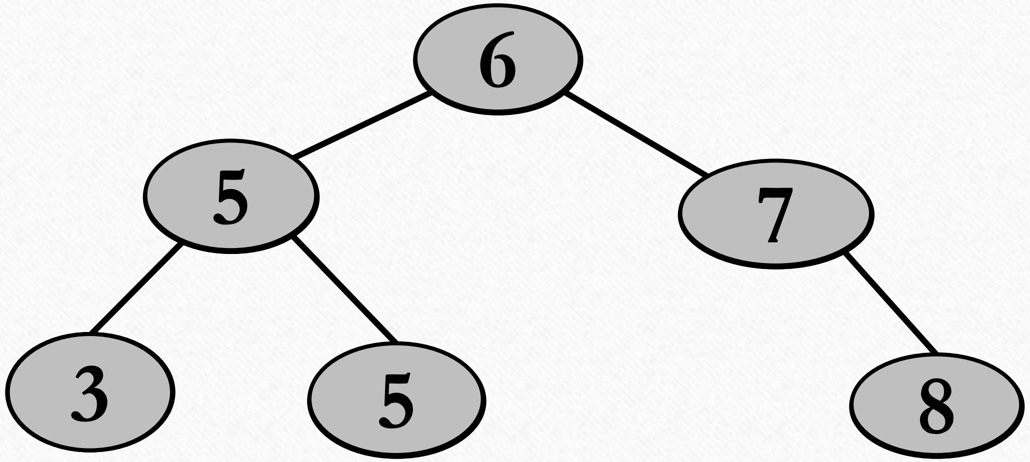
V. Types of the Tree Data Structure

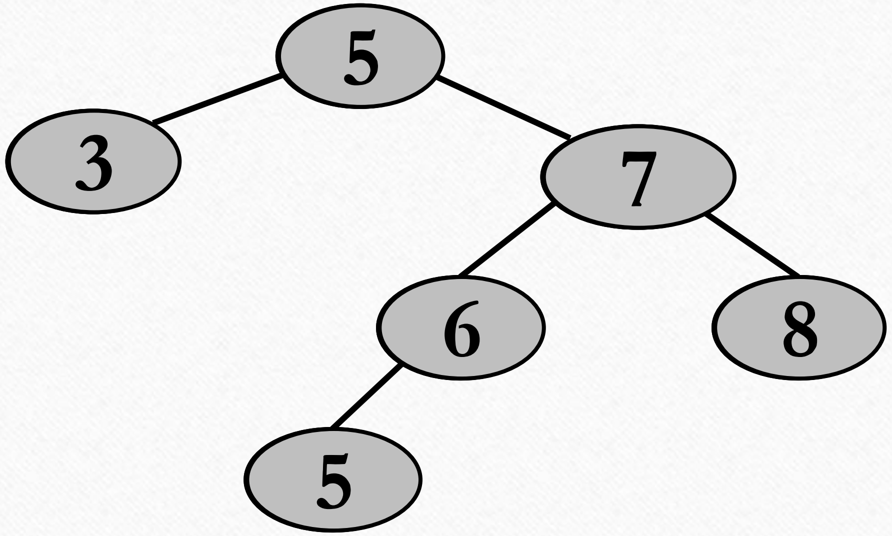
Binary Search Trees (BST)

- Since keys in a binary search tree are ordered in a way such that all keys to the left are smaller and all keys to the right are larger,
- the **inorder traversal** of a binary search tree will always visit its keys in **sorted order**.

V. Types of the Tree Data Structure

Binary Search Trees (BST)





inorder traversal
3, 5, 5, 6, 7, 8

inorder traversal
3, 5, 5, 6, 7, 8

V. Types of the Tree Data Structure

Operations in Binary Search Trees

- Binary search trees are useful for efficient **searching, insertion,** and **deletion operations**.
- Searching for a value in a **balanced binary** search tree has a **time complexity** of $\boldsymbol{O(logn)}$, where $\boldsymbol{n}$ is the number of nodes in the tree.
- This is because each comparison of the search value with a node's value **eliminates half of the remaining nodes** in the tree, remember in this case we assume that the binary tree is not left or right-skewed.
- In a binary search tree, many operations can be performed. Some of the operations are mentioned below in detail.

V. Types of the Tree Data Structure

Searching in a Binary Search Tree

Binary search trees support efficient searching since you will only need to look down one side of the tree at each level.

- If you want to search for a target key ***k***, and ***k*** is smaller than the current root element you are looking at, then ***k*** must be in the left subtree if it exists.
- Otherwise, if ***k*** is larger than the current root element, then k must be in the right subtree if it exists.

V. Types of the Tree Data Structure

Searching in a Binary Search Tree

- You can think of this as a "binary search" of the tree's elements since you are removing half of the search space every time; hence why this data structure is called a binary search tree!
- The searching algorithm can be implemented both iteratively and recursively. The following iterative implementation returns a pointer to the node with key ***k*** if it exists; otherwise, it returns **NULL** (**nullptr**).

V. Types of the Tree Data Structure

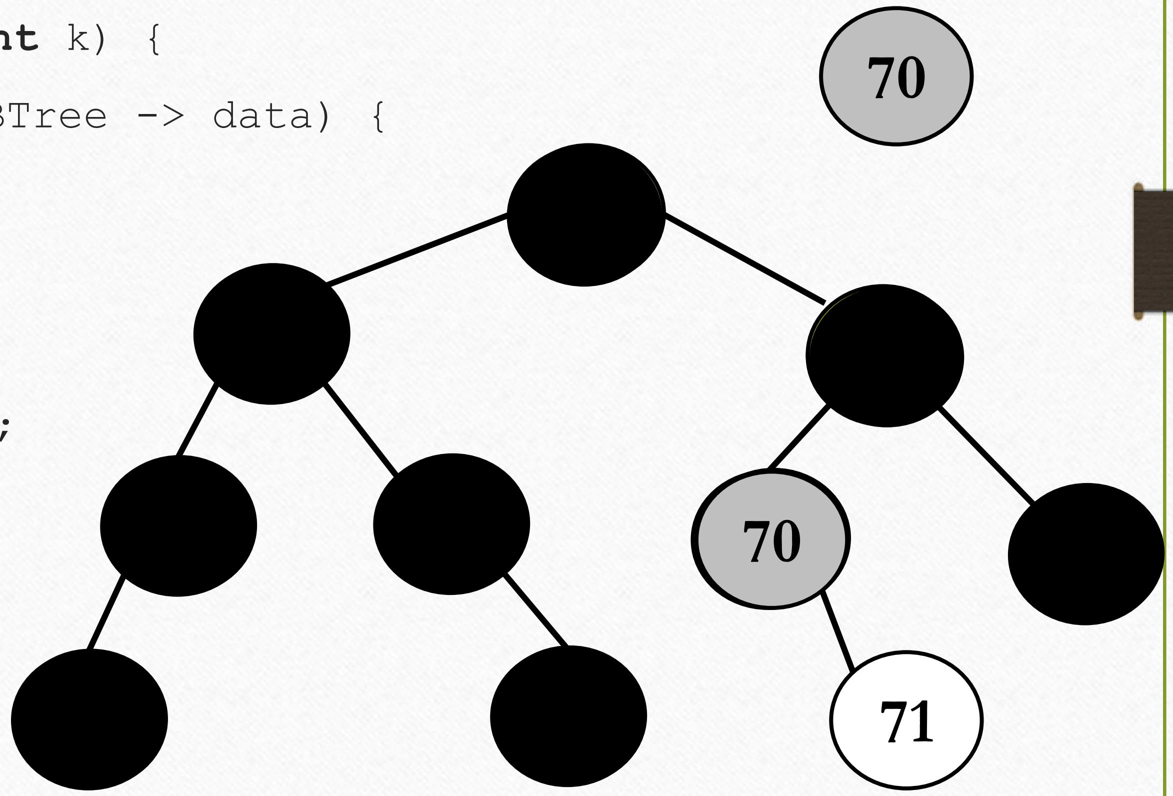
Searching in a Binary Search Tree

```
BTNode* search_BST(BTREE Btree, int k) {
    while (BTree != NULL && k != BTree -> data) {
        if (k < BTree -> data)
            BTree = BTree -> left;
        else
            BTree = BTree -> right;
    }
    return BTree;
}
```

V. Types of the Tree Data Structure

Searching in a Binary Search Tree

```
BTNode* search_BST(BTREE Btree, int k) {
    while (BTree != NULL && k != BTree -> data) {
        if (k < BTree -> data)
            BTree = BTree -> left;
        else
            BTree = BTree -> right;
    }
    return BTree;
}
```



V. Types of the Tree Data Structure

Searching in a Binary Search Tree

- The following implementation does the same thing, but with recursion instead of iteration.

- Note that this implementation is tail recursive, since the recursive call is always done last.

V. Types of the Tree Data Structure

Searching in a Binary Search Tree

```
BTNode* search_BST(BTREE BTree, int k) {
    if (BTree == NULL || BTree -> data == k) {
        return BTree;
    }
    if (k < BTree -> data) {
        return search_BST(BTree -> left, k);
    }
    return search_BST(BTree -> right, k);
}
```

V. Types of the Tree Data Structure

Searching in a Binary Search Tree

- **What is the time complexity of searching for a key in a binary search tree?**
- In the worst case, the binary search tree may take the form of a **stick**, and you may have to search the entire tree to find a given key.
- This results in $\boldsymbol{O(n)}$, runtime, where $\boldsymbol{n}$ is the number of nodes in the tree.

V. Types of the Tree Data Structure

Searching in a Binary Search Tree

The Maximum and the Minimum

- To find the minimum identify the leftmost node, i.e. the farthest node you can reach by following only left branches.
- To find the maximum identify the rightmost node, i.e. the farthest node you can reach by following only right branches.

V. Types of the Tree Data Structure

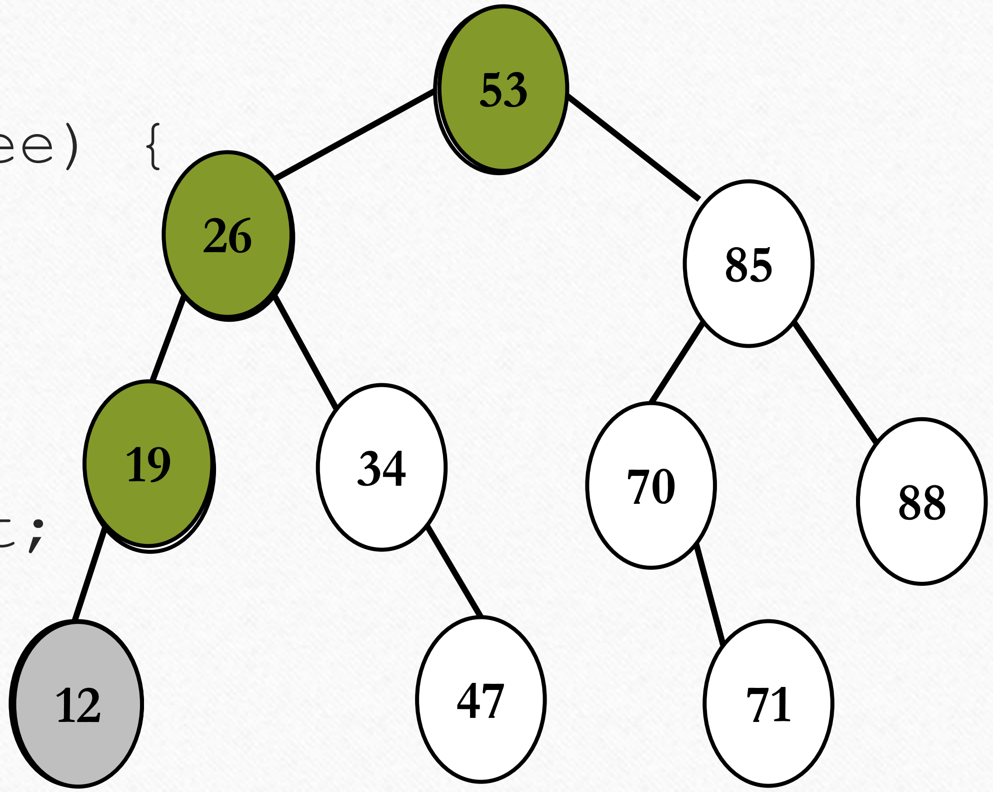
Searching in a Binary Search Tree

```
BTNode* Min_BST(BTREE BTree) {
   if (BTree == NULL) {
      return NULL;
   }
   while (BTree -> left) {
      BTree = BTree -> left;
   }
   return BTree;
}
```

V. Types of the Tree Data Structure

Searching in a Binary Search Tree

```
BTNode* Min_BST(BTREE BTree) {
    if (BTree == NULL) {
        return NULL;
    }
    while (BTree -> left) {
        BTree = BTree -> left;
    }
    return BTree;
}
```



V. Types of the Tree Data Structure

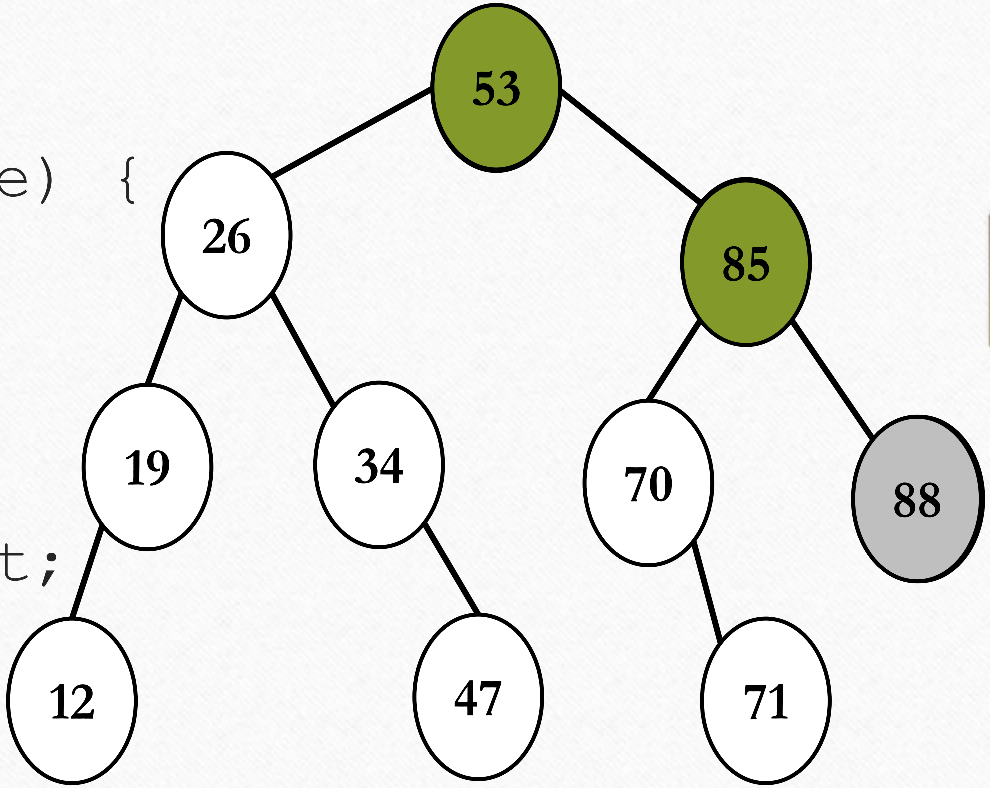
Searching in a Binary Search Tree

```
BTNode* Max_BST(BTREE BTree) {
  if (BTree == NULL) {
    return NULL;
  }
  while (BTree -> right) {
    BTree = BTree -> right;
  }
  return BTree;
}
```

V. Types of the Tree Data Structure

Searching in a Binary Search Tree

```
BTNode* Max_BST(BTREE BTree) {
    if (BTree == NULL) {
        return NULL;
    }
    while (BTree -> right) {
        BTree = BTree -> right;
    }
    return BTree;
}
```



V. Types of the Tree Data Structure

Inserting into a Binary Search Tree

- To insert a key into a binary search tree, one needs to maintain the binary property of the tree.
- To achieve this, the appropriate location for the inserted element is obtained and then the element is inserted into the tree. Therefore, We follow a procedure that is similar to search:
- we start at the root and trace a path downwards, but this time we want to look for a **NULL** (**nullptr**) position to append the node.

V. Types of the Tree Data Structure

Inserting into a Binary Search Tree

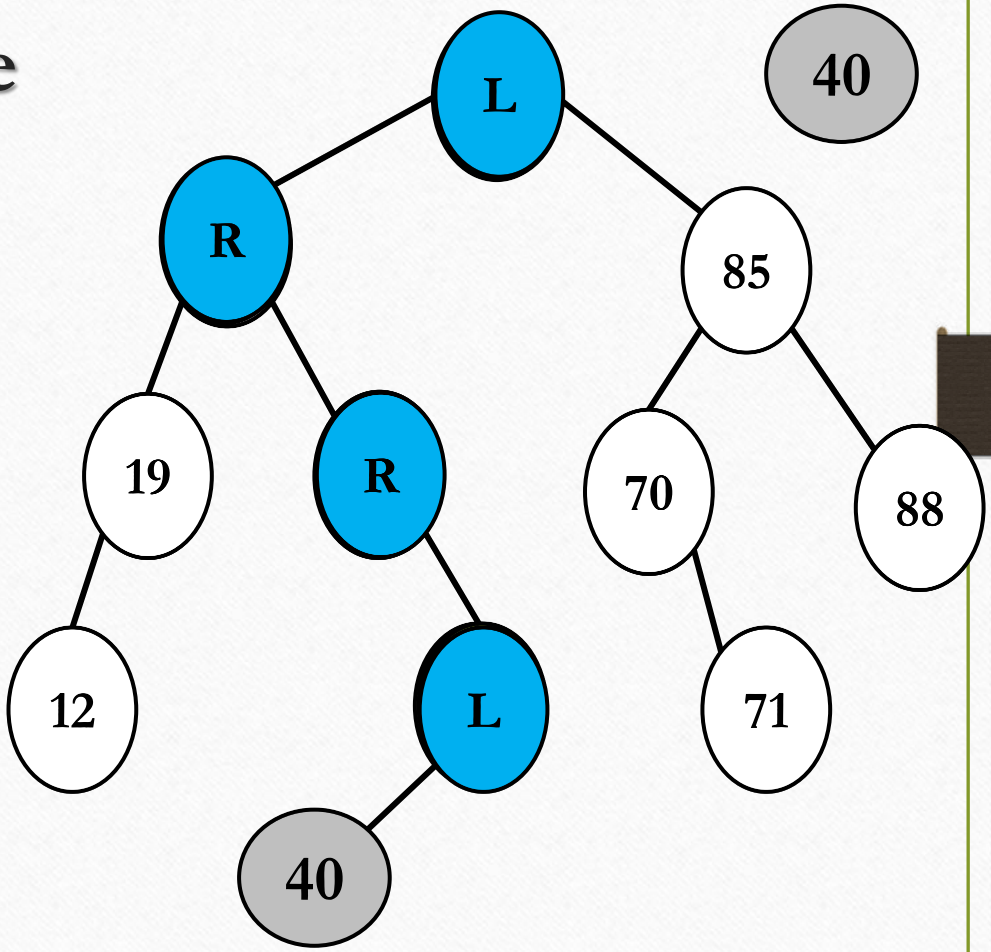
- The implementation for BST insertion is shown below.

```
void insert_BST(BTREE &BTree, int k) {
   if (BTree == NULL)
      BTree = Create_BTNode(k);
   else
      if (k < BTree -> data)
         insert_BST(BTree -> left, k);
      else
         insert_BST(BTree -> right, k);
}
```

V. Types of the Tree Data Structure

Inserting into a Binary Search Tree

```
void insert_BST(BTREE &BTree, int k) {
    if (BTree == NULL)
        BTree = Create_BTNode(k);
    else
        if (k < BTree -> data)
            insert_BST(BTree -> left, k);
        else
            insert_BST(BTree -> right, k);
}
```



V. Types of the Tree Data Structure

Inserting into a Binary Search Tree

- Similar to search, the time complexity of insert depends on the height of the tree.
- In the worst case, the binary search tree could be in the form of a **stick**, requiring the algorithm to traverse the entire tree before finding the position to insert the new node — this would take $\boldsymbol{O(n)}$ time.

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

- So far, we have covered search and insertion.
- However, **what if we wanted to remove a key from a binary search tree?**
- Removal is slightly more complicated, since we will have to detach a node while still maintaining the sorted property of a binary search tree.

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

- There are four different conditions we have to consider when removing a node from a binary search tree:

1) The node we want to remove has no children.
2) The node we want to remove has no left child.
3) The node we want to remove has no right child.
4) The node we want to remove has two children.

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

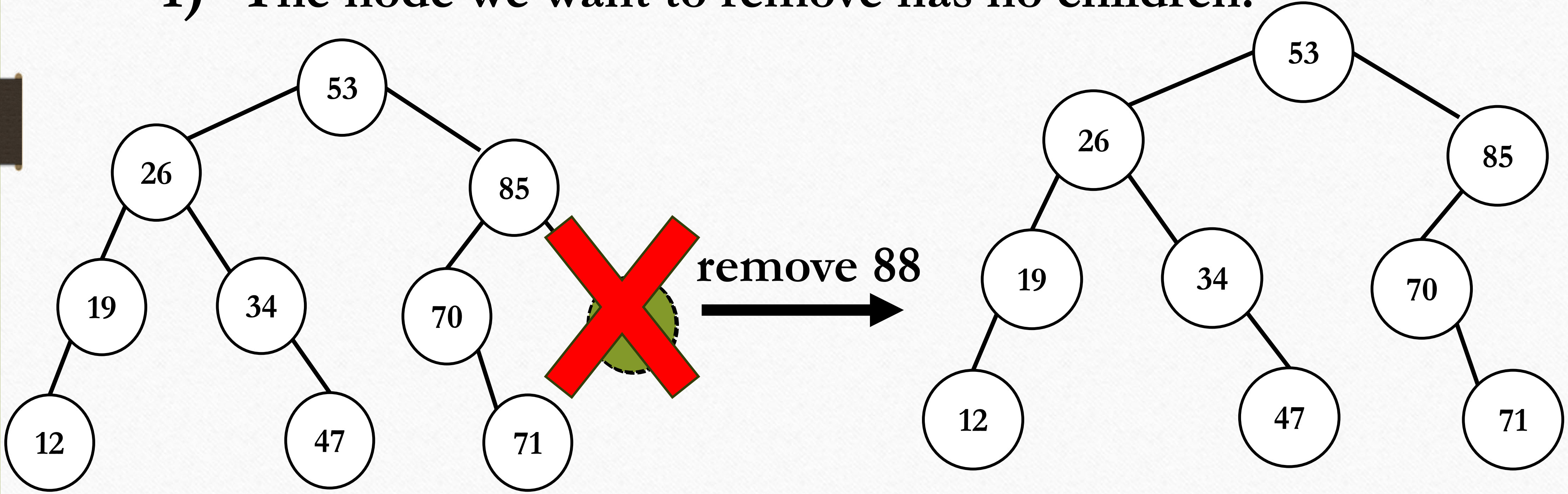
1) **The node we want to remove has no children.**

- This condition is trivial; if we want to remove a leaf node, we can just snip off the leaf.
- In the example below, **88** has no children, so we just delete it

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

1) The node we want to remove has no children.



V. Types of the Tree Data Structure

Removing from a Binary Search Tree

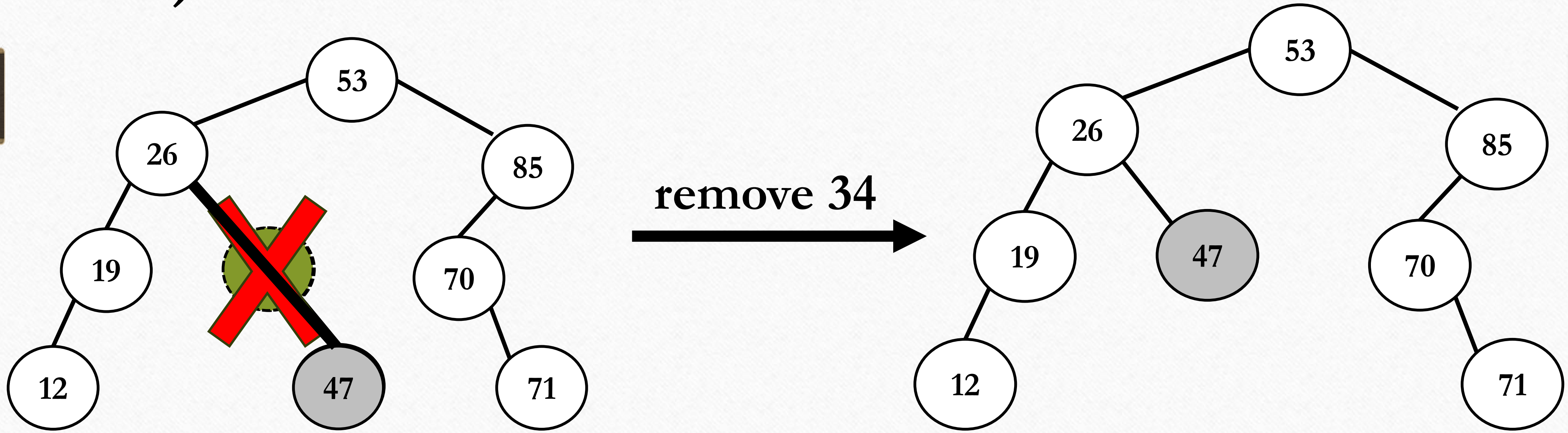
2) The node we want to remove has no left child.

- If the node we want to remove has no left child, we can just replace it with its right child.
- In the example below, **34** has no left child, so we just replace it with its right child, **47**.

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

2) The node we want to remove has no left child.



V. Types of the Tree Data Structure

Removing from a Binary Search Tree

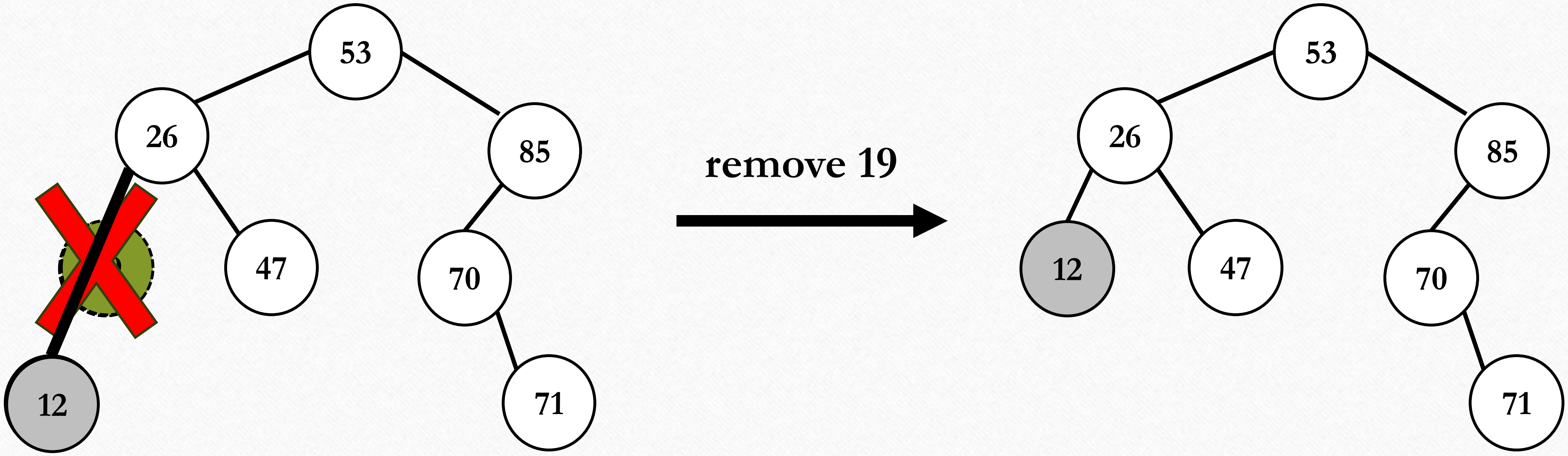
3) The node we want to remove has no right child.

- If the node we want to remove has no right child, we can just replace it with its left child.
- In the example below, **19** has no right child, so we just replace it with its left child, **12**

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

3) The node we want to remove has no right child.



V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) The node we want to remove has two children.

- This is a trickier case, since the node to remove points to **two things** instead of one.
- As a result, we will need a procedure for replacing the deleted node in a **way that preserves the binary search tree property.**

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) The node we want to remove has two children.

- A key observation to make here is that any node in the left subtree of the deleted node must be smaller than any node in the right subtree of the deleted node.
- Thus, we can preserve the binary search tree property by either

A. replacing the deleted node with the smallest node in its right subtree, or

B. replacing the deleted node with the largest node in its left subtree.

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

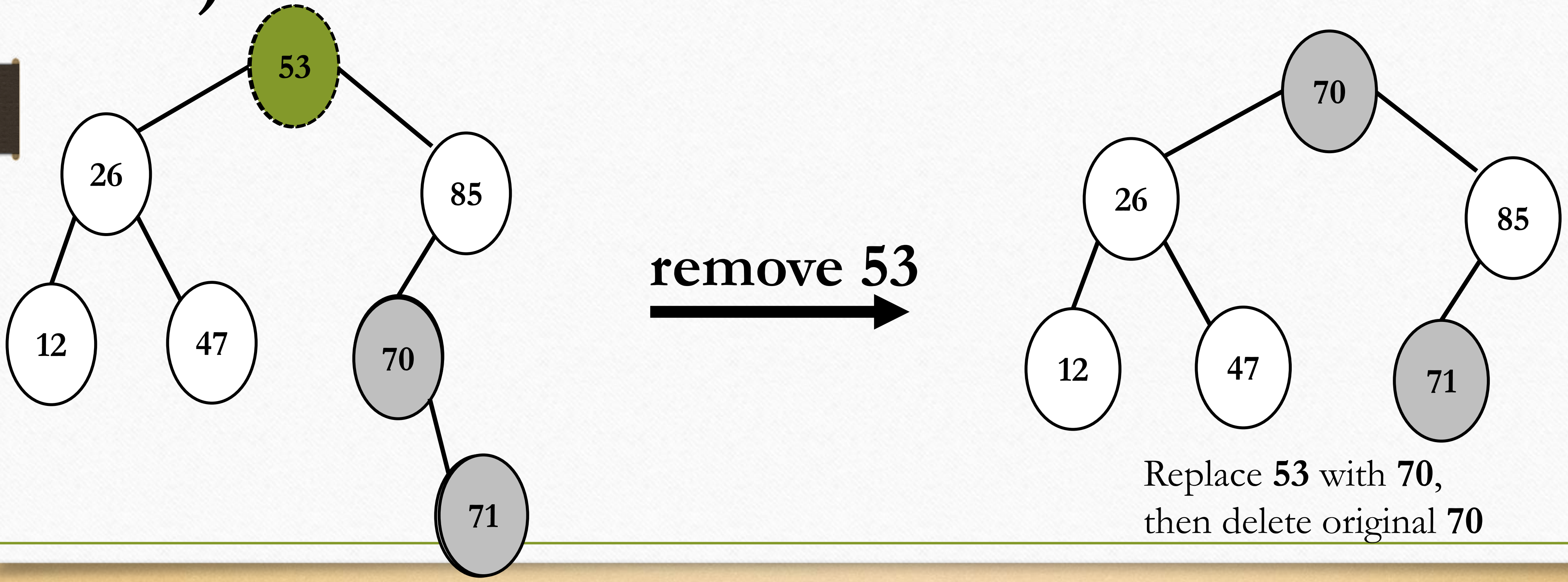
4) The node we want to remove has two children.

- For example, suppose we wanted to remove **53** from the following binary search tree.
- We can either replace **53** with the smallest element in its right subtree (**70**), or with the largest element in its left subtree (**47**).
- **Both approaches would maintain the binary search tree property:**

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) The node we want to remove has two children.

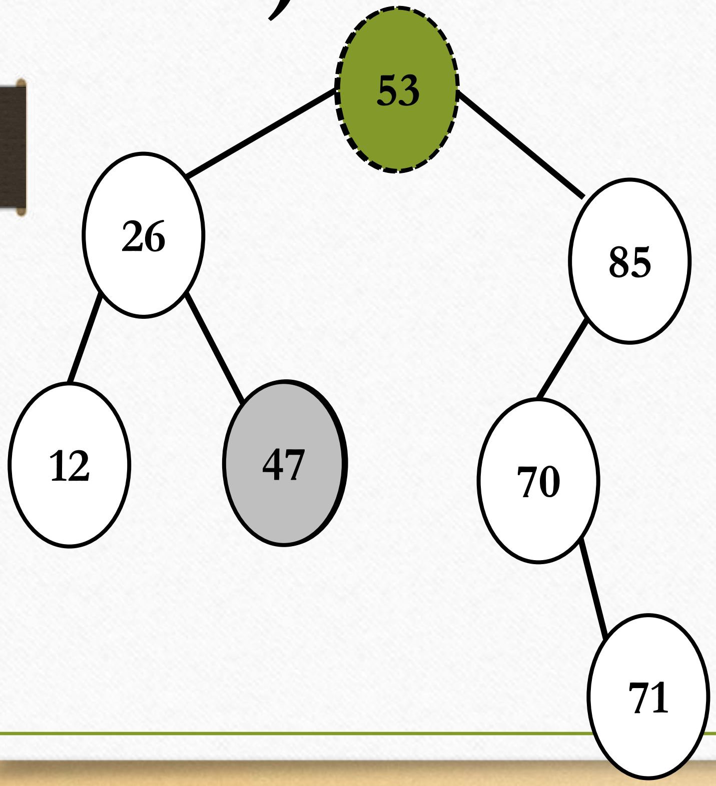


Replace **53** with **70**,
then delete original **70**

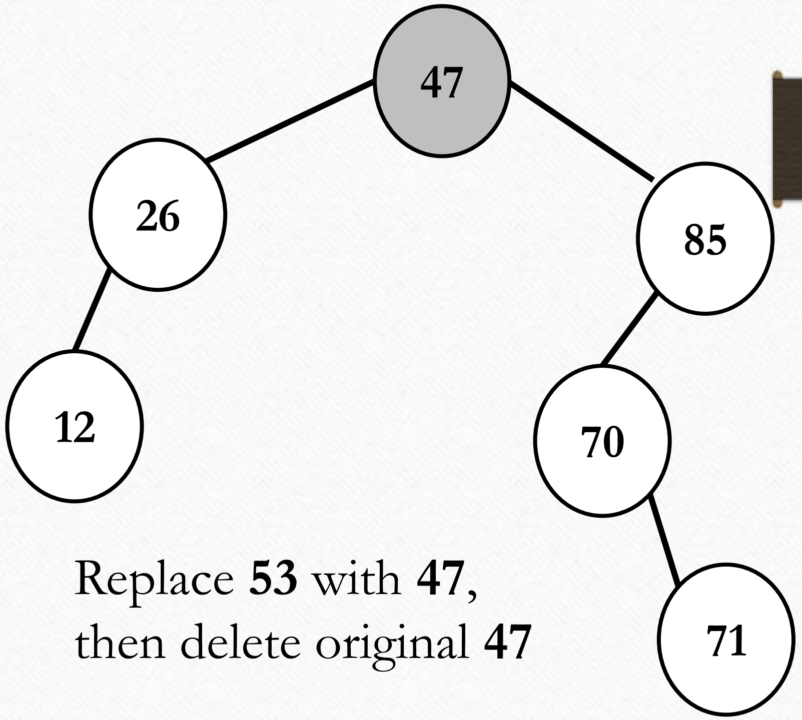
V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) The node we want to remove has two children.



remove 53 →



Replace **53** with **47**,
then delete original **47**

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) The node we want to remove has two children.

- The smallest node in the right subtree of a given node ***k*** is known as the inorder **successor** of node ***k***.
- Similarly, the largest node in the left subtree of a given node ***k*** is known as the inorder **predecessor** of node ***k***.
- This is because the inorder successor of ***k*** comes directly after (i.e., succeeds) ***k*** in an inorder traversal, and the inorder predecessor of ***k*** comes directly before (i.e., precedes) ***k*** in an inorder traversal

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) The node we want to remove has two children.

Inorder traversal: 12, 26, **47**, 53, **70**, 71, 85

Inorder predecessor of **53**

Inorder successor of **53**

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) The node we want to remove has two children.

- Because the **inorder successor and predecessor** are directly **adjacent** to the node we want to delete, it is safe to replace the deleted node with either of these values.
- The code for remove is shown below.
- In this version, the removed node is substituted with the inorder successor (the logic for the inorder predecessor version is analogous but not displayed here).

```
void remove_BST(BTREE &BTree, int data){
  BTNode* node_to_delete = BTree;
  BTNode* inorder_successor;
  if (BTree == NULL) {
   return;
  }
  else if (data < BTree -> data) {
      remove_BST(BTree -> left, data);
   }
 else if (BTree -> data < data) {
   remove_BST(BTree -> right, data);
  }
   else {
      if (BTree ->left == NULL) {
         BTree = BTree ->right;
         delete node_to_delete;
       }
      else if (BTree -> right == NULL) {
            BTree = BTree -> left;
            delete node_to_delete;
         }
```

```
   else {
   inorder_successor = BTree -> right;
   while (inorder_successor -> left) {
        inorder_successor =
        inorder_successor -> left;
   } // while
  node_to_delete        ->        data        =
 inorder_successor -> data;
            remove_BST(BTree    ->    right,
          inorder_successor -> data);
         }
      } // else
} // tree_remove()
```

```
void remove_BST(BTREE &BTree, int data) {
    BTNode* node_to_delete = BTree;
    BTNode* inorder_successor;
    if (BTree == NULL) {
        return;
    }
    else if (data < BTree -> data) {
        remove_BST(BTree -> left, data);
    }
    else if (BTree -> data < data) {
        remove_BST(BTree -> right, data);
    }
```

```
void remove_BST(BTREE &BTree, int data) {
  /******************Part 1 *******************/
  else {
    if (BTree ->left == NULL) {
      BTree = BTree ->right;
      delete node_to_delete;
     }
    else if (BTree -> right == NULL) {
        BTree = BTree -> left;
        delete node_to_delete;
      }
```

```
void remove_BST(BTREE &BTree, int data) {
  /********************Part 1 ********************/
  /********************Part 2 ********************/
 else {                    // Node to delete has both left and right subtrees
      inorder_successor = BTree -> right;
      while (inorder_successor -> left)
         inorder_successor = inorder_successor -> left;
      } // while
      node_to_delete -> data = inorder_successor -> data;
      remove_BST(BTree -> right, inorder_successor -> data);
      }
   } // else
} // tree_remove()
```

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) The node we want to remove has two children.

- Replacing a deleted node **with either its inorder successor or predecessor** is the simplest way to approach deletion, as it maintains the BST property without requiring you to reconfigure the other elements in the tree.
- The process for **finding the inorder successor or predecessor of any node** is also relatively straightforward.

V. Types of the Tree Data Structure

Removing from a Binary Search Tree

4) **The node we want to remove has two children.**

- To **identify the inorder successor** of a node, **go right** once, then go as far left as possible until you reach a node without a left child.
- To **identify the inorder predecessor** of a node, **go left** once, then go as far right as possible until you reach a node without a right child.

V. Types of the Tree Data Structure

Summary of Binary Search Tree Time Complexities

Operation	Best-Case Time	Average-Case Time	Worst-Case Time
Searching	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$
Inserting	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$
Deleting	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$

Questions ?