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Variational methods for nonlinear elliptic problems

A Course with Exercises and Key Topics

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Preface

What is the main objective of this course? Anyone interested in studying ordinary and partial differential equations will undoubtedly need some tools from functional analysis, as well as familiarity with certain methods used to prove the existence and multiplicity of solutions and to determine some of their properties. Among the most important of these methods are the so-called *variational methods and critical point theory*. We try to find solutions to a given nonlinear problem by looking for critical points of a functional defined on the functional space in which we want the solution to lie. So, for a given problem, one usually proceeds as follows: Firstly, we find the real valued functional that is naturally associated with the problem. Secondly, we find or construct a theorem that ensures our functional has a critical point. Finally, we prove that a critical point of our functional is a solution to the problem.

This course is primarily intended for Master's students in Partial Differential Equations (PDEs) and Functional Analysis of PDEs, as well as related specialties. It can also serve as a reference for beginner PhD students who are looking for a simplified introduction to this field. In these lectures, I have aimed to simplify the most important variational methods by selecting simple model problems that help the learner grasp the objectives smoothly, without delving into complexities that might hinder their understanding and deeper exploration. Therefore, while it may not offer new results, its value lies in the rephrasing of the lectures, the careful selection of examples, and the presentation of key problems in a way that is accessible and comfortable for the reader.

I would also like to draw the reader's attention to the fact that I do not recall any work ever being entirely accurate; thus, the discovery of an error should come as no surprise. I would be sincerely grateful to anyone who brings such issues to my attention.

MOKHTARI Abdelhak
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Contents

1	Basic concepts in functional analysis	5
1.1	Reflexive space	5
1.1.1	Dual space	5
1.1.2	Some notions of convergence	6
1.2	Operators on Banach spaces	8
1.2.1	Continuity, convexity and coercivity of operators	8
1.2.2	Lower and Upper Semi-continuity	9
1.3	Functional spaces	10
1.3.1	L^p Spaces	10
1.3.2	Some convergence criteria	11
1.3.3	$W^{1,p}(\Omega)$ Spaces	12
1.3.4	Sobolev Embeddings	13
1.3.5	$W_0^{1,p}(\Omega)$ Space	13
1.4	Exercices	14
2	Differentiability of Functionals and Critical Points	15
2.1	Gâteaux differentiability	15
2.2	Fréchet differentiability	18
2.3	Basic examples	19
2.4	Exercices	21
3	Direct Methods in the Calculus of Variations	24
3.1	Extremum and critical points	24
3.2	Weak solutions and critical points	25
3.2.1	Example of a linear problem	25
3.2.2	Example of a nonlinear problem	26
3.3	Study of problems with a associated functional convex	27
3.3.1	Global minimum of a convex functional	27
3.3.2	Applications	30
3.4	Minimization Techniques for Problems involving a nonlinear term satisfies the growth condition	33

3.4.1	Sublinear case in the growth condition	35
3.4.2	Eigenvalue and eigenfunction of the Laplacian operator	36
3.4.3	Linear case in the growth condition	38
3.4.4	Existence of nontrivial solutions	38
3.5	Elliptic problem involving the p -Laplacian operator	40
3.5.1	Some important results	40
3.5.2	Application	41
3.6	Ekeland Variational Principle	44
3.6.1	Palais-Smale condition	44
3.6.2	Ekeland's Lemma and consequences	45
3.7	Exercises	49
4	Min-Max Methods	52
4.1	Mountain Pass Theorem	52
4.1.1	Min-max principle	52
4.1.2	Application: Solutions for an elliptic problem involving superlinear term	53
4.2	Rabinowitz's Saddle-Point Theorem and application	63
4.3	Exercises	70
5	Exam Problems and Topics	73
5.1	Topic 1: Final exam (M'sila, 2022-2023)	73
5.2	Topic 2: Catch-up exam (M'sila, 2022-2023)	75
5.3	Topic 3: Final exam (M'sila, 2023-2024)	76
5.4	Topic 4: Catch-up exam (M'sila, 2023-2024)	77
5.5	Topic 5: Final exam (M'sila, 2024-2025)	79
5.6	Topic 6: Catch-up exam (M'sila, 2024-2025)	80
5.7	Topic 7	81
5.8	Topic 8	82
5.9	Topic 9	83
6	Solutions of Some Topics	84
6.1	Solution to Topic 1	84
6.2	Solution of Topic 5	86
6.3	Solution of Topic 6	89
6.4	Solution of Topic 7	91
6.5	Solution of Topic 8	93
6.6	Solution of Topic 9	94