Chapter 5

Exam Problems and Topics

5.1 Topic 1: Final exam (M'sila, 2022-2023)

Let Ω be a bounded regular open set in $\mathbb{R}^N$ with $N \geq 3$. We consider the following nonlinear Dirichlet problem:

$$\begin{cases} -\Delta u = |u|^{q-2}u, & \text{dans } \Omega; \\ u = 0, & \text{sur } \partial\Omega. \end{cases} \tag{5.1}$$

where $q \in [2, 2^*[$ and $2^* = \frac{2N}{N-2}$. The goal of this exercise is to study the existence of a nontrivial weak solution in the space $H_0^1(\Omega)$ using the Mountain Pass Theorem.

1. Provide the variational formulation associated with problem (5.1).
2. Consider the functional $\varphi : H_0^1(\Omega) \to \mathbb{R}$ defined by:
$$\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{q}\int_\Omega |u|^q\, dx.$$
   (a) Justify that the functional φ is well-defined.
   (b) Assume that $\varphi \in C^1(H_0^1(\Omega), \mathbb{R})$; find the expression for $\langle \varphi'(u), v\rangle$ for all $u, v \in H_0^1(\Omega)$, and verify that the critical points of φ are exactly the weak solutions of (5.1).
3. We want to prove the first geometric condition that ensures the existence of a strict local minimum of φ.
   (a) Show that for all $u \in H_0^1(\Omega)$, we have:
$$\varphi(u) \geq \frac{1}{2}\|u\|_{H_0^1}^2 - C_1\|u\|_{H_0^1}^q.$$
   (b) Deduce that there exist $\rho, \alpha > 0$ such that:
$$\varphi(u) \geq \alpha \quad \text{for all} \quad u \in H_0^1(\Omega) \text{ with } \|u\|_{H_0^1} = \rho.$$
4. Now, we turn to the second geometric condition. Fix $v \in H_0^1(\Omega)$ such that $\|v\|_{H_0^1} = 1$ and $v \geq 0$, and define:
$$\theta = \int_\Omega |v|^q\, dx.$$

(a) Verify that:
$$\lim_{t\to+\infty}\varphi(tv) = -\infty.$$

(b) Deduce that there exists $\psi \in H_0^1(\Omega)$ such that $\varphi(\psi) < 0$ and $\|\psi\|_{H_0^1} > \rho$.

5. In this part, we show that φ satisfies the Palais-Smale condition (P.S).

(a) State the Palais-Smale condition.

(b) Let (u_n) be a sequence satisfying the Palais-Smale condition. Explain what this means.

i. Verify that for sufficiently large n, we have:
$$|\langle \varphi'(u_n), u_n\rangle| \leq \|u_n\|_{H_0^1},$$
and deduce that:
$$\varphi(u_n) - \frac{1}{q}\langle \varphi'(u_n), u_n\rangle \leq C + \frac{1}{q}\|u_n\|_{H_0^1}.$$

ii. By contradiction and using the relation above, show that the sequence (u_n) is bounded in $H_0^1(\Omega)$.

(c) i. Justify the existence of a subsequence (u_{n_k}) of (u_n) (for simplicity, we keep the notation (u_n)) and a function $u \in H_0^1(\Omega)$ such that:
$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } H_0^1(\Omega);\\ u_n \to u, & \text{strongly in } L^q(\Omega) \text{ for all } q \in [2, 2^*);\\ u_n(x) \to u(x), & \text{pointwise almost everywhere in } \Omega.\end{cases}$$

ii. Justify that:
$$\langle \varphi'(u_n), u_n - u\rangle \to 0, \quad \langle \varphi'(u), u_n - u\rangle \to 0 \quad \text{as} \quad n\to\infty.$$

(d) Calculate explicitly:
$$\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle.$$

(e) Show the following estimates:
$$\left|\int_\Omega |u_n|^{q-2}u_n(u_n-u)\,dx\right| \leq C_2\|u_n - u\|_{L^q},$$
$$\left|\int_\Omega |u|^{q-2}u(u_n-u)\,dx\right| \leq C_3\|u_n - u\|_{L^q}.$$

(f) Deduce that the sequence (u_n) converges strongly to u in $H_0^1(\Omega)$.

6. Conclude.

5.2 Topic 2: Catch-up exam (M'sila, 2022-2023)

Exercise 1:

Let Ω be a bounded open set in $\mathbb{R}^N$. Consider the following problem:

$$\begin{cases} -\Delta u = \cos u, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

1. Provide the variational formulation associated with the problem (5.2). Show that if $u \in H_0^1(\Omega)$, then each term in the variational formulation is well-defined.

2. Find a functional $J : H_0^1(\Omega) \to \mathbb{R}$, whose critical points are weak solutions of the problem (5.2).

3. Show that J is coercive and weakly lower semicontinuous. What can we conclude from this?

Exercise 2:

Let E be a Banach space and let $I : E \to \mathbb{R}$ be a differentiable functional. Assume that:

$$\langle I'(u) - I'(v), u - v\rangle > 0, \quad \forall u, v \in E,\ u \neq v.$$

Fix $u, v \in E$ and define the function $\psi : \mathbb{R} \to \mathbb{R}$ by: $\psi(t) = I\Big(u + t(u - v)\Big)$.

1. (a) Compute $\psi'(t)$.

   (b) Fix $s < t$, show that

   $$\psi'(t) - \psi'(s) = \frac{1}{t-s}\langle I'\Big(u + t(u-v)\Big) - I'\Big(u + s(u-v)\Big), t(u-v) - s(u-v)\rangle.$$

   (c) Deduce that ψ is strictly convex, and in particular, we have:

   $$\psi(t) < t\psi(1) + (1-t)\psi(0).$$

   (d) Conclude that I is strictly convex.

   Recall that any strictly convex, differentiable, and coercive functional has a unique critical point.

2. *Application:*

   Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, let $q \in L^\infty(\Omega)$ satisfy $q(x) \geq 0$ almost everywhere in Ω. Let $A(x)$ be a symmetric matrix with components in $L^\infty(\Omega)$. Assume there exists $\lambda > 0$ such that:

   $$(A(x)y)y \geq \lambda |y|^2, \quad \text{for almost every } x \in \Omega, \quad \text{and for all } y \in \mathbb{R}^N.$$

For any $f \in L^2(\Omega)$, consider the following problem:

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + q(x)u = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Consider the associated functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega (A(x)\nabla u) \cdot \nabla v \, dx + \int_\Omega q(x)uv \, dx - \int_\Omega fv \, dx.$$

(a) Write the weak formulation associated with (5.3).

(b) Show that J is strictly convex.

(c) Show that J is coercive.

(d) What can we conclude from this?

5.3 Topic 3: Final exam (M'sila, 2023-2024)

I) Let $(E, \|\cdot\|)$ be a Banach space and let $J : E \to \mathbb{R}$ be a functional.

1. (a) Give a definition of the Fréchet differentiability and Gâteaux differentiability.

   (b) Give the relationship between them, then give an appropriate condition so that there is an equivalence between them.

2. (a) Give a definition of the critical point of J.

   (b) State the Palais-Smale condition.

II) The following result is a consequence of Ekeland's variational principle.

Proposition: *Let $(E, \|\cdot\|)$ be a Banach space and let $J \in C^1(E, \mathbb{R})$. We assume that J is bounded from blow (i.e. $\exists M > 0 : J(u) \geq M, \forall u \in E$) and satisfies the Palais-Smale condition at level $c = \inf_{u \in E} J(u)$. Then there exists $u_0 \in E$ such that $J(u_0) = \min_{u \in E} J(u)$.*

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be an open bounded. We consider the following problem

$$(P) \begin{cases} -\Delta u + g(u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfies

(g_1) $\exists a, b > 0 : |g(t)| \leq a + b|t|^{q-1}$, $q \in (1, 2^*)$ $(2^* = \frac{2N}{N-2})$.

(g_2) $G(s) \geq 0, \forall s \in \mathbb{R}$, where $G(s) = \int_0^s g(t)\, dt$.

The goal is to prove that Problem (P) admits at least a weak solution in $H_0^1(\Omega)$ using the previous proposition.

1. Define the notion of weak solution for this Problem.

2. Define $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, dx + \int_\Omega G(u(x)) \, dx - \int_\Omega f(x)u(x)dx.$$

(a) Verify that J is well defined.

(b) Justify, without detailed proof, that J is differentiable and calculate $J'(u)$.

3. (a) Verify that there exist $c_1 > 0$ such that

$$J(u) \geq \frac{1}{2}\|u\|^2_{H^1_0} - c_1\|u\|_{H^1_0}. \tag{5.4}$$

(b) Deduce that J is bounded from below.

4. Let $(u_n) \subset H^1_0(\Omega)$ be a sequence of Palais-Smale.

(a) Using (5.4), show that (u_n) is bounded in $H^1_0(\Omega)$.

(b) Justify the existence of a subsequence (u_{n_k}) (for simplify we keep the notation (u_n)) and $u \in H^1_0(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } H^1_0(\Omega); \\ u_n \to u, & \text{strongly in } L^q(\Omega),\ q \in (1, 2^*); \\ u_n(x) \to u(x), & \text{a.e. } x \in \Omega; \\ \exists h \in L^q(\Omega) : |u_(x)| \leq h(x), & \text{a.e. } x \in \Omega. \end{cases}$$

5. (a) Check without calculation that $\langle J'(u_n) - J'(u), u_n - u\rangle \to 0$ as $n \to +\infty$.

(b) Calculate explicitly $\langle J'(u_n) - J'(u), u_n - u\rangle$.

(c) Show that

$$\begin{cases} \int_\Omega g(u_n)u_n\, dx \to \int_\Omega g(u)u\, dx, \\ \int_\Omega g(u_n)u\, dx \to \int_\Omega g(u)u\, dx, \\ \int_\Omega g(u)u_n\, dx \to \int_\Omega g(u)u\, dx. \end{cases}$$

(d) Deduce that J satisfies the Palais-Smale condition.

6. What do you conclude?

5.4 Topic 4: Catch-up exam (M'sila, 2023-2024)

I)

1. (a) Give a definition of the Fréchet differentiability and Gâteaux differentiability in a Banach space.

(b) Give the relationship between them, then give an appropriate condition so that there is an equivalence between them.

2. We consider the following functional:

$$\begin{aligned} \varphi : W^{1,p}_0(\Omega) &\to \mathbb{R} \\ u &\mapsto \varphi(u) = \|u\|^p_{W^{1,p}_0} = \int_\Omega |\nabla u|^p\, dx. \end{aligned}$$

(a) Show that $\lim_{t\to 0}\int_\Omega \frac{|\nabla u+t\nabla v|^p-|\nabla u|^p}{t}\,dt = p\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx.$

(b) Show that φ is Fréchet differentiable and we have that

$$\langle \varphi'(u), v\rangle = p\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx.$$

II) Let $\Omega\subset\mathbb{R}^N$, with $N\geq 3$, be an open and bounded. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that

(f_1) $|f(t)|\leq a+b|t|^{p^*-1}$, $p^*=\frac{Np}{N-p}$ with $p<N$.

(f_2) $f(t)t\leq 0$ and $(f(t)-f(s))(t-s)\leq 0$. $\forall t,s\in\mathbb{R}$.

Let $h\in L^q(\Omega)$ ($\frac{1}{p}+\frac{1}{q}=1$), we consider the problem

$$(P)\begin{cases} -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = f(u)+h(x), & \text{in } \Omega;\\ u=0, & \text{on } \partial\Omega.\end{cases}$$

Let $F(t)=\int_0^t f(s)\,ds$ and we define a differentiable functional $J: W_0^{1,p}(\Omega)\to\mathbb{R}$ by

$$J(u)=\frac{1}{p}\int_\Omega |\nabla u|^p\,dx-\int_\Omega F(u)\,dx-\int_\Omega h(x)u\,dx.$$

1. Justify that J is well defined in $W_0^{1,p}(\Omega)$.

2. Justify that that a critical point of J is a weak solution of (P).

3. (a) Using the following elementary inequalities

$$\begin{cases} (|x|^{p-2}x-|y|^{p-2}y)(x-y)\geq C_p|x-y|^p, & \text{if } p\geq 2;\\ (|x|^{p-2}x-|y|^{p-2}y)(x-y)\geq \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1<p<2.\end{cases}$$

Show that $\langle J'(u)-J'(v), u-v\rangle\geq C\|u-v\|^p_{W_0^{1,p}}$ for all $u,v\in W_0^{1,p}(\Omega)$ and for some $C>0$.

(b) Deduce that J is strictly convex.

4. (a) Using (f_2), verify that $F(t)\leq 0$ for all $t\in\mathbb{R}$.

(b) Show that J is coercive.

5. What do you conclude?

6. Let $f(t)=-|t|^{\sigma-2}t$ with $\sigma\in(1,p^*)$. Verify that f satisfies the assumptions (f_1) and (f_2).

5.5 Topic 5: Final exam (M'sila, 2024-2025)

Let $\Omega \subset \mathbb{R}^N$ be a smooth open bounded set with $N \geq 3$. Let $a > 0$, $b \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We consider the following Kirchhoff-type problem:

$$(\mathrm{P}) \qquad \begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We say that $u \in H_0^1(\Omega)$ is a weak solution of (P) if:

$$\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega f(u)v\,dx = 0, \quad \forall v \in H_0^1(\Omega).$$

The energy functional corresponding to problem (P) is defined as follows:

$$J(u) = \frac{a}{2}\int_\Omega |\nabla u|^2\,dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2\,dx\right)^2 - \int_\Omega F(u)\,dx,$$

where $F(t) = \int_0^t f(s)\,ds$. Assume that there exist $1 < \theta < 2$ and $\lambda > 0$ such that:

$$\lim_{|t|\to\infty} \frac{|F(t)|}{|t|^\theta} < \infty. \qquad (H)$$

1. Show that the assumption (H) implies that:

$$\exists C_1, C_2 > 0 : \quad |F(t)| \leq C_1|t|^\theta + C_2, \quad \forall t \in \mathbb{R},$$

   for some constants $C_1, C_2 > 0$.

2. Under the hypotheses (H), show that the functional J is well-defined.

3. We set:

$$\psi(u) = \left(\int_\Omega |\nabla u|^2\,dx\right)^2, \quad u \in H_0^1(\Omega).$$

   Show that ψ is differentiable, and we have:

$$\langle \psi'(u), v\rangle = 4\|u\|^2_{H_0^1(\Omega)}\int_\Omega \nabla u \cdot \nabla v\,dx, \quad \forall u, v \in H_0^1(\Omega).$$

4. Justify, without detail, that J is differentiable and compute $J'(u)$. What do you notice?

5. Show that there exist constants $C_3, C_4 > 0$ such that, for any $u \in H_0^1(\Omega)$, we have:

$$J(u) \geq \frac{a}{2}\|u\|^2_{H_0^1} + \frac{b}{4}\|u\|^4_{H_0^1} - C_3\|u\|^\theta_{H_0^1} - C_4.$$

   Deduce that J is coercive.

6. Let $(u_n) \subset H_0^1(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$.

   (a) Show that:

$$\int_\Omega F(u_n)\,dx \to \int_\Omega F(u)\,dx.$$

   (b) Deduce that J is weakly lower semi-continuous.

7. Conclude.

5.6 Topic 6: Catch-up exam (M'sila, 2024-2025)

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$. We consider the following problem:

$$(P)\begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(0) = 0$ and satisfies:

$$(\mathbf{H_1}) : \exists a, b > 0,\ \exists \sigma \in [1, 2[: |F(t)| \leq a|t| + b|t|^\sigma, \forall t \in \mathbb{R}.$$

with: $F(t) = \int_0^t f(s)\, ds$.

We consider the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega F(u)\, dx.$$

We define:

$$J_1(u) = \frac{1}{2}\|u\|^2_{H_0^1} = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx, \quad J_2(u) = \int_\Omega F(u)\, dx.$$

1. Write the weak formulation of (P).

2. (a) Show that J_1 is differentiable in $H_0^1(\Omega)$ and that:

$$\langle J_1'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v\, dx, \quad \forall u, v \in H_0^1(\Omega).$$

   (b) It is given that J_2 is differentiable and that $\langle J_2'(u), v\rangle = \int_\Omega f(u)v\ dx$. Justify that J is differentiable on $H_0^1(\Omega)$ and what can we conclude?

3. Show that there exist $c_1, c_2 > 0$ such that:

$$J(u) \geq \frac{1}{2}\|u\|^2_{H_0^1} - c_1\|u\|_{H_0^1} - c_2\|u\|^\sigma_{H_0^1}. \tag{5.5}$$

4. Let $(u_n) \subset H_0^1(\Omega)$ be a minimizing sequence of J, i.e.,

$$\lim_{n\to\infty} J(u_n) = \inf_{u\in H_0^1(\Omega)} J(u).$$

   Using (5.5), show that the sequence $(u_n)_n$ is bounded in $H_0^1(\Omega)$.

5. Deduce that there exists a subsequence of $(u_n)_n$ (still denoted by $(u_n)_n$) and $u^* \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u^*$ in $H_0^1(\Omega)$. Then show that:

$$\lim_{n\to+\infty}\int_\Omega F(u_n)\ dx = \int_\Omega F(u^*)\ dx. \tag{5.6}$$

6. Knowing that $\|u^*\|_{H_0^1} \leq \liminf_{n\to+\infty}\|u_n\|_{H_0^1}$, show that u^* is a minimizer of J; i.e., $J(u^*) = \inf_{u\in H_0^1} J(u)$. What can we conclude?

7. Let $\varphi_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the first positive eigenfunction of the operator Δ, that is:

$$\int_\Omega \nabla\varphi_1 \nabla v \, dx = \lambda_1 \int_\Omega \varphi_1 v \, dx, \quad \forall v \in H_0^1(\Omega).$$

where $\lambda_1 > 0$. Suppose that f satisfies

$$\mathbf{H_2} : \exists \delta > 0, \exists \beta > \lambda_1 : f(t) \geq \beta t, \quad \forall t \in [0, \delta].$$

(a) Show that, for $\varepsilon > 0$ sufficiently small, we have:

$$J(\varepsilon\varphi_1) \leq \frac{\varepsilon^2}{2}(\lambda_1 - B) \int_\Omega \varphi_1^2 \, dx.$$

(b) Deduce that $J(u^*) < 0$. Conclude.

5.7 Topic 7

Let's consider the following problem (P):

$$(\mathrm{P}) \qquad \begin{cases} -\Delta_p u = |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$. We assume that $1 < q < p$.

We admit that the functional

$$\begin{array}{rcl} L^r(\Omega) & \to & \mathbb{R} \\ u & \longmapsto & \varphi(u) := \|u\|^r_{L^r(\Omega)}, \quad r > 1 \end{array}$$

is differentiable, and

$$\langle \varphi'(u), v \rangle = r \int_\Omega |u|^{r-2} uv \, dx, \quad \forall u, v \in L^r(\Omega).$$

We define

$$J(u) := \frac{1}{p}\|\nabla u\|^p_{L^p} - \frac{1}{2}\|u\|^q_{L^q}.$$

1. Show that J is differentiable on $W_0^{1,p}(\Omega)$ and compute $J'(u)$.

2. Show that for all $\varepsilon > 0$, there exists $c_1 > 0$ such that

$$\|u\|_{L^q} \leq \varepsilon \|u\|^p_{W_0^{1,p}} + c_1.$$

3. Deduce that J is bounded below.

4. Let $(u_n)_n$ be a minimizing sequence of J, that is,

$$\lim J(u_n) = \inf_{v \in W_0^{1,p}(\Omega)} J(v).$$

(a) Show that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.

(b) Deduce that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u, \text{ in } W_0^{1,p}(\Omega), \qquad u_n \to u \text{ in } L^q(\Omega).$$

(c) Show that $J(u) = \inf\limits_{v \in W_0^{1,p}(\Omega)} J(v)$.

(d) Deduce that u is a weak solution of (P).

(e) Let $\varphi_1 \in W_0^{1,p}(\Omega)$ be the first positive eigenfunction of Δ_p (i.e., $-\Delta_p \varphi_1 = \lambda_1 \varphi_1^{p-1}$, $\lambda_1 > 0$). Show that there exists $t > 0$ such that $J(t\varphi_1) < 0$.

(f) Show that $u \neq 0$.

5.8 Topic 8

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set, and let $1 < p < N$. We define:

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}.$$

We set:

$$I(u) = \|u\|^p_{W_0^{1,p}(\Omega)} := \int_\Omega |\nabla u|^p \, dx, \quad J(u) = \|u\|^p_{L^p(\Omega)} := \int_\Omega |u|^p \, dx,$$

and we define the quotient functional $Q : W_0^{1,p}(\Omega) \setminus \{0\} \to \mathbb{R}$ by:

$$Q(u) = \frac{I(u)}{J(u)}.$$

Thus, $\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)} Q(u)$.

1. Using Poincaré's inequality, verify that $\lambda_1 > 0$.

2. Let (u_n) be a minimizing sequence of Q, that is,

$$\lim_{n\to\infty} Q(u_n) = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} Q(u).$$

   We can normalize (u_n) by setting $\|u_n\|_{L^p} = 1$. Show that (u_n) is bounded in $W_0^{1,p}(\Omega)$.

3. Deduce that there exists a subsequence (denoted again by $(u_n)_n$) and $u^* \in W_0^{1,p}(\Omega)$ such that:

$$u_n \rightharpoonup u^* \text{ in } W_0^{1,p}(\Omega), \quad u_n \to u^* \text{ in } L^p(\Omega).$$

4. Knowing that $\|u^*\|_{W_0^{1,p}(\Omega)} \leq \liminf\limits_{n\to\infty} \|u_n\|_{W_0^{1,p}}$, show that $Q(u^*) = \lambda_1$.

5. It is assumed that every minimum point of a differentiable functional is a critical point. Given that Q is differentiable, we have:

$$\langle Q'(u), v\rangle = \frac{\lambda_1}{J(u)} \left[\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx - Q(u) \int_\Omega |u|^{p-2} uv \, dx \right].$$

   Deduce that u^* satisfies

$$\int_\Omega |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla v \, dx = \lambda_1 \int_\Omega |u^*|^{p-2} u^* v \, dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

5.9 Topic 9

Let Ω be a bounded and regular open set in $\mathbb{R}^N$ $(N \geq 3)$ containing the origin $(0 \in \Omega)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following conditions:

$(\mathbf{H_1})$ $(f(t) - f(s))(t - s) \leq 0, \quad \forall t, s \in \mathbb{R}.$

$(\mathbf{H_2})$ $\exists a, b > 0$, $\exists \sigma \in [1, 2[: \quad |F(t)| \leq a|t| + b|t|^\sigma, \quad \forall t \in \mathbb{R},$

where $F(t) = \int_0^t f(s)\, ds$. We consider the following problem:

$$(P) \begin{cases} -\Delta u = \mu \dfrac{u}{x^2} + f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \quad \mu > 0$$

We recall Hardy's inequality:

$$\int_\Omega \frac{|u(x)|^2}{|x|^2}\, dx \leq \frac{1}{H} \int_\Omega |\nabla u|^2\, dx, \quad \forall u \in H_0^1(\Omega), \qquad \text{where } H = \left(\frac{N-2}{2}\right)^2.$$

We consider the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by:

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2\, dx - \frac{\mu}{2} \int_\Omega \frac{u^2}{|x|^2}\, dx - \int_\Omega F(u)\, dx.$$

We define

$$\Psi_1(u) = \frac{1}{2} \int_\Omega |\nabla u|^2\, dx - \frac{\mu}{2} \int_\Omega \frac{u^2}{|x|^2}\, dx, \quad \Psi_2(u) = \int_\Omega F(u)\, dx.$$

1. Verify that J is well-defined.

2. Suppose that $0 < \mu < H$. Show that:

$$\Psi_1(u) \geq \left(\frac{H-\mu}{2H}\right) \|u\|_{H_0^1}^2, \quad \text{where } \|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}.$$

3. (a) Show that ψ_1 is differentiable (in the Fréchet sense) on $H_0^1(\Omega)$ and that we have:

$$\langle \psi_1'(u), v\rangle = \int_\Omega \nabla u \nabla v\, dx - \mu \int_\Omega \frac{uv}{|x|^2}\, dx, \quad \forall u, v \in H_0^1(\Omega).$$

   (b) It is assumed that ψ_2 is differentiable on $H_0^1(\Omega)$ and

$$\langle \psi_2'(u), v\rangle = \int_\Omega f(u) v\, dx.$$

   Deduce that J is differentiable on $H_0^1(\Omega)$ and that:

$$\langle J'(u), v\rangle = \langle \psi_1'(u) - \psi_2'(u), v\rangle.$$

   What can we observe?

4. Show that J' is is strictly monotone.

5. Show that there exist constants $c_1, c_2 > 0$ such that:

$$J(u) \geq \frac{H-\mu}{2H} \|u\|_{H_0^1}^2 - c_1 \|u\|_{H_0^1} - c_2 \|u\|_{H_0^1}^\sigma.$$

   Deduce that J is coercive.

6. Conclude.

Chapter 6

Solutions of Some Topics

6.1 Solution to Topic 1

1. A function $u \in H_0^1(\Omega)$ is a weak solution of (P) if and only if:

$$\int_\Omega \nabla u \nabla v \, dx - \int_\Omega |u|^{q-2} uv \, dx = 0, \quad \forall v \in H_0^1(\Omega).$$

2. (a) For all $u \in H_0^1(\Omega)$, we have $\int_\Omega |\nabla u|^2 \, dx = \|u\|_{H_0^1}^2 < \infty$, and by the continuous embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ ($q \in [2, 2^*[\subset [1, 2^*]$), there exists $C_0 > 0$ such that:

$$\|u\|_{L^q} \leq C_0 \|u\|_{H_0^1}, \forall u \in H_0^1(\Omega). \tag{6.1}$$

thus $\int_\Omega |u|^q \, dx < \infty$.

(b) We have $\varphi \in C^1(H_0^1(\Omega), \mathbb{R})$, moreover:

$$\langle \varphi'(u), v \rangle = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega |u|^{q-2} uv \, dx, \quad \forall u, v \in H_0^1(\Omega).$$

It is easy to see that:

$$\begin{aligned} u \in H_0^1(\Omega) \text{ is a critical point of } \varphi \quad &\Leftrightarrow \quad \varphi'(u) = 0 \text{ in } \left(H_0^1(\Omega)\right)' \\ &\Leftrightarrow \quad \langle \varphi'(u), v \rangle = 0, \quad \forall v \in H_0^1(\Omega) \\ &\Leftrightarrow \quad \int_\Omega \nabla u \nabla v \, dx - \int_\Omega |u|^{q-2} uv \, dx = 0 \\ &\Leftrightarrow \quad u \text{ is a weak solution of } (P) \end{aligned}$$

3. (a) Using (6.1), we get $\varphi(u) \geq \frac{1}{2}\|u\|_{H_0^1}^2 - C_1 \|u\|_{H_0^1}^q$, with $C_1 = C_0^q$.

(b) Since $q > 2$, there exists $\rho > 0$ small enough so that $\frac{1}{2}\rho^2 - C_1 \rho^q > 0$, let $\alpha = \frac{1}{2}\rho^2 - C_1\rho^q$, then we can write:

$$\forall u \in H_0^1(\Omega), \|u\|_{H_0^1} = \rho : \varphi(u) \geq \frac{1}{2}\rho^2 - C_1 \rho^q = \alpha > 0.$$

4. (a) We have $\varphi(tv) = \frac{1}{2}t^2 \int_\Omega |\nabla v|^2 \, dx - \frac{1}{q} t^q \int_\Omega |v|^q \, dx = \frac{1}{2}t^2 - \frac{1}{q}\theta t^q$, since $q > 2$, we get $\lim\limits_{t \to +\infty} \varphi(tv) = \lim\limits_{t \to +\infty} \dfrac{1}{2}t^2 - \dfrac{1}{q}\theta t^q = -\infty$.

(b) According to question (4-a), there exists $t_0 >> 0$ large enough such that $\varphi(t_0 v) < 0$ and $\|t_0 v\|_{H_0^1} = t_0 > \alpha$. Thus we take $\psi = t_0 v$.

5. (a) We say that φ satisfies the Palais-Smale (P.S) condition if: for any sequence $(u_n) \subset H_0^1(\Omega)$ such that $|\varphi(u_n)| \leq C$ and $\varphi'(u_n) \to 0$ in $\Big(H_0^1(\Omega)\Big)'$, we can extract a subsequence that converges strongly in $H_0^1(\Omega)$.

(b) Let $(u_n) \subset H_0^1(\Omega)$ be a (P.S) sequence; i.e.,

$$|\varphi(u_n)| \leq C, \quad \varphi'(u_n) \to 0 \quad \text{in } \Big(H_0^1(\Omega)\Big)'. \tag{6.2}$$

i. It is easy to see that $|\langle \varphi'(u_n), u_n\rangle| \leq \|\varphi'(u_n)\|_{(H_0^1)'}\|u_n\|_{H_0^1}$. According to (6.2), $\|\varphi'(u_n)\|_{(H_0^1)'} \to 0$, so for n large enough we have $\|\varphi'(u_n)\|_{(H_0^1)'} \leq 1$, thus $|\langle \varphi'(u_n), u_n\rangle| \leq \|\varphi'(u_n)\|_{(H_0^1)'}\|u_n\|_{H_0^1} \leq \|u_n\|_{H_0^1}$.
We deduce that:

$$\varphi(u_n) - \frac{1}{q}\langle \varphi'(u_n), u_n\rangle \leq |\varphi(u_n)| + \frac{1}{q}|\langle \varphi'(u_n), u_n\rangle| \leq C + \frac{1}{q}\|u_n\|_{H_0^1}.$$

ii. Suppose that (u_n) is not bounded in $H_0^1(\Omega)$; then there exists a subsequence (u_{n_k}) (denoted (u_n)) such that $\|u_n\|_{H_0^1} \to +\infty$ as $n \to +\infty$. We have:

$$\begin{aligned} C + \frac{1}{q}\|u_n\|_{H_0^1} \geq \varphi(u_n) - \frac{1}{q}\langle \varphi'(u_n), u_n\rangle &= \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx - \frac{1}{q}\int_\Omega |u_n|^q\,dx \\ &\quad -\frac{1}{q}\Big[\int_\Omega |\nabla u_n|^2\,dx - \int_\Omega |u_n|^q\,dx\Big] \\ &= \Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_n\|_{H_0^1}^2 \end{aligned}$$

which implies that $\Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_n\|_{H_0^1}^2 - \frac{1}{q}\|u_n\|_{H_0^1} \leq C$; this is impossible because $\Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_n\|_{H_0^1}^2 - \frac{1}{q}\|u_n\|_{H_0^1} \to +\infty$ as $\|u_n\|_{H_0^1} \to +\infty$, and this is due to $q > 2$, i.e., $\frac{1}{2} - \frac{1}{q} > 0$. Therefore, the conclusion is that the sequence (u_n) is bounded in $H_0^1(\Omega)$.

(c) i. Since (u_n) is bounded in $H_0^1(\Omega)$, which is reflexive, we can extract a subsequence (u_{n_k}) (denoted (u_n)) and a function $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$.
By the compact embedding $H_0^1(\Omega) \hookrightarrow_{comp} L^q(\Omega)$ ($q \in [2, 2^*[\subset [1, 2^*[$), we then have $u_n \to u$ strongly in $L^q(\Omega)$; ($\|u_n - u\|_{L^q} \to 0$, as $n \to +\infty$).
Since $u_n \to u$ strongly in $L^q(\Omega)$, by T.C.D diverse, there exists a subsequence of (u_n) (still denoted (u_n)) such that $u_n(x) \to u(x)$ a.e. $x \in \Omega$.

ii. Since $\varphi'(u) \in \Big(H_0^1(\Omega)\Big)'$ and $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, then $\langle \varphi'(u), u_n\rangle \to \langle \varphi'(u), u\rangle$, hence $\langle \varphi'(u), u_n - u\rangle \to 0$ as $n \to +\infty$.
Similarly, since $\varphi'(u_n) \in \Big(H_0^1(\Omega)\Big)'$ for all n and $\varphi'(u_n) \to 0$ in $\Big(H_0^1(\Omega)\Big)'$ and $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, we have $\langle \varphi'(u_n), u_n - u\rangle \to 0$ as $n \to +\infty$.

(d)

$$\begin{aligned}
\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle &= \langle \varphi'(u_n), u_n - u \rangle - \langle \varphi'(u), u_n - u \rangle \\
&= \int_\Omega \nabla u_n \nabla (u_n - u)\, dx - \int_\Omega |u_n|^{q-2} u_n (u_n - u)\, dx \\
&\quad - \int_\Omega \nabla u \nabla (u_n - u)\, dx + \int_\Omega |u|^{q-2} u (u_n - u)\, dx \\
&= \|u_n - u\|_{H_0^1} - \int_\Omega (|u_n|^{q-2} u_n - |u|^{q-2} u)(u_n - u)\, dx.
\end{aligned}$$

(e) We apply Hölder's inequality

$$\begin{aligned}
\left| \int_\Omega |u_n|^{q-2} u_n (u_n - u)\, dx \right| &\leq \left(\int_\Omega |u_n - u|^q\, dx \right)^{\frac{1}{q}} \left(\int_\Omega (|u_n|^{q-1})^{\frac{q}{q-1}}\, dx \right)^{\frac{q-1}{q}} \\
&= \|u_n\|_{L^q}^{q-1} \|u_n - u\|_{L^q} \\
&\leq C_2 \|u_n - u\|_{L^q}.
\end{aligned}$$

The last estimate holds because $u_n \to u$ strongly in $L^q(\Omega)$, so the real sequence $\|u_n\|_{L^q}^{q-1}$ is bounded.
Similarly, we have

$$\begin{aligned}
\left| \int_\Omega |u|^{q-2} u (u_n - u)\, dx \right| &\leq \left(\int_\Omega |u_n - u|^q\, dx \right)^{\frac{1}{q}} \left(\int_\Omega (|u|^{q-1})^{\frac{q}{q-1}}\, dx \right)^{\frac{q-1}{q}} \\
&= \|u\|_{L^q}^{q-1} \|u_n - u\|_{L^q} \\
&= C_3 \|u_n - u\|_{L^q}.
\end{aligned}$$

with $C_3 = \|u\|_{L^q}^{q-1}$.

(f) On the one hand, from question (5-c-ii) we have: $\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle \to 0$ as $n \to +\infty$.
On the other hand, by (5-e) and since $\|u_n - u\|_{L^q} \to 0$ as $n \to +\infty$, we have: $\int_\Omega (|u_n|^{q-2} u_n - |u|^{q-2} u)(u_n - u)\, dx \to 0$ as $n \to +\infty$.
Thus, question (5-d) ensures that $\|u_n - u\|_{H_0^1} \to 0$ as $n \to +\infty$; i.e., the sequence (u_n) converges strongly in $H_0^1(\Omega)$.
We deduce that the functional φ satisfies the (P.S) condition.

6. We have $\varphi \in C^1(H_0^1(\Omega), \mathbb{R})$, the two geometric conditions are satisfied, and the (P.S) condition is satisfied. Therefore, by the Mountain Pass Theorem, the functional φ has at least one critical point, and consequently, our problem has at least one nontrivial weak solution in $H_0^1(\Omega)$.

6.2 Solution of Topic 5

(I) 1. Let $(E, \|\cdot\|)$ be a Banach space. E is said to be **reflexive** if the canonical injection $i : E \to E^{**}$ is surjective, i.e., $i(E) = E^{**}$, where:

$$\begin{aligned}
i : E &\to E^{**} \\
x &\mapsto i(x) : E^* \to \mathbb{R} \\
&\qquad\qquad x \mapsto \langle i(x), f \rangle_{E^{**},E^*} = \langle f, x \rangle_{E^*,E}.
\end{aligned}$$

2. Let $\mathcal{I} : E \to \mathbb{R}$ be a differentiable functional. Suppose that u_0 is a point minimum of $\mathcal{I}$; that is $\mathcal{I}(u_0) = \min_{u \in E} \mathcal{I}(u)$. Let $v \in E$ be fixed and consider $R : \mathbb{R} \to \mathbb{R}$ defined by:
$$R(t) = \mathcal{I}(u_0 + tv).$$
We can see that 0 is a point minimum of R as u_0 is a point minimum of $\mathcal{I}$. Then $R'(0) = 0$ (R is differentiable at 0). On the other hand:
$$R'(0) = \lim_{t \to 0} \frac{R(t) - R(0)}{t} = \lim_{t \to 0} \frac{\mathcal{I}(u_0 + tv) - \mathcal{I}(u_0)}{t} = \langle \mathcal{I}'(u_0), v \rangle.$$
Thus, $\langle \mathcal{I}'(u_0), v \rangle = 0$, $\forall v \in E$. This implies that $\mathcal{I}'(u_0) = 0$, i.e., u_0 is a critical point of $\mathcal{I}$.

(II) (15pt)

1. Hypothesis (H) means that: $\forall \varepsilon > 0, \exists A > 0, \forall t, |t| > A \implies \left| \frac{|F(t)|}{|t|^\theta} - l \right| \leq \varepsilon$.
So, $l - \varepsilon \leq \frac{|F(t)|}{|t|^\theta} \leq l + \varepsilon$. Then, there exists $C_1 > 0$ such that:
$$|F(t)| \leq (l + \varepsilon)|t|^\theta \leq C_1 |t|^\theta, \quad \forall t, |t| > A.$$
The function F is continuous on $[-A, A]$, so it is bounded, then:
$$\exists C_2 > 0 : |F(t)| \leq C_2, \quad \forall t \in [-A, A].$$
Thus, combining these results we get:
$$|F(t)| \leq C_1 |t|^\theta + C_2, \quad \forall t \in \mathbb{R}. \tag{6.3}$$

2. Let $u \in H_0^1(\Omega)$, we have:
$$\int |\nabla u|^2 \, dx = \|u\|_{H_0^1}^2 < \infty.$$
From (6.3) and using the continuous embedding $H_0^1(\Omega) \hookrightarrow L^\theta(\Omega)$ (as $\theta \in (1, 2)$) we have:
$$\left| \int F(u) \, dx \right| \leq \int |F(u)| \, dx \leq C_1 \|u\|_{H_0^1}^\theta + C_2 |\Omega|.$$
Hence, $\mathcal{J}$ is well defined.

3. We set $\langle u, v \rangle_{H_0^1} = \int \nabla u \cdot \nabla v \, dx$ and $\|u\|_{H_0^1}^2 = \int |\nabla u|^2 \, dx$. So, $\psi(u) = \|u\|_{H_0^1}^4$. Let $u, v \in H_0^1(\Omega)$, we have:
$$\begin{aligned} \psi(u + v) - \psi(u) &= \|u + v\|_{H_0^1}^4 + \|u\|_{H_0^1}^4 \\ &= 4\|u\|_{H_0^1}^2 \langle u, v \rangle_{H_0^1} + o(\|u\|), \end{aligned}$$
where
$$o(\|u\|) = 2\|u\|_{H_0^1}^2 \|v\|_{H_0^1}^2 + \|v\|_{H_0^1}^4 + 2\|v\|_{H_0^1}^2 \langle u, v \rangle_{H_0^1},$$
satisfying $\lim\limits_{\|v\| \to 0} o(\|v\|) = 0$. Obviously that the map $H_0^1 \ni v \mapsto A_u(v) = 4\|u\|_{H_0^1}^2 \langle u, v \rangle_{H_0^1}$ is linear and continuous, then we conclude that ψ is differentiable and we have:
$$\langle \psi'(u), v \rangle = 4\|u\|_{H_0^1}^2 \int_\Omega \nabla u \nabla v \, dx.$$

4. F satisfies the growth condition, so $u \mapsto \int_\Omega F(u)\,dx$ is differentiable, we know that ψ and the norm $\|\cdot\|^2_{H^1_0}$ are differentiable. Hence J is differentiable and we have

$$\begin{aligned}\langle J'(u), v\rangle &= a\int_\Omega \nabla u\nabla v\;dx + b\|u\|^2_{H^1_0}\int_\Omega \nabla u\nabla v\;dx - \int_\Omega f(u)v\;dx\\ &= (a + b\|u\|^2_{H^1_0})\int_\Omega \nabla u\nabla v\;dx - \int_\Omega f(u)v\;dx.\end{aligned}$$

   Then, we can notice that any critical point of J is a weak solution of Problem (P).

5. (a) Let $u_n \rightharpoonup u$ weakly in $H^1_0(\omega)$. Since $H^1_0(\omega) \subset L^\theta(\omega)$ is compact as $\theta \in [1, 2^*]$, we have:

$$u_n \to u \text{ in } L^\theta.$$

   Consequently, by the converse of D.C.T., there exists a subsequence (still denoted u_n) such that:

$$u_n(x) \to u(x) \quad \text{a.e. in } \Omega \tag{$\star$}$$

   and

$$\exists g \in L^\theta(\omega), \quad |u_n(x)| \le g(x) \quad \text{a.e. in } \Omega.$$

   Since F is continuous, we have $F(u_n(x)) \to F(u(x))$ for all $x \in \Omega$.
   Moreover:

$$|F(u_n(x))| \le C_1|u_n(x)|^\theta + C_2 \le C_1 g^\theta(x) + C_2.$$

   We set $h = C_1 g^\theta + C_2 \in L^1(\Omega)$. By the Dominated Convergence Theorem, we get:

$$\int_\Omega F(u_n)\,dx \to \int_\Omega F(u)\,dx. \tag{6.4}$$

   (b) We know that $\|\cdot\|_{H^1_0}$ is weakly lower semicontinuous (w.l.s.c.), i.e. for any $(u_n) \subset H^1_0(\Omega)$ such that $u_n \rightharpoonup u$ we have

$$\|u\|_{H^1_0} \le \liminf_{n\to\infty} \|u_n\|_{H^1_0}. \tag{6.5}$$

   From (6.4) and (6.5), we obtain:

$$\begin{aligned}J(u) &= \frac{a}{2}\|u\|^2_{H^1_0} + \frac{b}{4}\|u\|^4_{H^1_0} - \int_\Omega F(u)\,dx\\ &\le \frac{a}{2}\liminf_{n\to\infty}\|u_n\|^2_{H^1_0} + \frac{b}{4}\liminf_{n\to\infty}\|u_n\|^4_{H^1_0} - \lim_{n\to\infty}\int_\Omega F(u_n)\,dx\\ &= \liminf_{n\to\infty} J(u_n).\end{aligned}$$

   This proves that J is w.l.s.c.

6. **Conclusion:** J is coercive and w.l.s.c. Then J has a global minimum. Since J is differentiable, J has at least a critical point, which is means that Problem (P) has at least a weak solution in $H^1_0(\Omega)$.

6.3 Solution of Topic 6

1. (a) Let $u \in H_0^1(\Omega)$ be fixed. We show that there exists a linear and continuous mapping $L_u : H_0^1(\Omega) \to \mathbb{R}$ such that:

$$J_1(u+v) - J_1(u) = L_u(v) + o(v), \quad \forall v \in H_0^1(\Omega).$$

Indeed, let $v \in H^1(\Omega)$, we have:

$$\begin{aligned} J_1(u+v) - J_1(u) &= \int_\Omega |\nabla u + \nabla v|^2 \, dx - \int_\Omega |\nabla u|^2 \, dx \\ &= 2\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega |\nabla v|^2 \, dx. \end{aligned}$$

It is evident that $L_u : H^1(\Omega) \to \mathbb{R}$ defined by: $v \mapsto A_u(v) = 2\int_\Omega \nabla u \cdot \nabla v \, dx$ is linear. Moreover, it is continuous because, for all $v \in H_0^1(\Omega)$ and using the Cauchy-Schwarz inequality, we obtain:

$$|A_u(v)| = 2\left|\int_\Omega \nabla u \cdot \nabla v \, dx\right| \leq 2\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} = 2\|u\|_{H_0^1}\|v\|_{H_0^1}.$$

We also have: $\displaystyle \lim_{\|v\|_{H_0^1} \to 0} \frac{o(v)}{\|v\|_{H_0^1}} = \lim_{\|v\|_{H_0^1} \to 0} \|v\|_{H_0^1} = 0.$

It follows that J_1 is differentiable, and we have:

$$\langle J_1'(u), v\rangle = L_u(v) = 2\int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall v \in H_0^1(\Omega).$$

(b) Since J_1 and J_2 are differentiable, we can see that J is differentiable on $H_0^1(\Omega)$, and we have:

$$\langle J'(u), v\rangle = \frac{1}{2}\langle J_1'(u), v\rangle - \langle J_2'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega f(u)v \, dx.$$

It follows that every critical point of J is a weak solution of problem (P).

2. Using assumption (f_1), we obtain:

$$\forall u \in H^1(\Omega), \quad \int_\Omega F(u) \, dx \leq a\int_\Omega |u| \, dx + b\int_\Omega |u|^p \, dx.$$

As the injections $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^\sigma(\Omega)$ are continuous (since $1 < \sigma < 2$), there exist $\alpha, \beta > 0$ such that:

$$\|u\|_{L^1} \leq \alpha\|u\|_{H_0^1}, \quad \|u\|_{L^\sigma} \leq \beta\|u\|_{H_0^1}.$$

Thus:

$$\int_\Omega F(u) \, dx \leq a\alpha\|u\|_{H^1} + b\beta^p\|u\|_{H^1}^\sigma.$$

Consequently:

$$J(u) \geq \frac{1}{2}\|u\|_{H^1}^2 - a\alpha\|u\|_{H_0^1} - b\beta^\sigma\|u\|_{H_0^1}^\sigma.$$

Taking $c_1 = a\alpha > 0$ and $c_2 = b\beta^\sigma > 0$.

3. Suppose that (u_n) is not bounded, then there exists a subsequence (u_{n_k}) such that:

$$\lim_{n_k \to +\infty} \|u_{n_k}\|_{H_0^1} = +\infty.$$

However, according to (1) and since $\sigma < 2$, we obtain:

$$\lim_{n_k \to +\infty} J(u_{n_k}) = +\infty,$$

which contradicts the fact that:

$$\lim_{n_k \to +\infty} J(u_{n_k}) = \inf_{u \in H_0^1(\Omega)} J(u).$$

Thus, (u_n) is bounded in $H_0^1(\Omega)$.

4. Since $H_0^1(\Omega)$ is reflexive, there exists a subsequence (u_{n_k}) denoted as (u_n) such that:

$$u_n \rightharpoonup u^* \quad \text{weakly in } H^1(\Omega).$$

Since $H_0^1(\Omega)$ is compactly embedded in $L^\sigma(\Omega)$, we get $u_n \to u^*$ strongly in $L^\sigma(\Omega)$. Consequently, there exists a subsequence (u_n) such that:

$$u_n \to u^* \quad \text{almost everywhere in } \Omega.$$

and

$$\exists h \in L^\sigma(\Omega), \quad |u_n| \leq h(x) \quad \text{almost everywhere in } \Omega.$$

Since F is continuous, we have: $F(u_n) \to F(u)$ almost everywhere in Ω. Moreover:

$$|F(u_n)| \leq a|u_n| + b|u_n|^\sigma \leq ah + bh^\sigma.$$

Since $ah + bh^\sigma \in L^1(\Omega)$, by the dominated convergence theorem, we deduce that:

$$\int_\Omega F(u_n)\, dx \to \int_\Omega F(u)\, dx.$$

5. We have:

$$\begin{aligned} J(u^*) &= \frac{1}{2}\|u^*\|_{H^1}^2 - \int_\Omega F(u^*)\, dx. \\ &\leq \frac{1}{2}\liminf \|u_n\|_{H^1}^2 - \lim_{n\to+\infty} \int_\Omega F(u_n)\, dx = \liminf_{n\to+\infty} J(u_n) = \inf_{u\in H^1(\Omega)} J(u) \leq J(u^*). \end{aligned}$$

We deduce that:

$$J(u^*) = \inf_{u \in H^1(\Omega)} J(u).$$

Thus, u^* is a minimum point, so (P) admits a solution u^*, which is a critical point of J.

6. (a) For sufficiently small ε, we have $0 < \varepsilon\varphi_1 < \delta$. According to (f_2), we obtain:

$$F(\varepsilon\varphi_1) \geq \frac{\beta}{2}\varepsilon^2\varphi_1^2. \tag{6.6}$$

Taking $v = \varphi_1$ in the formulation, we obtain:

$$\int_\Omega |\nabla\varphi_1|^2\,dx = \lambda_1 \int_\Omega \varphi_1^2\,dx. \tag{6.7}$$

Using (6.6) and (6.7), we get:

$$\begin{aligned} J(\varepsilon\varphi_1) &= \frac{\varepsilon^2}{2}\int_\Omega |\nabla\varphi_1|^2\,dx - \int_\Omega F(\varepsilon\varphi_1)\,dx. \\ &\leq \frac{\varepsilon^2\lambda_1}{2}\int_\Omega \varphi_1^2\,dx - \frac{\beta}{2}\varepsilon^2\int_\Omega \varphi_1^2\,dx \\ &= \frac{\varepsilon^2}{2}(\lambda_1 - \beta)\int_\Omega \varphi_1^2\,dx < 0 \qquad (\lambda_1 < \beta) \end{aligned}$$

(b) We deduce that:

$$J(u^*) = \inf J(u) \leq J(\varepsilon\varphi_1) < 0.$$

Since $J(0) = 0$ and $J(u^*) < 0$, then $u^* \neq 0$, and consequently, the weak solution u^* is nontrivial.

6.4 Solution of Topic 7

1. We have:

- For $r = q$: $\|u\|^q_{L^q(\Omega)} = \phi(u)$. Thus, $u \mapsto \|u\|_{L^q(\Omega)}$ is differentiable and

$$\left\langle \left(\|u\|^q_{L^q(\Omega)}\right)', v \right\rangle = q\int_\Omega |u|^{q-2}uv dx, \quad \forall u, v \in L^q(\Omega)$$

- For $r = p$: $\|u\|^p_{W_0^{1,p}} = \phi(\nabla u)$ and $u \mapsto \|u\|^p_{W_0^{1,p}}$ is differentiable and

$$\left\langle \left(\|u\|^p_{W_0^{1,p}}\right)', v \right\rangle = p\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \quad \forall u, v \in W_0^{1,p}$$

.

Thus, J is differentiable and

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \int_\Omega |u|^{q-2}uv dx, , \quad \forall u, v \in W_0^{1,p}(\Omega)$$

2. We apply Young's inequality:

$$\forall\alpha > 0, \exists C > 0 : ab \leq \alpha a^r + Cb^{r'}, \quad 1/r + 1/r' = 1.$$

Let $\varepsilon > 0$, we have:

$$\begin{aligned} \|u\|^q_{L^q(\Omega)} &\leq \alpha\|u\|^p_{L^q(\Omega)} + C, \text{ with } a = \|u\|^q_{L^q(\Omega)},\ b = 1,\ r = p/q > 1 \\ &\leq \alpha C_0\|u\|^p_{L^q(\Omega)} + C, \text{ where we use } W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \\ &\leq \varepsilon\|u\|^p_{L^q(\Omega)} + C, \quad \text{for } \alpha = \varepsilon/C_0 \end{aligned}$$

3. For $\varepsilon < \frac{1}{p}$, we have:

$$J(u) := \frac{1}{p}\|u\|^p_{W_0^{1,p}} - \frac{1}{q}\|u\|^q_{L^q(\Omega)} \geq (\frac{1}{p} - \varepsilon)\|u\|^p_{W_0^{1,p}} - C \geq -C$$

4. (a) We have: $J(u_n) \to \inf J$ Thus $|J(u_n)| \leq C \quad \forall u$ for a positive constant C. Using the previous inequality, we obtain

$$\frac{1}{p}\|u_n|^p_{W_0^{1,p}} \leq \varepsilon\|u_n\|^p_{W_0^{1,p}} + C', C' = C+$$

hence

$$(\frac{1}{p} - \varepsilon)\|u_n\|^p_{W_0^{1,p}} \leq C'$$

For $\varepsilon < 1/p$, we deduce that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.

(b) Since $\{u_n\}$ is bounded in $W_0^{1,p}$ and $W_0^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for $(q < p)$, then, for a subsequence:

$$u_n \rightharpoonup u \quad \text{in } L^q \quad \text{and} \quad u_n \rightharpoonup u \quad \text{in } W_0^{1,p}.$$

(c) We have:

$$\|u\|_{L^q} = \lim \|u\|_{L^q(\Omega)}, \quad \|u\|_{W_0^{1,p}} \leq \liminf_{n \to \inf} \|u_n\|_{W_0^{1,p}}$$

Thus

$$J(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}} - \frac{1}{q}\|u\|^q_{L^q} \leq \lim \left(\frac{1}{p}\|u_n\|^p_{W_0^{1,p}} - \frac{1}{q}\|u_n\|^q_{L^q}\right) = \inf J \leq J(u).$$

Thus, $J(u) = \inf J$.

(d) Since J is differentiable and u is minimal for J, then $J'(u) = 0$, that is,

$$\langle J'(u), u\rangle = 0 \quad \forall u \in W_0^{1,p}.$$

Thus,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \; dx \int_\Omega u^{q-2}uv \; dx.$$

(e)

$$J(t\varphi) = \frac{t^p}{p}\|\varphi\|^p_{W_0^{1,p}} - \frac{t^q}{q}\|\varphi\|_{L^q} = t^q\left(\frac{t^{p-q}}{p}\|\varphi\|^p_{W_0^{1,p}} - \frac{1}{q}\|u\|_{L^q}\right) < 0$$

for $t > 0$ sufficiently small, we have $J(t\varphi) < 0$ because $q < p$.

(f) We notice that $J(0) = 0$ and

$$J(u) = \inf J \leq J(t\varphi) < 0.$$

Therefore, we deduce that $u \neq 0$.

6.5 Solution of Topic 8

1. Let $u \in W_0^{1,p}(\Omega)$. According to Poincaré's inequality,

$$\exists C > 0 : \|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

Thus,

$$\frac{1}{C} \leq \frac{\|\nabla u\|_{L^p(\Omega)}}{\|u\|_{L^p(\Omega)}} := Q(u)$$

Consequently,

$$0 < \frac{1}{C} \leq \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} := \lambda_1$$

2. Let $(u_n)_{n\geq 1}$ be a minimizing sequence in $W_0^{1,p}(\Omega)$, i.e.,

$$Q(u_n) := \frac{\|\nabla u_n\|_{L^p(\Omega)}}{\|u_n\|_{L^p(\Omega)}} \to \lambda_1 \quad \text{as } n \to +\infty.$$

We can normalize $(u_n)_{n\geq 1}$ by setting $\|u_n\|_{L^p(\Omega)} = 1$ for all $n \geq 1$, then

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} := \int_\Omega |\nabla u_n|^p dx \to \lambda_1.$$

Thus, $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.

3. Since $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$, it follows that there exists a subsequence, still denoted by (u_n), such that:

$$u_n \rightharpoonup u^* \text{ in } W_0^{1,p}(\Omega).$$

By the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow_c L^p(\Omega)$ (for a subsequence),

$$u_n \to u^* \text{ in } L^p(\Omega).$$

4. We have $J(u_n) = 1$. Thus,

$$Q(u^*) = I(u^*) \leq \lim_{n\to+\infty} \inf I(u_k) = \lim_{n\to+\infty} I(u_n) = \lambda_1 := \inf Q(u) \leq Q(u^*).$$

Consequently, $Q(u^*) = \lambda_1$.

5. Since u^* is a minimum point of Q, we have $Q'(u^*) = 0$, and therefore

$$\langle Q'(u^*), v\rangle = \frac{\lambda_1}{J(u^*)}\left[\int_\Omega |\nabla u^*|^{p-2}\nabla u^* \nabla v \, dx - Q(u)\int_\Omega |u^*|^{p-2}u^* v \, dx\right] = 0$$

for all $v \in W_0^{1,p}(\Omega)$. Thus,

$$\int_\Omega |\nabla u^*|^{p-2}\nabla u^* \cdot \nabla v \, dx = \lambda_1 \int_\Omega |u^*|^{p-2}u^* v \, dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

6.6 Solution of Topic 9

1. Let $u \in H_0^1(\Omega)$, we have:
$$\int_\Omega |\nabla u|^2\,dx = \|u\|_{H^1}^2 < \infty,$$
Hardy's inequality ensures that:
$$\int_\Omega \frac{u^2}{|x|^2}\,dx \le \frac{1}{H}\|u\|_{H_0^1}^2 < \infty.$$
$$\int_\Omega |F(u)|\,dx \le a\|u\|_{L^1} + b\|u\|_{L^\sigma}^\sigma < \infty.$$
Since $H^1(\Omega) \subset L^q(\Omega)$ for $q = 1$ or $q = \sigma \in [1, 2[$.

Therefore, we deduce that J is well-defined.

2. Using Hardy's inequality, we obtain:
$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\mu}{2}\int_\Omega \frac{u^2}{|x|^2}\,dx &\ge \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\mu}{2H}\int_\Omega |\nabla u|^2\,dx.\\ &= \left(\frac{H-\mu}{2H}\right)\int_\Omega |\nabla u|^2\,dx.\end{aligned}$$
Hence, $\Psi_1(u) \ge \left(\frac{H-\mu}{2H}\right)\|u\|_{H_0^1}^2$.

3. (a) Ψ_1 is differentiable. Indeed, let $u, v \in H^1(\Omega)$, we have:
$$\begin{aligned}\Psi_1(u+v) - \Psi_1(u) &= \frac{1}{2}\int_\Omega |\nabla u + \nabla v|^2\,dx - \frac{\mu}{2}\int_\Omega \frac{(u+v)^2}{|x|^2}\,dx\\ &\quad -\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\mu}{2}\int_\Omega \frac{u^2}{|x|^2}\,dx\\ &= \int_\Omega \nabla u\cdot\nabla v\,dx - \mu\int_\Omega \frac{uv}{|x|^2}\,dx\\ &\quad +\frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{\mu}{2}\int_\Omega \frac{v^2}{|x|^2}\,dx.\end{aligned}$$
For fixed $u \in H_0^1$, we define:
$$A_u(v) = \int_\Omega \nabla u\cdot\nabla v\,dx - \mu\int_\Omega \frac{uv}{|x|^2}\,dx, \quad o(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{\mu}{2}\int_\Omega \frac{v^2}{|x|^2}\,dx.$$
We note that $\lim_{\|v\|_{H_0^1}\to+\infty} o(v) = 0$; indeed, by Hardy's inequality, we have:
$$|o(v)| \le \frac{1}{H}\|v\|_{H_0^1}^2 + \frac{\mu}{2}\|v\|_{H_0^1}^2 = \left(\frac{1}{2} + \frac{\mu}{2}\right)\|v\|_{H_0^1}^2.$$
Furthermore, A_u is linear (evident) and continuous since for all $v \in H_0^1(\Omega)$ and using Cauchy-Schwarz inequality followed by Hardy's inequality, we obtain:
$$\begin{aligned}|A_u(v)| &\le \left|\int_\Omega \nabla u\nabla v\,dx\right| + \mu\left|\int_\Omega \frac{u}{|x|}\frac{v}{|x|}\,dx\right|.\\ &\le \|u\|_{H_0^1}\|v\|_{H_0^1} + \mu\left(\int_\Omega \frac{u^2}{|x|^2}\,dx\right)^{1/2}\left(\int_\Omega \frac{v^2}{|x|^2}\,dx\right)^{1/2}\\ &\le \|u\|_{H_0^1}\|v\|_{H_0^1} + \frac{\mu}{H}\|u\|_{H_0^1}\|v\|_{H_0^1}\\ &= \left(1 + \frac{\mu}{H}\right)\|u\|_{H_0^1}\|v\|_{H_0^1}.\end{aligned}$$

We set $C = \left(1 + \frac{\mu}{H}\right) \|u\|_{H^1} > 0$. Thus, A_u is continuous.

Consequently, Ψ_1 is (Fréchet) differentiable, and we have:

$$\langle \Psi_\lambda'(u), v\rangle = A_u(v) = \int_\Omega \nabla u \nabla v \, dx - \mu \int_\Omega \frac{uv}{|x|^2} \, dx.$$

(b) Knowing that $J = \Psi_1 - \Psi_2$ and since Ψ_1, Ψ_2 are differentiable on $H_0^1(\Omega)$, then J is differentiable on $H^1(\Omega)$, and therefore we have:

$$\langle J'(u), v\rangle = \langle \Psi_1'(u) - \Psi_2'(u), v\rangle = \int_\Omega \nabla u \nabla v \, dx - \mu \int_\Omega \frac{uv}{|x|^2} \, dx - \int_\Omega f(u) v \, dx.$$

We observe that every critical point of J is a weak solution of problem (P).

4. Let $u, v \in H_0^1(\Omega)$, we have:

$$\langle J'(u), u - v\rangle = \int_\Omega \nabla u \cdot (\nabla u - \nabla v) \, dx - \mu \int_\Omega \frac{u(u-v)}{|x|^2} \, dx - \int_\Omega f(u)(u-v) \, dx.$$

$$\langle J'(v), u - v\rangle = \int_\Omega \nabla v \cdot (\nabla u - \nabla v) \, dx - \mu \int_\Omega \frac{v(u-v)}{|x|^2} \, dx - \int_\Omega f(v)(u-v) \, dx.$$

From which we conclude that J' satisfies (1); indeed, for all $u, v \in H_0^1(\Omega)$ with $u \neq v$, we have:

$$\begin{aligned}
\langle J'(u) - J'(v), u - v\rangle &= \langle J'(u), u - v\rangle - \langle J'(v), u - v\rangle \\
&= \int_\Omega |\nabla u - \nabla v|^2 \, dx - \mu \int_\Omega \frac{(u-v)^2}{|x|^2} \, dx \\
&\quad - \int_\Omega (f(u) - f(v))(u - v) \, dx.
\end{aligned}$$

We can see that

$$\int_\Omega |\nabla u - \nabla v|^2 \, dx - \mu \int_\Omega \frac{(u-v)^2}{|x|^2} \, dx = \Psi_1\big(2(u-v)\big) \geq \Big(\frac{H-\mu}{H}\Big)\|u - v\|_{H_0^1}^2.$$

According to hypothesis (H_1), we have $(f(u) - f(v))(u - v) \leq 0$ so that

$$-\int_\Omega (f(u) - f(v))(u - v) \, dx \geq 0.$$

Which implies that:

$$\langle J'(u) - J'(v), u - v\rangle \geq \Big(\frac{H-\mu}{H}\Big)\|u - v\|_{H_0^1}^2 > 0, \quad \forall u, v \in H_0^1(\Omega), u \neq v.$$

5. First, we know that:

$$\Psi_1\big(u\big) \geq \Big(\frac{H-\mu}{2H}\Big)\|u\|_{H_0^1}^2. \tag{6.8}$$

On the other hand, using (H_1), we obtain:

$$\int_\Omega |F(u)| \, dx \leq a\|u\|_{L^1} + b\|u\|_{L^\sigma}^\sigma$$

Since the injections $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^\sigma(\Omega)$ are continuous, there exist $\alpha, \beta > 0$ such that

$$\|u\|_{L^1} \leq \alpha \|u\|_{H_0^1}, \quad \|u\|_{L^\sigma}^\sigma \leq \beta \|u\|_{H_0^1}^\sigma,$$

consequently:

$$\int_\Omega |F(u)|\, dx \leq a\|u\|_{L^1} + b\|u\|_{L^\sigma}^\sigma \leq a\alpha\|u\|_{H_0^1} + b\beta\|u\|_{H_0^1}^\sigma. \tag{6.9}$$

Relations (6.8) and (6.9) imply that:

$$J(u) \geq \frac{H-\mu}{2H}\|u\|_{H_0^1}^2 - c_1\|u\|_{H_0^1} - c_2\|u\|_{H_0^1}^\sigma,$$

with $c_1 = \alpha a$ and $c_2 = \beta b$. Since $\sigma < 2$ and $\frac{H-\mu}{2H} > 0$, then:

$$\lim_{\|u\|_{H_0^1} \to +\infty} \frac{H-\mu}{2H}\|u\|_{H_0^1}^2 - c_1\|u\|_{H_0^1} - c_2\|u\|_{H_0^1}^\sigma = +\infty,$$

therefore $\lim_{\|u\|_{H_0^1} \to +\infty} J(u) = +\infty$, J is coercive.

6. Since J is differentiable, coercive, and J' is strictly monotone (see Proposition 3.3), we deduce that J admits a unique critical point in $H_0^1(\Omega)$. Consequently, problem (P) admits a unique weak solution in $H_0^1(\Omega)$.