2. Given $v \in H_0^1(\Omega)$ with $v \neq 0$, show that there exists $\rho > 0$ such that $\psi(\rho_0 v) \leq 0$.

- Show that the functional ψ satisfies the Palais-Smale condition.
- Deduce that Problem (4.37) has a nontrivial weak solution.

Exercise 2:

Let $\lambda > 0$. Show that the following problem:

$$\begin{cases} -u'' + \lambda u = u^3, & t \in (0, \pi); \\ u'(0) = u'(\pi) = 0. \end{cases}$$

has a solution $u \in C^2[0, \pi]$ which is a mountain-pass critical point of the corresponding functional.

Exercise 3:

Verify that the "Palais-Smale condition" implies the "Cerami condition" where the Cerami condition is the following compactness condition:

If (u_n) is such that $J(u_n) \to c$ and $(1 + \|u_n\|)\|J'(u_n)\| \to 0$, then (u_n) has a convergent subsequence.

Exercise 4:

Consider the Dirichlet problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (D)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, with $f(0) = 0$, $f(x, s) = o(|s|)$ as $s \to 0$ (uniformly for $x \in \Omega$), f satisfying the growth condition and

$$\liminf_{|s| \to \infty} \frac{f(x, s)}{s} > \lambda_1, \quad \text{uniformly for } x \in \Omega.$$

Moreover, assume that

$$\lim_{|s| \to \infty} [sf(x, s) - 2F(x, s)] = +\infty, \quad \text{uniformly for } x \in \Omega.$$

where, as usual $F(x, s) = \int_0^s f(x, t)\, dt$.

1. Show that (D) has a nontrivial solution. *[Hint: Use the Fatou lemma to verify that, in view of the above condition, the pertinent functional satisfies the Cerami condition which introduced in Exercise 2.*
2. We take $f(s) := F'(s)$, where $F(s) = s^2 \ln(1 + s^2)$. Check that

$$\lim_{|s| \to \infty} [sf(s) - 2F(s)] = +\infty.$$

Exercise 5:

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded, smooth domain and consider the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda_1 u + g(u) - h(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \qquad (4.38)$$

where $h : \overline{\Omega} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions with g increasing and such that

$$g(-\infty) < g(s) < g(+\infty) \quad \forall s \in \mathbb{R},$$

where $g(\pm\infty) := \lim_{s \to \pm\infty} g(s)$.

1. Show that, if $\phi_1 > 0$ is a λ_1-eigenfunction and the following condition

$$g(-\infty) \int_{\Omega} \phi_1 \, dx < \int_{\Omega} h(x)\phi_1 \, dx < g(+\infty) \int_{\Omega} \phi_1 \, dx \tag{4.39}$$

   is satisfied, then Problem (4.38) has a solution.

2. State a corresponding Neumann problem where existence of a solution can be proved in a manner similarly to the one used in the previous part.