Chapter 4

Min-Max Methods

4.1 Mountain Pass Theorem

4.1.1 Min-max principle

Lemma 4.1 *Let $(E, \|\cdot\|)$ a Banach space and be $J : E \to \mathbb{R}$ a function in $\mathcal{C}^1$*

- *if there exists a sequence of $(P.S)$ forJ and J satisfied $(P.S)$ then J admits a critical point.*
- *if there exists a sequence of $(P.S)$ at level c and J satisfied $(P.S)$ then J admits a critical point at level c.*

Definition 4.1 *Let E be a Banach space. and let $J : E \to \mathbb{R}$ be a functional. A family of subset E noted Γ is called class of Min-max value:*

$$c = \inf_{A \in \Gamma} \sup_{u \in A} J(u)$$

is called the value of Min-max associated with Γ.

The basic idea behind so-called Min-max method is to find a critical value of a functional $\varphi \in C^1(E, \mathbb{R})$ as a Min-max value $c \in \mathbb{R}$ of φ over a suitable class of subsets of E which can be chosen to be invariant under the deformation $\tau(t, \cdot)$ given in the deformation theorem (see for example Theorem 3.2.3 of [5]).

we present an illustration of the Min-max method which has proven to be a powerful tool in the attack of many problems on differential equations. It is the celebrated mountain-pass theorem of Ambrosetti and Rabinowitz [2].

Theorem 4.1 (Mountain-pass theorem) *Let E be a Banach space, $J \in \mathcal{C}^1(E, \mathbb{R})$ satisfying the Palais-Smale condition. We assume that $J(0) = 0$ and that :*

(1) *there are $\rho > 0$ and $\alpha > 0$ such that $\|u\|_E = \rho$, then $J(u) \geq \alpha$*

(2) *there exists $v \in E$, such that $\|v\|_E > \rho$ and $J(v) < \alpha$.*

Then J has a critical value c such that $c \geq \alpha$. More precisely, if we put

$$\Gamma := \{\gamma([0,1]); \gamma \in C([0,1], E), \gamma(0) = 0, \gamma(1) = v\},$$

And

$$c := \inf_{A \in \Gamma} \max_{u \in A} J(v),$$

Then it is a critical value of J, and $c \geq \alpha$.

4.1.2 Application: Solutions for an elliptic problem involving super-linear term

We consider the following problem

$$\begin{cases} -\Delta u + q(x)u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $q \in L^\infty(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let us define:

$$F(t) = \int_0^t f(s)\ ds.$$

We consider the the following hypotheses

- (f_1) $\exists\ p \in (2, 2^*)$, $\exists\ a, b > 0 : |f(t)| \leq a + b\,|t|^{p-1} \quad \forall t \in \mathbb{R}$.
- (f_2) f is super-linear at zero, *i.e.*
$$\lim_{t \to 0} \frac{f(t)}{t} = 0$$
- (f_3) $\exists\ M > 0, \exists \mu > 2 : f(t)t \geq \mu F(t) \quad \forall\ |t| \geq M$.
- (f_4) $\exists\ t_0 \in \mathbb{R}, |t_0| \geq M : F(t_0) > 0$.

Definition 4.2 *We say that $u \in H_0^1(\Omega)$ is a weak solution of Problem (4.1), if and only if*

$$\int_\Omega \nabla u \nabla v\ dx + \int_\Omega q(x)v\ dx - \int_\Omega f(u)v\ dx = 0.$$

for any $v \in H_0^1(\Omega)$.

We consider the functional $J : H_0^1(\Omega) \to \mathbb{R}$, given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\ dx - \frac{1}{2}\int_\Omega q(x)u^2\ dx - \int_\Omega F(u)\ dx.$$

Remark 4.1 *If the function f satisfies hypothesis (f_1) and (f_2), then*

$$\forall \epsilon > 0, \exists c_\epsilon > 0 : F(t) \leq \epsilon t^2 + c_\epsilon\,|t|^p, \forall t \in \mathbb{R}. \tag{4.2}$$

Let us check it out. By definition, (f_2) means that

$$\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall t \in \mathbb{R} : |t| \leq \delta_\epsilon \Rightarrow \frac{|f(t)|}{|t|} \leq \epsilon,$$

then,

$$|F(t)| \leq \frac{\epsilon}{2} |t|^2, \quad \forall t \in [-\delta_\epsilon, \delta_\epsilon]. \tag{4.3}$$

From (f_1) we get

$$\frac{|f(t)|}{|t|^{p-1}} \leq \frac{a}{|t|^{p-1}} + b \text{ if } |t| > \delta_\epsilon.$$

If $|t| \geq \delta_\epsilon$ we have,

$$\frac{1}{|t|^{p-1}} \leq \frac{1}{\delta_\epsilon^{p-1}}.$$

So, we deduce that

$$\frac{|f(t)|}{|t|^{p-1}} \leq \frac{a}{\delta_\epsilon^{p-1}} + b,$$

which implies that

$$|f(t)| \leq \left(\frac{a}{\delta_\epsilon^{p-1}} + b\right) |t|^{p-1}$$

Integrating both sides we get

$$|F(t)| \leq \frac{1}{p}\left(\frac{a}{\delta_\epsilon^{p-1}} + b\right) |t|^p,$$

we choose $c_\epsilon = \frac{1}{p}\left(\frac{a}{\delta_\epsilon^{p-1}} + b\right) > 0$, then

$$|F(t)| \leq c_\epsilon |t|^p, \quad |t| > \delta_\epsilon \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$|F(t)| \leq \frac{\epsilon}{2} t^2 + c_\epsilon |t|^p, \quad \forall t \in \mathbb{R}. \tag{F_1}$$

This ends the justification.

The main result is:

Theorem 4.2 *Let the hypotheses $(f_1), (f_2), (f_3)$ and (f_4) hold true. Then, Problem (4.1) admits at least non trivial weak solution.*

The proof of Theorem 4.2 is based on the application of Mountain-Pass Theorem. We follow the following steps:

Step 1: We have

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{2}\int_\Omega q(x) u^2 \, dx - \int_\Omega F(u) \, dx.$$

From (f_1) we know that the nonlinear term $u \mapsto F(u)$ is differentiable, which implies that J is differentiable and of class $\mathcal{C}^1$ on $H_0^1(\Omega)$. In addition we have

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \nabla v \, dx = \int_\Omega q(x) v \, dx + \int_\Omega f(u) v \, dx.$$

Hence, the weak solutions of Problem (4.1) are exactly the critical points of the functional J. On the other hand, notice that $J(0) = 0$.

Step 2: The functional J verifies the $(P.S)$ condition. To make this step easier to understand, we formulate it as an exercise.

Exercice 4.3 *Suppose that $\{u_n\} \subset H_0^1(\Omega)$ is Palais-Smale Sequence .*

1. *Shows that*
$$\exists c > 0 : \quad J(u_n) - \frac{1}{\mu} \langle J'(u_n), u_n \rangle \leq c(1 + \|u_n\|_{H_0^1}).$$

2. (a) *Using the hypothesis(f_3)) show that there exists $c_1 > 0$ such that*
$$\int_\Omega (\frac{1}{\mu} f(u_n)u_n - F(u_n)) \, dx \geq -c_1.$$
   (b) *By calculating*
$$J(u_n) - \frac{1}{\mu} \langle J'(u_n), u_n \rangle,$$
   show that
$$c(1 + \|u_n\|_{H_0^1}) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|_{H_0^1}^2 - c_1.$$

3. (a) *Using question $(2-b)$, verify that $\{u_n\}$ is bounded sequence in H_0^1.*
   (b) *Deduce that there exists a subsequence $\{u_{nk}\}$ noted $\{u_n\}$ such that:*
$$\begin{cases} u_n \rightharpoonup u & in \quad H_0^1(\Omega), \\ u_n \to u & in \quad L^p(\Omega), \forall p \in [2, 2^*] \\ u_n \to u & a.e. \quad x \in \Omega. \end{cases}$$

4. *Show that*
$$\int_\Omega f(u_n)u_n \, dx \to \int_\Omega f(u)u \, dx$$
   and,
$$\int_\Omega f(u_n)u \, dx \to \int_\Omega f(u)u \, dx \tag{4.5}$$

5. *Justify that:*
$$\langle J'(u_n), u_n - u \rangle \to 0, \quad as \;\; n \to \infty,$$
   and,
$$\langle J'(u), u_n - u \rangle \to 0, \quad as \;\; n \to \infty.$$

6. (a) *Calculate the quantity $\langle J'(u_n) - J'(u), u_n - u \rangle$. Then conclude that $u_n \to u$ in $H_0^1(\Omega)$.*
   (b) *What can we deduce?*

Solution:

1. Let $\{u_n\}$ be a (P.S) Sequence; that is:

$$\begin{cases} |J(u_n)| \leq c_1 \\ J'(u_n) \to 0 \quad \text{in } \left(H_0^1(\Omega)\right)' \end{cases} \tag{4.6}$$

By the definition of duality's norm, we have

$$|\langle J'(u_n), u_n\rangle| \leq \|J'(u_n)\|_{\left(H_0^1\right)'} \|u_n\|_{H_0^1}$$

Using (4.6), there exists $c_2 > 0$ such that:

$$\|J'(u_n)\|_{\left(H_0^1\right)'} \leq c_2$$

consequently,

$$\begin{aligned} J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle &\leq \left| J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \right| \\ &\leq |J(u_n)| + \frac{1}{\mu}|\langle J'(u_n), u_n\rangle| \\ &\leq c_1 + \frac{c_2}{\mu}\|u_n\|_{H_0^1} \\ &\leq \max(c_1, \frac{c_2}{\mu})\|u_n\|_{H_0^1} \\ &\leq c(1 + \|u_n\|_{H_0^1}). \end{aligned} \tag{4.7}$$

$2-(a)$ We write Ω as follow

$$\Omega = \{x \in \Omega : |u_n| > M\} \cup \{x \in \Omega : |u_n| \leq M\}$$

then, by (f_3) we can see that

$$\int_{|u_n|>M} \frac{1}{\mu} f(u_n)u_n - F(u_n)\ dx \geq 0.$$

Therefore, we get

$$\begin{aligned} \int_\Omega \frac{1}{\mu} f(u_n)u_n - F(u_n)\ dx &= \int_{|u_n|>M} \frac{1}{\mu} f(u_n)u_n - F(u_n)\ dx \\ &\quad + \int_{|u_n|\leq M} \frac{1}{\mu} f(u_n)u_n - F(u_n)\ dx \\ &\geq \int_{|u_n|\leq M} \frac{1}{\mu} f(u_n)u_n - F(u_n)\ dx. \end{aligned}$$

Using (f_1), In the set $\{|u_n| > M\}$ we have the estimate:

$$|f(u_n)u_n| \leq (a + bM^{p-1})M \tag{4.8}$$

which implies:

$$f(u_n)u_n \geq -(a + bM^{p-1})M.$$

On the other hand,

$$F(u_n) \leq a|u_n| + \frac{b}{p}|u_n|^p \leq aM + \frac{b}{p}M^p. \tag{4.9}$$

So, from (4.8) and (4.9), we deduce that

$$\begin{aligned}\int_\Omega \frac{1}{\mu} f(u_n)u_n - F(u_n)\, dx &\geq -\left[(a+bM^{p-1})M + aM + \frac{b}{p}M^p\right] \\ &\geq -c_1 \qquad (4.10)\end{aligned}$$

where, $c_1 = 2aM + \frac{b}{p}M^p > 0$.

(b) By a simple calculation and using (4.10) we have:

$$\begin{aligned}J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2_{H^1_0} + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega q(x)u_n^2\, dx + \int_\Omega \frac{1}{\mu}f(u_n)u_n - F(u_n)\, dx \\ &\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2_{H^1_0} - c_1.\end{aligned}$$

since $\left(\frac{1}{2}-\frac{1}{\mu}\right)$ as $\mu > 2$ and using (4.7), we get

$$c\left(1+\|u_n\|_{H^1_0}\right) \geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2_{H^1_0} - c_1. \qquad (4.11)$$

2.

(a) Arguing by absurd we show that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Firstly by (4.11) we have:

$$\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2_{H^1_0} - c\|u_n\|_{H^1_0} \leq c + c_1 \qquad (4.12)$$

Supposes that $\{u_n\}$ is not bounded in $H^1_0(\Omega)$ which means that:

$$\lim_{n\to\infty}\|u_n\| = +\infty.$$

hence,

$$\lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2_{H^1_0} - c\|u_n\|_{H^1_0}\right\} = +\infty \leq c$$

This is a clear contradiction. Consequently $\{u_n\}$ is bounded in $H^1_0(\Omega)$.

(b) The sequence $\{u_n\}$ is bounded in $H^1_0(\Omega)$ which is reflexive space, then there exist a subsequence $\{u_{nk}\}$ (still denoted by $\{u_n\}$) such that

$$u_n \rightharpoonup u \ \ in \ \ H^1_0(\Omega). \qquad (4.13)$$

Thanks to the compactly embedding of $H^1_0(\Omega)$ into $L^p(\Omega)$, we have

$$u_j \to u \quad in \quad L^p(\Omega), \quad \forall p \in [2, 2^*).$$

By the converse of Lebesgue's Dominated Convergence Theorem, There exist a subsequence denoted by $\{u_n\}$ and a function $h \in L^p$, such that

$$u_n(x) \to u(x) \quad \text{a.e.} \ \ in \ \ \Omega, asn \to \infty, \qquad (4.14)$$

and,

$$|u_n(x)| \leq h(x) \quad \text{a.e.} \ \ in \ \ \Omega, \qquad (4.15)$$

3. We show that
$$\int_\Omega f(u_n)u_n \, dx \to \int_\Omega f(u)u \, dx.$$
From (4.14) and the continuity of f, we have
$$f(u_n) \to f(u) \text{ a.e. } in \ \ \Omega,$$
then,
$$f(u_n)u_n \to f(u)u\text{a.e. } in \ \ \Omega,$$
from (f_1), we get
$$\begin{aligned} |f(u_n)u_n| &\leq (a + b\,|u_n(x)|^{p-1})\,|u_n(x)| \\ &\leq (ah + b\ h^p) \in L^1(\Omega). \end{aligned}$$
by Dominated Convergence Theorem, we obtain
$$\int_\Omega f(u_n)u_n \, dx \to \int_\Omega f(u)u \, dx.$$
In the same way, we show that (4.5)is fulfilled.

4. Firstly we show that
$$\langle J'(u), u_n - u\rangle \to 0.$$
from (4.13) and the fact that $J'(u) \in (H_0^1(\Omega))'$ we have
$$\langle J'(u), u_n\rangle \to \langle J'(u), u\rangle\,.$$
Then, from (4.13) and (4.6) we have
$$\langle J'(u_n), u_n - u\rangle \to 0.$$

5.

(a) By a direct calculation we get:
$$\begin{aligned} \langle J'(u_n) - J'(u), u_n - u\rangle &= \langle J'(u_n), u_n - u\rangle - \langle J'(u), u_n - u\rangle \\ &= \|u_n - u\|^2_{H_0^1} + \int_\Omega q(x)(u_n - u)^2 \, dx - \int_\Omega (f(u_n) - f(u))(u_n - u) \, dx. \end{aligned}$$
We have proved that
$$\int_\Omega (f(u_n) - f(u))(u_n - u) \, dx \to 0. \tag{4.16}$$
Using the compactly embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ we get
$$\int_\Omega q(x)(u_n - u)^2 \, dx \leq \|q\|_{L^\infty}\|u_n - u\|_{L^2} \to 0, \tag{4.17}$$
as $n \to +\infty$. From (4.16), (4.17) and the fact that
$$\langle J'(u_n) - J'(u), u_n - u\rangle \to 0, \quad \text{as} \quad n \to \infty,$$
we conclude that
$$\|u_n - u\|_{H_0^1} \to 0,$$
as $n \to +\infty$.

(b) The functional J satisfies (P.S) condition.

This concludes the exercise solution.

Step 3: Now, the following result shows the first geometric conditions of Mountain Pass Theorem

Lemma 4.2 *Under the assumptions of Theorem 4.2, there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ for any $u \in H_0^1(\Omega)$, with $\|u\|_{H_0^1} = \rho$.*

Proof. Let $u \in H_0^1(\Omega)$, since $q(x) \geq 0$ and using (4.2) and the continuous embeddings of $H_0^1(\Omega)$ into $L^2(\Omega)$ and $L^p(\Omega)$, we get

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_{H_0^1}^2 + \frac{1}{2}\int_\Omega q(x)u^2\,dx - \int_\Omega F(u)\,dx \\ &\geq \frac{1}{2}\|u\|_{H_0^1}^2 - \epsilon\int_\Omega u^2\,dx - c_\epsilon\int_\Omega u^p\,dx \\ &\geq \frac{1}{2}\|u\|_{H_0^1}^2 - \epsilon c_1\|u\|_{H_0^1}^2 - c_\epsilon\,c_2\|u\|_{H_0^1}^p \\ &= \left(\frac{1}{2} - \epsilon\,c_1\right)\|u\|_{H_0^1}^2 - c_\epsilon\,c_2\|u\|_{H_0^1}^p \end{aligned}$$

taking $\epsilon = \frac{1}{4c_1}$

$$J(u) \geq \frac{1}{4}\|u\|_{H_0^1}^2 - c\,c_2\|u\|_{H_0^1}^p.$$

Since $p > 2$, then there exist $\rho > 0$ small enough such that

$$\frac{1}{4}\rho^2 - c\,c_2\rho^p > 0$$

we take

$$\alpha = \frac{1}{4}\rho^2 - c\,c_2\rho^p > 0$$

Consequently, there exist $\alpha, \rho > 0$ such that

$$\forall u \in H_0^1(\Omega) : \|u\|_{H_0^1} = \rho : J(u) \geq \frac{1}{4}\rho^2 - c\,c_2\rho^p = \alpha > 0.$$

■ In the follow, we need some preliminary results, which we present as an exercise.

Exercice 4.4 *1. Show that the hypotheses (f_3) implies that the function $t \to \frac{F(t)}{|t|^\mu}$ is increasing for $t \geq M$ and decreasing for $t \leq -M$.*

2. Assume that (f_3) and (f_4) are satisfied, Show that there exists $c > 0$ such that

$$F(t) \geq c\,|t|^\mu \quad \forall\, t \geq t_0 \quad \textit{if } t_0 > M \quad \textit{and} \quad \forall\, t \leq t_0 \quad \textit{if } t_0 < -M. \tag{F_2}$$

Solution

1. We consider the function defined by $g(t) = \frac{F(t)}{|t|^\mu}$, $|t| > M$, the derivative of g is

$$g'(t) = \frac{(f(t)t - \mu F(t))}{t^{\mu+1}}, \forall t > M.$$

Then, from (f_3) we have

$$\begin{cases} f(t)t - \mu F(t) \geq 0 \\ t^{\mu+1} > 0, & \forall t > M > 0, \end{cases}$$

then, $g'(t) > 0$ for any $t > M$ consequently g is increasing on $[M, +\infty[$.
we have $g(t) = \frac{F(t)}{(-t)^\mu}$ $\quad \forall t < -M$, then the derivative of g is

$$g'(t) = \frac{-(f(t)t + \mu F(t))}{(-t)^{\mu+1}}.$$

then $g'(t) < 0, \forall t < -M$ consequently g is decreasing for $t \in]-\infty, -M]$.

2. We distinguish two cases:

- if $t_0 > M$: Let $t \geq t_0$ we have $t \geq t_0 > M$, so $f(t)t \geq \mu F(t)$
$t \to \frac{F(t)}{t^\mu}$ is increasing for $[t_0, +\infty[$

$$\frac{F(t)}{t^\mu} > \frac{F(t_0)}{t_0^\mu} > 0 \quad \forall t \geq t_0$$

which means $F(t) > 0$ for any $t \geq t_0$. Then using (f_4), we get

$$\frac{f(t)}{F(t)} \geq \frac{\mu}{t}, \quad \forall t \geq t_0$$

by integrate the both sides, we obtain

$$\begin{aligned} \int_{t_0}^{t} \frac{f(s)}{F(s)}\, ds &\geq \mu \int_{t_0}^{t} \frac{1}{s}\, ds \\ \Rightarrow ln\left(\frac{F(t)}{F(t_0)}\right) &\geq ln\left(\frac{t}{t_0}\right)^\mu. \end{aligned}$$

then,

$$\frac{F(t)}{F(t_0)} \geq \frac{t^\mu}{t_0^\mu},$$

consequently,

$$F(t) \geq c\, t^\mu, \quad \forall t \geq t_0.$$

- If $t_0 < -M$: Let $t \leq t_0$, we have

$$\frac{F(t)}{t^\mu} > \frac{F(t_0)}{t_0^\mu} > 0 \quad \forall t \leq t_0$$

which means $F(t) > 0$, for any $t \leq t_0$. From (f_4), we get

$$\frac{f(t)}{F(t)} \leq \frac{\mu}{t}, \quad \forall t \leq t_0$$

by integrate, we get

$$\begin{aligned}\int_t^{t_0} \frac{f(s)}{F(s)}\,ds &\geq \mu \int_t^{t_0} \frac{1}{s}\,ds \\ \Rightarrow ln\left(\frac{F(t_0)}{F(t)}\right) &\geq ln\left(\frac{|t_0|}{|t|}\right)^{\mu}.\end{aligned}$$

then,

$$\frac{F(t_0)}{F(t)} \leq \frac{|t_0|^{\mu}}{|t|^{\mu}},$$

consequently,

$$F(t) \geq c\ |t|^{\mu}.$$

Next, we prove the second geometric condition of Mountain Pass Theorem.

Lemma 4.3 *Under the assumptions of Theorem 4.2, there exists* $v \in H_0^1(\Omega)$ *such that* $J(v) < 0$ *and* $\|v\|_{H_0^1} > \rho$.

We formulate the proof as an exercise.

Exercice 4.5 *Let* $\varphi \in \mathcal{D}(\Omega)$ *such that*

$$\begin{cases}\varphi(x) \geq 0 \quad \forall\ x \in \Omega, \\ \varphi(x) \geq 1 \quad \forall\ x \in B(x_0, r) \quad \textit{for certain} \quad x_0 \in \Omega \quad \textit{and} \quad r > 0,\end{cases}$$

1. *Show that for all* $t \geq 0$, *we have:*

(a) $\exists c > 0,\ F(t\varphi(x)) \geq ct^{\mu}\varphi^{\mu}(x) \quad \forall t \geq t_0$ *and* $\forall x \in B(x_0, r)$.

(b) $\exists c_1 > 0\ |F(t\varphi)| \leq c_1$ *if* $0 \leq t\varphi(x) \leq t_0$

2. *Writing:*

$$\begin{aligned}\Omega &= \{x \in \Omega : 0 \leq t\ \varphi(x) \leq t_0\} \cup \{x \in \Omega : t\ \varphi(x) > t_0\} \\ &= \{0 \leq t\ \varphi \leq t_0\} \cup \{t\ \varphi > t_0\}\end{aligned}$$

show that

$$\int_{\Omega} F(t\varphi)\ dx \geq -c_2 + c_3 t^{\mu}$$

3. *Deduce that*

$$J(t\ \varphi) \leq c_4\ t^2 - c_3\ t^{\mu} + c_2.$$

4. *Deduce the result of the Lemme 4.3 .*

1. Show that :

(a) Let $t \geq t_0$ and let $x \in B(x_0, r)$, we have

$$t\varphi(x) \geq t \geq t_0,$$

Then, from (F_2) we get

$$F(t\varphi(x)) \geq c\ t^{\mu}\varphi^{\mu}(x).$$

(b) $\forall\ t > 0$ such that $t\ \varphi(x) \in [0, t_0]$:$|F(t\ \varphi(x))| \leq c_1$
For φ is fixed, the function: $F(t\ \varphi(x))$ is continuous on compact $[0, t_0]$, then it is bounded i.e;

$$|F(t\varphi)| \leq c_1 \quad 0 \leq t\varphi(x) \leq t_0.$$

2. Writing,

$$\int_{\Omega} F(t\varphi)\ dx = \int_{\{0 \leq t\varphi \leq t_0\}} F(t\varphi)\ dx + \int_{\{t\ \varphi > t_0\}} F(t\varphi)\ dx$$

We set $\Omega_1 = \{0 \leq t\varphi \leq t_0\}$ and $\Omega_2 = \{t\varphi > t_0\}$.
We know that

$$\forall x \in \Omega_1 : -c_1 \leq F(t\ \varphi) \leq c_1$$

$$\forall x \in \Omega_2 : \quad |F(t\varphi)| \geq c\ t^{\mu}\varphi^{\mu}(x)$$

then,

$$\begin{aligned} \int_{\Omega} F(t\varphi)\ dx &\geq -\int_{\Omega_1} c_1\ dx + \int_{\Omega_2} c\ t^{\mu}\varphi^{\mu}(x)\ dx \\ &= -c_1\,|\Omega_1| + t^{\mu}\int_{\Omega_2} c\varphi^{\mu}\ dx \\ &= -c_2 + c_3 t^{\mu}. \end{aligned}$$

3. Using the precedent question we obtain,

$$\begin{aligned} J(t\varphi) &= \frac{1}{2}t^2\int_{\Omega} |\nabla\varphi|^2\ dx + \frac{1}{2}t^2\int_{\Omega} q(x)\varphi^2\ dx - \int_{\Omega} F(t\varphi)\ dx \\ &\leq \frac{1}{2}t^2\,\|\varphi\|^2_{H^1_0} + \frac{1}{2}t^2\,\|q\|_{L^\infty} C_\Omega\,\|\varphi\|^2_{H^1_0} + c_2 - c_3 t^{\mu} \\ &= \left(\frac{1}{2}\,\|\varphi\|^2_{H^1_0} + \frac{1}{2}\,\|q\|_{L^\infty} C_\Omega\,\|\varphi\|^2_{H^1_0}\right) t^2 + c_2 - c_3 t^{\mu} \\ &= c_4 t^2 - c_3 t^{\mu} + c_2. \end{aligned}$$

4. Since $\mu > 2$, Then $\lim\limits_{t\to+\infty} J(t\varphi) = -\infty$, hence we choose $t > 0$ enough large such that $J(t\varphi) < 0$ and $\|t\varphi\|_{H^1_0} > \rho$ and we take $v = t\varphi \in H^1_0(\Omega)$, so $\|v\|_{H^1_0} > \rho$ and $J(v) < 0$. Then, we conclude that the second geometric condition is satisfied.

Proof of Theorem 4.2.
The functional J satisfies all conditions of Mountain Pass Theorem, then J admits at least non-trivial critical point, So the existence of a nontrivial weak solution of problem (4.1). ∎

4.2 Rabinowitz's Saddle-Point Theorem and application

In this section we present a second important illustration of the min-max method, the saddle-point theorem of Rabinowitz. The interested reader can see the details in [10].

Theorem 4.6 *Let E be Banach spaces of infinite-dimensional, and let $J \in \mathcal{C}^1(E, \mathbb{R})$ which satisfies the Palais-Smale Condition, suppose that:*

$$E = V \oplus W \qquad \text{with dim } V < +\infty,$$

If D is a is a neighborhood bounded by 0 in V such that

$$a = \max_{\partial D} J < \inf_{W} J = b. \tag{4.18}$$

Then,

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{D}} \varphi\Big(h(u)\Big),$$

$$(\Gamma = \{h \in C(\overline{D}, X), h(u) = u, \forall u \in \partial\Omega\})$$

is a critical value of φ with $c \geq b$.

Remark 4.2 *It should be noted that the condition 4.18 is satisfied when we have*

- $\lim\limits_{\|v\| \to +\infty} J(v) = -\infty, \quad \forall v \in V.$
- $\lim\limits_{\|w\| \to +\infty} J(w) = +\infty, \quad \forall w \in W.$

Recall that for λ_k be the eigenvalues of the operator Δ and φ_k be the eigenfunctions associated, that is

$$\begin{cases} -\Delta\varphi_k = \lambda_k \varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

we know that $\{\varphi_k\}_{k \in \mathbb{N}^*}$ is a Hilbertian basis of $L^2(\Omega)$. Moreover,

1. $\forall u \in L^2(\Omega)$, $u = \sum\limits_{k=1}^{\infty} \alpha_k \varphi_k$ and $\|u\|_{L^2} = \sum\limits_{k=1}^{\infty} \alpha_k^2$, where $\alpha_k = \int_\Omega u\varphi_k \, dx$.

2. $\forall u \in H_0^1(\Omega)$, $u = \sum\limits_{k=1}^{\infty} \dfrac{\beta_k}{\sqrt{\lambda_k}} \varphi_k$ and $\|u\|_{H_0^1} = \sum\limits_{k=1}^{\infty} \beta_k^2$, where

$$\beta_k = \left\langle u, \frac{\varphi_k}{\sqrt{\lambda_k}} \right\rangle = \frac{1}{\sqrt{\lambda_k}} \int_\Omega \nabla u \nabla \varphi_k \, dx$$

   α_k and β_k are the Fourier coefficients of u belong to $L^2(\Omega)$ and $H_0^1(\Omega)$.

3. The relationship between α_k and β_k is

$$\beta_k = \sqrt{\lambda_k} \alpha_k. \tag{4.19}$$

Application

Let us consider the following problem:

$$\begin{cases} -\Delta u = \lambda u + f(u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.20}$$

where h is a function in $L^2(\Omega)$, and f satisfies the following hypotheses:

(f) Suppose that $f \in \mathcal{C}^1(\mathbb{R})$, bounded and there exists $c > 0$ such that:

$$|f'(t)| \leq c(1 + |t|^{2^*-2})$$

Definition 4.3 *We say that $u \in H_0^1(\Omega)$ is a weak solution of Problem (P_λ), if and only if*

$$\int_\Omega \nabla u \nabla v \; dx - \lambda \int_\Omega u \; v \; dx - \int_\Omega f(u) v \; dx - \int_\Omega h(x) v \; dx = 0.$$

for any $v \in H_0^1(\Omega)$.

We consider the functional $J : H_0^1(\Omega) \to \mathbb{R}$,

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \; dx - \frac{\lambda}{2} \int_\Omega u^2 \; dx - \int_\Omega F(u) \; dx - \int_\Omega h(x) u \; dx.$$

where

$$F(t) = \int_0^t f(s) \; ds$$

Let $0 < \lambda_1 < \lambda_2 \leq \ldots$ the sequence of eigenvalue of the operator $-\Delta$. Let $\{\varphi_1, \varphi_2 ...\}$ the eigenfunctions associated with $\{\lambda_1, \lambda_2, \ldots\}$, we consider

$$E = \text{vect}\{\varphi_1, \ldots, \varphi_v\} \text{ and } W = E^\perp = \overline{\text{vect}\{\varphi_k \backslash k \geq v+1\}}.$$

we have $H_0^1(\Omega) = E \oplus W$. We have the following result:

Lemma 4.4 *Let $\lambda \in \mathbb{R}$ such that $\lambda_v < \lambda < \lambda_{v+1}$. We have*

$$\|u\|_{L^2}^2 = \int_\Omega u^2 \; dx \geq \frac{1}{\lambda_v} \|u\|_{H_0^1}^2, \quad \forall u \in E.$$

and,

$$\|v\|_{L^2}^2 = \int_\Omega v^2 \; dx \geq \frac{1}{\lambda_{v+1}} \|v\|_{H_0^1}^2, \quad \forall v \in W.$$

Proof. Let $u \in E$, we have

$$\begin{aligned} \int_\Omega u^2 \; dx &= \sum_{k=1}^\infty \alpha_k^2 = \sum_{k=1}^\infty \left(\frac{\beta_k}{\sqrt{\lambda_k}} \right)^2 \\ &\geq \frac{1}{\lambda_v} \sum_{k=1}^\infty \beta_k^2 \\ &= \frac{1}{\lambda_v} \|u\|_{H_0^1}^2 . \end{aligned}$$

Let $v \in W = E^{\perp}$, we have

$$\begin{aligned}
\int_{\Omega} v^2\ dx &= \sum_{k=v+1}^{\infty} \alpha_k^2 = \sum_{k=v+1}^{\infty} \left(\frac{\beta_k}{\sqrt{\lambda_k}}\right)^2 = \sum_{k=v+1}^{\infty} \frac{\beta_k^2}{\lambda_k} \\
&\leq \frac{1}{\lambda_{v+1}} \sum_{k=v+1}^{\infty} \beta_k^2 \\
&= \frac{1}{\lambda_{v+1}} \|v\|_{H_0^1}^2 .
\end{aligned}$$

■

The main result is:

Theorem 4.7 *Let $\lambda > \lambda_1$ and $\lambda \neq \lambda_k$, for any $k \in \mathbb{N}^*$. Assume that (f) holds true. Then, Problem 4.20 admits at least a weak solution in $H_0^1(\Omega)$.*

We need some lemmas.

Lemma 4.5 *Under assumptions of Theorem 4.7, the functional J satisfies the Palais-Smale Condition.*

Proof. Let $n \in \mathbb{N}^*$, and let λ such that $\lambda_n < \lambda < \lambda_{n+1}$. Let $\{u_n\}$ is a Palais-Smale Sequence i.e:

$$\begin{cases} |J(u_n)| \leq c, \\ J'(u_n) \to 0 \quad \text{in } \left(H_0^1(\Omega)\right)' . \end{cases} \tag{4.21}$$

Step 1: The sequence (u_n) is bounded. We reformulate this step as an exercise:

Exercice 4.8 *By contradiction, we supposes that $\{u_n\}$ is not bounded in $H_0^1(\Omega)$ which means that*

$$\lim_{n\to\infty} \|u_n\| = +\infty, \tag{4.22}$$

as $n \to +\infty$. We put $\psi_n = \frac{u_n}{\|u_n\|}$.

1. *Show that for any $v \in H_0^1(\Omega)$ with $\|v\| = 1$, we have*

$$\int_{\Omega} \nabla\psi_k \nabla v\ dx - \lambda \int_{\Omega} \psi_k v\ dx - \frac{1}{\|u_k\|} \int_{\Omega} f(u_k)v - \frac{1}{\|u_k\|} \int_{\Omega} hv\ dx = 0. \tag{4.23}$$

2. *From the boundedness of Ψ_n show that there exists $\psi \in H_0^1(\Omega)$ such that*

$$\int_{\Omega} \nabla\psi \nabla v\ dx - \lambda \int_{\Omega} \psi v\ dx = 0,$$

   for any $v \in H_0^1(\Omega)$.

3. *We want to show that $\Psi \neq 0$.*

(a) *For any $n \in \mathbb{N}$, verify that*

$$\int_{\Omega} \nabla u_n \nabla\psi_n\ dx - \lambda \int_{\Omega} u_n\psi_n\ dx - \int_{\Omega} f(u_n)\psi_n - \int_{\Omega} h\psi_n\ dx = 0.$$

(b) *Deduce that*

$$\|\psi_n\|^2_{H_0^1} - \lambda \|\psi_n\|^2_{L^2} \to 0,$$

as $n \to +\infty$.

(c) *Justified why* $\|\psi_n\|_{H_0^1} \to 0,$ *as* $n \to +\infty$, *and deduce that* $\Psi \neq 0$.

4. *Show that* λ *is an eigenvalue of the operator* Δ *and deduce that* $\{u_n\}$ *is bounded in* $H_0^1(\Omega)$.

Solution:

1. Let $n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{\|u_n\|}\langle J'(u_n), v\rangle &= \int_\Omega \frac{\nabla u_n}{\|u_n\|}\nabla v \; dx - \lambda \int_\Omega \frac{u_n}{\|u_n\|} v \; dx - \frac{1}{\|u_n\|}\int_\Omega f(u_n) v \; dx - \frac{1}{\|u_n\|}\int_\Omega hv \; dx \\ &= \int_\Omega \nabla\psi_n \nabla v \; dx - \lambda \int_\Omega \psi_n v \; dx - \frac{1}{\|u_n\|}\int_\Omega f(u_n) v \; dx - \frac{1}{\|u_n\|}\int_\Omega hv \; dx,\end{aligned}$$

From (4.21), we deduce that

$$\langle J'(u_n), v\rangle \to 0, \forall v \in H_0^1(\Omega) : \|v\| = 1,$$

hence, by (4.22) we have

$$\frac{1}{\|u_n\|}\langle J'(u_n), v\rangle \to 0,$$

as $n \to +\infty$. Since $\|\psi_n\|_{H_0^1} = 1$, the sequence ψ_n is bounded in $H_0^1(\Omega)$, so there exist a subsequence $\{\psi_{n_k}\}$ denoted by $\{\psi_n\}$ such that

$$\psi_n \rightharpoonup \psi \text{ in } H_0^1(\Omega), \tag{4.24}$$

2. which means that

$$\langle \psi_n, v\rangle \to \langle \psi, v\rangle .$$

In particular, we have

$$\int_\Omega \nabla\psi_n \nabla v \; dx \to \int_\Omega \nabla\psi \nabla v \; dx.$$

Moreover, using (4.24) and the compactly embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ we obtain that

$$\Psi_n \to \Psi \text{ in } L^2(\Omega).$$

By Cauchy Schwartz inequality, we have

$$\begin{aligned}\left|\int_\Omega \Psi_n v \; dx - \int_\Omega \Psi v \; dx\right| &= \left|\int_\Omega (\Psi_n - \Psi) v \; dx\right| \\ &\leq \|\Psi_n - \Psi\|_{L^2} \|v\|_{L^2} .\end{aligned}$$

then,

$$\lim_{n\to+\infty} \int_\Omega \Psi_n v \; dx = \int_\Omega \Psi v \; dx$$

On the other hand, since f is bounded, there exists $c_0 > 0$ such that

$$\left|\frac{1}{\|u_n\|}\int_\Omega f(u_n) v \; dx\right| \leq \frac{c}{\|u_n\|}\int_\Omega v \; dx$$

then, because (4.22) holds true we deduce that

$$\lim_{n\to+\infty}\frac{1}{\|u_n\|}\int_\Omega f(u_n)v\ dx=0.$$

Obviously,

$$\lim_{n\to+\infty}\frac{1}{\|u_n\|}\int_\Omega hv\ dx=0.$$

Therefore, the uniqueness of the limit implies that:

$$\int_\Omega \nabla\psi\nabla v\ dx-\lambda\int_\Omega \psi v\ dx=0,\quad \forall v\in H_0^1(\Omega). \tag{4.25}$$

3. We show that $\psi\neq 0$. By contradiction, we suppose that $\psi=0$,

(a) It is enough to take $v=\psi_n\in H_0^1(\Omega)$ in (4.23).

(b) By dividing both sides of $(3-a)$ over $\|u_n\|$, we get

$$\int_\Omega \nabla\left(\frac{u_n}{\|u_n\|}\right)\nabla\psi_n\ dx-\lambda\int_\Omega \frac{u_n}{\|u_n\|}\psi_n\ dx-\frac{1}{\|u_n\|}\int_\Omega f(u_n)\psi_n-\frac{1}{\|u_n\|}\int_\Omega h\psi_n\ dx\to 0,$$

as $n\to+\infty$. Then we deduce that

$$\|\psi_n\|^2_{H_0^1}-\lambda\|\psi_n\|^2_{L^2}\to 0,$$

as $n\to+\infty$.

(c) Since$\psi_n\to\psi=0$ in $L^2(\Omega)$ which means,

$$\|\psi_n\|_{L^2}\to 0,\ \text{as } n\to+\infty.$$

Then, by $(3-b)$ we deduce that

$$\|\psi_n\|_{H_0^1}\to 0,\ \text{as } n\to+\infty,$$

But we know that

$$\|\psi_n\|=\frac{\|u_n\|}{\|u_n\|}=1,$$

which leads to a contradiction, so $\psi\neq 0$.

4. From (4.25), we see that ψ is a weak solution of the following problem,

$$\begin{cases}-\Delta\psi=\lambda\psi & \text{in } \Omega,\\ \psi=0 & \text{on } \partial\Omega,\end{cases} \tag{4.26}$$

Since Ψ and it is a nontrivial weak solution of Problem 4.26, we deduce that λ is an eigenvalue of $(-\Delta)$, but we have assumed that $\lambda\neq\lambda_k$ so there is a contradiction, then we deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Which prove the first Step of the Palais-Smale Condition.

Step 2: Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, there exist a subsequence $\{u_{n_k}\}$ still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \ \ in \ \ H_0^1(\Omega), \tag{4.27}$$

Then, thanks to the compactly embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ as $2 \in [1, 2^*)$, we have

$$\|u_n - u\|_{L^2} \to 0, \tag{4.28}$$

as $n \to +\infty$. By (4.21), we can verify that

$$\langle J'(u_n) - J'(u), u_n - u\rangle \to 0, \text{ as } n \to +\infty. \tag{4.29}$$

On other hand, by simple calculating we get,

$$\begin{aligned} \langle J'(u_n) - J'(u), u_n - u\rangle &= \langle J'(u_n), u_n - u\rangle - \langle J'(u), u_n - u\rangle \\ &= \|u_n - u\|_{H_0^1}^2 - \lambda \|u_n - u\|_{L^2}^2 - \int_\Omega (f(u_n) - f(u))(u_n - u) \ dx, \end{aligned} \tag{4.30}$$

From (4.28) we obtain

$$\|u_n - u\|_{L^1} \to 0, \text{as} n \to +\infty.$$

and,

$$\begin{aligned} \left|\int_\Omega (f(u_n) - f(u))(u_n - u) \ dx\right| &\leq \int_\Omega (|f(u_n)| + |f(u)|)|u_n - u| \ dx \\ &\leq C\|u_n - u\|_{L^1} \to 0. \end{aligned}$$

consequently,

$$\int_\Omega (f(u_n) - f(u))(u_n - u) \ dx \to 0, \tag{4.31}$$

as $n \to +\infty$. Combining the relations (4.28), (4.29), (4.30) and (4.31), we get

$$\|u_n - u\|_{H_0^1} \to 0, \text{ as } n \to +\infty.$$

We conclude that the functional J satisfies Palais-Smale condition $(P.S)_c$. ■

Now, we prove the geometric conditions of Saddle Point Theorem. Let φ_k be the eigenfunctions, We consider

$$E_n = \text{ vect } \{\varphi_1, \varphi_2, \dots, \varphi_n\}$$

and,

$$W = E_n^\perp = \overline{\text{ vect } \{\varphi_k \backslash k \geq n+1\}}$$

Recall $H_0^1(\Omega) = E_n \oplus W$, and by Lemma 4.4, we have

$$\|u\|_{H_0^1}^2 \leq \lambda_n \|u\|_{L^2}^2, \quad \forall u \in E_n. \tag{4.32}$$

and,

$$\|u\|_{H_0^1}^2 \leq \lambda_{n+1} \|u\|_{L^2}^2, \quad \forall u \in W. \tag{4.33}$$

Then, we have.

Lemma 4.6 *psps1,condgeops We have the following results:*

1. $\lim_{\|u\|\to\infty} J(u) = -\infty$, $\forall u \in E_n$.

2. $\lim_{\|w\|\to\infty} J(w) = +\infty$, $\forall w \in W$.

Proof. Let $u \in E_n$, we have The boundedness of f implies that,

$$|F(t)| \le c_0 |t|, \quad \forall t \in \mathbb{R}$$

then,

$$\int_\Omega F(u)\ dx \ge -c_0 \int_\Omega |u|\ \ dx$$

By using the continuous embedding of $H_0^1(\Omega)$ in $L^1(\Omega)$, we have

$$\int_\Omega F(u)\ dx \ge -c_0 c_1 \|u\|_{H_0^1}.$$

Next, by applying Cauchy-Schwartz inequality and Poincaré inequality we get

$$\left|\int_\Omega h\ u\ dx\right| \le C_\Omega \|h\|_{L^2} \|u\|_{L^2}$$

Thus,

$$\int_\Omega h\ u\ dx \ge -C_\Omega \|h\|_{L^2} \|u\|_{L^2}$$

Consequently,

$$\int_\Omega F(u)\ dx + \int_\Omega h\ u\ dx \ge -\left(c_0 c_1\ + C_\Omega \|h\|_{L^2}\right) \|u\|_{H_0^1}. \tag{4.34}$$

we set $c = c_0 c_1\ + C_\Omega \|h\|_{L^2}$. Then, by Lemma 4.4 and (4.34) we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{\lambda}{2}\|u\|_{L^2}^2 - \int_\Omega F(u)\ dx - \int_\Omega h(x) u\ dx. \\ &\le \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{\lambda}{2\lambda_n}\|u\|_{H_0^1}^2 + c\|u\|_{H_0^1} \\ &= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_n}\right)\|u\|_{H_0^1}^2 + c\|u\|_{H_0^1}, \end{aligned}$$

We have $\left(1 - \frac{\lambda}{\lambda_n}\right) < 0$ as $\lambda_n < \lambda < \lambda_{n+1}$, thus

$$\lim_{\|u\|\to+\infty} \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_n}\right)\|u\|_{H_0^1}^2 + c\|u\|_{H_0^1} = -\infty, \quad \forall u \in E_n.$$

This completes prove of the first relation.

Now, we prove of the second relation. Let $w \in W$. The boundedness of f implies that,

$$\int_\Omega F(w)\ dx \ge -c_0 \int_\Omega |w|\ \ dx,$$

for some $c_0 > 0$. By using the continuous embedding of $H_0^1(\Omega)$ into $L^1(\Omega)$, we have

$$\int_\Omega F(w)\ dx \ge -c_1 c_2 \|w\|_{H_0^1}. \tag{4.35}$$

for some $c_1 > 0$. Next, by applying Cauchy-Schwartz inequality and Poincaré inequality we get

$$\int_\Omega h\ w\ dx \ge -C_\Omega \|h\|_{L^2} \|w\|_{L^2} \tag{4.36}$$

Therefore, by Lemma 4.4 and the relations (4.35) and (4.36), we get

$$\begin{aligned} J(w) &= \frac{1}{2}\|w\|_{H_0^1}^2 - \frac{\lambda}{2}\|w\|_{L^2}^2 - \int_\Omega F(w)\,dx - \int_\Omega h(x)w\,dx \\ &\geq \frac{1}{2}\|w\|_{H_0^1}^2 - \frac{\lambda}{2}\|w\|_{L^2}^2 - c_1c_2\|w\|_{H_0^1} - C_\Omega\|h\|_{L^2}\|w\|_{L^2} \\ &\geq \frac{1}{2}\|w\|_{H_0^1}^2 - \frac{\lambda}{2\lambda_{n+1}}\|w\|_{H_0^1}^2 - c\|w\|_{H_0^1} \\ &= \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{n+1}}\right)\|w\|_{H_0^1}^2 - c\|w\|_{H_0^1}, \end{aligned}$$

We have $\left(1-\frac{\lambda}{\lambda_{n+1}}\right) > 0$ as $\lambda < \lambda_{n+1}$. Therefore,

$$\lim_{\|w\|\to+\infty} \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{n+1}}\right)\|w\|_{H_0^1}^2 - c\|w\|_{H_0^1} = +\infty.$$

The proof of the second relation above is completed. ■

We reformulate the first affirmation in Lemma 4.6 as follow.

Corollary 4.1 *We pose $\|u\| = R > 0$ i,e. $u \in \partial B(0,\mathbb{R})$ such that $B(0,\mathbb{R})$ is the ball in E_n and that means*

$$\max_{u\in\partial B(0,\mathbb{R})} J(u) \to -\infty, \text{ as } R \to +\infty$$

We reformulate the second affirmation in Lemma 4.6 as follow.

Corollary 4.2 *We have:*

$$\max_{u\in\partial B(0,R)} J(u) < \inf_{u\in W} J(u),$$

for all R sufficiently large.

Proof of Theorem 4.7. From Lemma 4.5 and Lemma 4.6, we deduce that all conditions of Saddle Point Theorem are satisfied, then the functional J admits at least critical point which means that Problem 4.20 admits at least one nontrivial weak solution in $H_0^1(\Omega)$. ■

4.3 Exercises

Exercise 1:

Let us show that the following nonlinear Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary possesses a classical nontrivial solution:

$$\begin{cases} -\Delta u = u^3, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.37}$$

We define the following functional: $\psi(u) = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 - \frac{1}{4}u^4\right] dx$.

- Show that ψ is well defined and of class C^1 on $H_0^1(\Omega)$.
- 1. Show that $u = 0$ is a strict local minimum of ψ.