(i) Show that the functional φ is bounded from below, hence (3.1) has a weak solution u_0 minimizing φ;

(ii) Show that u_0 is the unique solution of (3.1) [Hint: The function $s \mapsto s^3$ is increasing, i.e., $(s_1^3 - s_2^3)(s_1 - s_2) > 0 \, \forall \, s_1 \neq s_2$ in $\mathbb{R}$];

(iii) Exhibit a boundary value problem which is more general than (3.1) and where the existence of a unique solution (minimizing the corresponding functional) holds true;

(iv) Show that if we replace the nonlinear term u^3 by $-u^3$ in (3.1), then the corresponding functional φ is no longer bounded from below, so that we cannot apply the minimization principle.

Exercise 2:
Consider the following problem

$$\begin{cases} -\Delta u = \lambda \sin u + f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.64}$$

where $\lambda > 0$, $f \in L^2(\Omega)$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Show that, for $\lambda < \lambda_1$, Problem 3.64 has a weak solution minimizing the functional:

$$\varphi(u) = \int_\Omega \left[\frac{1}{2} |\nabla u|^2 + \lambda(\cos u - 1) - fu \right] dx, \quad u \in H_0^1(\Omega).$$

Exercise 3:
Consider the Dirichlet problem:

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) + c(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $c, f \in L^\infty(\Omega)$, and $A(x)$ is an $N \times N$ real symmetric matrix with components in $L^\infty(\Omega)$ which is uniformly positive definite, i.e., $A(x)\xi \cdot \xi \geq \delta |\xi|^2$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$ (and some $\delta > 0$).

(i) Show that the problem (3.3) is variational in the sense that its weak solutions are the critical points of a suitable functional $\varphi : H_0^1(\Omega) \to \mathbb{R}$.

(ii) Assume $\delta > \frac{\|c\|_{L^\infty}}{\lambda_1}$. Show that $\hat{u} \in H_0^1(\Omega)$ is a weak solution of (3.3) if and only if $\hat{u}$ is a global minimum of φ.

Exercise 4:
Let Ω be a bounded open subset of $\mathbb{R}^N$ and $p > 0$. Show that for $\lambda > 0$, the functional

$$J(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{\lambda}{p+1} \int_\Omega |v|^{p+1} \, dx$$

satisfies the Palais-Smale condition on $E := H_0^1(\Omega) \cap L^{p+1}(\Omega)$. Find all values of $q \geq 1$ such that for a fixed $f \in E'$, the functional

$$I(u) := J(u) - \frac{1}{q+1} \int_\Omega |u|^{q+1} \, dx - \langle f, u \rangle$$

satisfies the Palais-Smale condition on E.

Exercise 5:

Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 3$. Show that for $\lambda > 0$, the functional

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}\,dx$$

does not satisfy the Palais-Smale condition if $p := \frac{N+2}{N-2}$. When $(N-2)p > N+2$, does J satisfy the Palais-Smale condition on $E := H_0^1(\Omega) \cap L^{p+1}(\Omega)$?