Chapter 3

Direct Methods in the Calculus of Variations

3.1 Extremum and critical points

Definition 3.1 (Extremum points) *Let E be a Banach space, let I be a functional defined on E. A point u_0 is called an extremum of I if there exists a neighborhood $\mathcal{V}_{u_0}$ of u_0 such that*

$$I(u) \leq I(u_0), \forall u \in \mathcal{V}_{u_0};$$

In this case I is maximal at u_0 (or has a maximum at u_0).

Or

$$I(u) \geq I(u_0), \forall u \in \mathcal{V}_{u_0};$$

In this case I is minimal at u_0 (or has a minimum at u_0).

Definition 3.2 (Critical points) *Let $\mathcal{O}$ be an open set of Banach space E, assume that $I : \mathcal{O} \to \mathbb{R}$ is differentiable. We say that $u \in \mathcal{O}$ is a critical point of I, if*

$$I'(u) = 0.$$

Remark 3.1

1. *As $I'(u)$ is an element of the dual space E', then $I'(u) = 0$ means that $\langle I'(u), v\rangle$ for all $v \in E$.*
2. *If u is not critical point, then we say that u is regular point of I.*
3. *If $I(u) = c$ and $I'(u) = 0$ we say that u is a critical point for I at level c and we say that c is a critical level for I. If c is not a critical level, then c is called a regular level for I.*
4. *The equation $I'(u) = 0$ is called the Euler, or Euler-Lagrange equation associated to the functional I.*

The following theorem shows the relationship between extreme points and critical points.

Theorem 3.1 *Let I be a functional defined on a domain $\mathcal{O} \subset E$ and let u_0 be an interior of $\mathcal{O}$. Assume that I is Gâteaux differentiable at u_0. Then, If u_0 is an extremum of I, it is a critical point of I.*

Proof. Suppose that u_0 is a point of minimum of I; that is

$$I(u_0) = \min_{v \in E} I(v).$$

Let $v \in E$ arbitrary fixed and consider $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = I(u_0 + tv)$, for t in some neighborhood of 0, h is well defined since u_0 is an interior point in $\mathcal{O}$. Morover, h is differentiable function at 0 as I is Gâteaux differentiable at u_0:

$$h'(0) = \lim_{t \to 0} \frac{h(t) - h(0)}{t} = \lim_{t \to 0} \frac{I(u_0 + tv) - I(u_0)}{t} \langle I'_G(u_0), v \rangle.$$

Since u_0 is point of minimum of I, then 0 is a point minimum of h, that is $h'(0) = 0$. It follows that $\langle I'_G(u_0), v \rangle = 0$ for any $v \in E$ which means that u_0 is a critical point of I. ■

3.2 Weak solutions and critical points

The concept of a weak solution to a differential equation is not fixed and usually depends on what one is interested in, but here in this publication we will propose the most common definition through a simple typical differential problem.

3.2.1 Example of a linear problem

Let us consider the model problem

$$\begin{cases} -\Delta u + q(x)u = h(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

where Ω is an open of $\mathbb{R}^N$, $q \in L^\infty(\Omega)$ and $h \in L^2(\Omega)$.

Definition 3.3 *Let $q \in L^\infty(\Omega)$, $h \in L^2(\Omega)$. We say that $u \in H_0^1(\Omega)$ is a weak solution of (3.1) if and only if*

$$\int_\Omega \nabla u \nabla v \, dx + \int_\Omega q(x)uv \, dx - \int_\Omega h(x)v \, dx = 0,$$

for any $v \in H_0^1(\Omega)$.

We now try to give the relationship between the weak solution of Problem (3.1) and the critical point of a functional that we choose in an appropriate way. Define a functional $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega q(x)|u|^2 \, dx - \int_\Omega h(x)u \, dx. \tag{3.2}$$

This functional is called the energy functional associated of Problem (3.1). We have the following result.

Proposition 3.1 *the functional J defined in 3.2 is differentiable on $H_0^1(\Omega)$, , and*

$$\langle J'(u), v \rangle = \int_\Omega \nabla u \nabla v \, dx + \int_\Omega q(x)uv \, dx - \int_\Omega h(x)v \, dx. \tag{3.3}$$

Proof. The examples given in Chapter 2 can be followed. ∎

Now, we can clearly see the relationship between the weak solution of problem (3.1) and the critical points of functional J by a simple comparison between Definition 3.3 and (3.3). Indeed, we have

$$\begin{aligned}
(u \in H_0^1(\Omega)\text{is a critical point of} J) &\Leftrightarrow J'(u) = 0 \in H_0^1(\Omega) \\
&\Leftrightarrow \langle J'(u), v\rangle = 0, \quad \forall v \in H_0^1(\Omega) \\
&\Leftrightarrow \int_\Omega \nabla u \nabla v \, dx + \int_\Omega q(x)uv \, dx - \int_\Omega h(x)v \, dx = 0 \\
&\Leftrightarrow v \text{ is a weak solution of Problem (3.1).}
\end{aligned}$$

Remark 3.2
- *The correspondence between weak solutions and critical point of functionals outlined above is valid of course for more general nonlinear problems.*
- *The procedure to define the notion of weak solution for certain nonlinear equations follows exactly the same steps as in the linear case; one tests the equation with a smooth function vanishing on the boundary of Ω, integrates by parts by means of the Green's formula, reduces the regularity requirement on the functions, and tries to interpret the integral equality as the vanishing of the differential of a suitable functional.*

3.2.2 Example of a nonlinear problem

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open and bounded. Assume that $q \in L^\infty(\Omega)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies the **growth condition**

$$f(t) \leq a + b|t|^{2^*-1}, \tag{3.4}$$

for all $t \in \mathbb{R}$ and for some constants a, b. Recall that $2^* = \frac{2N}{N-2}$. We consider the following problem

$$\begin{cases} -\Delta u + q(x)u = f(u), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Definition 3.4 *A weak solution of Problem (3.5) is a function $u \in H_0^1(\Omega)$ such that*

$$\int_\Omega \nabla u \nabla v \, dx + \int_\Omega q(x)uv \, dx - \int_\Omega f(u)v \, dx = 0, \tag{3.6}$$

for any $v \in H_0^1(\Omega)$.

The term that interests us in (3.6) is the following nonlinear term

$$\int_\Omega f(u)v \, dx,$$

we should start with is why is it well defined? The answer will be through the following justification:

Let $u, v \in H_0^1(\Omega)$, using the growth condition (3.12) we get,

$$\begin{aligned}
\left| \int_\Omega f(u)v \, dx \right| &\leq \int_\Omega |f(u)||v| \, dx \\
&\leq \int_\Omega (a + b|u|^{2^*-1})|v| \, dx \\
&= a\int_\Omega |v| \, dx + b\int_\Omega |u|^{2^*-1})|v| \, dx.
\end{aligned}$$

By Cauchy Schwarz inequality, we get,

$$a\int_\Omega |v|\ dx \leq a\sqrt{|\Omega|}\|v\|_{L^2} < \infty.$$

Using Hölder's inequality we have

$$\int_\Omega |u|^{2^*-1})|v|\ dx \leq \|u\|_{L^{2^*}}^{2^*-1}\|v\|_{L^{2^*}},$$

we know that the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous, hence there are two constants $C_1, C_2 > 0$ such that,

$$\|u\|_{L^{2^*}}^{2^*-1}\|v\|_{L^{2^*}} \leq C_1C_2\|u\|_{H_0^1}^{2^*-1}\|v\|_{H_0^1} < \infty.$$

We conclude that $\int_\Omega f(u)v\ dx < \infty$.

Now we construct the energy functional associated to this problem. Let

$$F(t) = \int_0^t f(s)\ ds,$$

and consider the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\ dx + \frac{1}{2}\int_\Omega q(x)|u|^2\ dx - \int_\Omega F(u)\ dx. \tag{3.7}$$

The differentiability of the nonlinear term $\int_\Omega F(u)\ dx$ can be verified see exercise ??. The rest of the terms in (3.7) are clear, then J is differentiable on $H_0^1(\Omega)$ and:

$$\langle J'(u), v\rangle = \int_\Omega \nabla u\nabla v\ dx + \int_\Omega q(x)uv\ dx - \int_\Omega f(u)v\ dx.$$

for any $v \in H_0^1(\Omega)$. Therefore weak solutions of Problem (3.5) correspond to critical points of J.

3.3 Study of problems with a associated functional convex

3.3.1 Global minimum of a convex functional

Definition 3.5 *Let E be a vector space. A functional $J : E \to \mathbb{R}$ is called convex if for every $u, v \in E$ and every real $t \in [0,1]$ we have,*

$$J\Big(tu + (1-t)v\Big) \leq tJ(u) + (1-t)J(v).$$

and we say that a functional $J : E \to \mathbb{R}$ is strictly convex if for every $u, v \in E$, $u \neq v$, and every real $t \in (0,1)$, we have,

$$J\Big(tu + (1-t)v\Big) < tJ(u) + (1-t)J(v).$$

and we say that J is (strictly) concave if $-J$ is (strictly) convex.

The following classical result gives an important relationship between convex and weakly lower semicontinuous functionals (see [4]).

Theorem 3.2 *Let $J : E \to \mathbb{R}$ be a continuous convex functional on a Banach space E. Then J is weakly lower semicontinuous. In particular, for every sequence $(u_n)_{n\in\mathbb{N}}$ converging weakly to $u \in E$, we have*

$$J(u) \leq \liminf_{n\to+\infty} J(u_n).$$

Example 3.1 *Let $(E, \|\cdot\|)$ be a Banach space. The norm $\|\cdot\| : E \to \mathbb{R}$ is a continuous convex functional, therefore by applying Theorem 3.2 we obtain a fact that we will use repeatedly in the sequel: if $u_n \rightharpoonup u$ in E, then*

$$\|u\| \leq \liminf_{n\to+\infty} \|u_n\|.$$

Definition 3.6 *A minimizing sequence $(u_n)_{n\in\mathbb{N}}$ of a functional $J : E \to \mathbb{R}$ is a sequence that satisfies:*

$$\lim_{n\to+\infty} J(u_n) = \inf_{u\in E} J(u).$$

The existence of the minimizing sequence is guaranteed by the characterization of the supremum.

We are interested in finding sufficient conditions for the existence of a global minimum for a functional on a Banach space. The most famous result in this area is the following:

Theorem 3.3 *Let E be a Banach space and refexive, let $J : E \to \mathbb{R}$ be a functional. If J satisfies:*

(i) J is cœrcive; i.e. $\lim_{n\to+\infty} J(u_n) = +\infty$

(ii) J is w.l.s.c.

then J is bounded from below on E and achieves its infimum at some $u_0 \in E$. If moreover,

(iii) J is Gateaux differentiable (Fréchet differentiable) at u_0,

then $J'_G(u_0) = 0$ ($J'(u_0) = 0$; i.e. J admits a critical point at u_0).

Proof. by contradiction, let us suppose that J is not bounded from below; i.e. there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} J(u_n) = -\infty. \tag{3.8}$$

Since J is cœrcive, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded (because the unboundedness of $(u_n)_{n\in\mathbb{N}}$ implies that $\|u_n\|_E \to +\infty$, hence by the cœrcivity of J we have $J(u_n) \to +\infty$ which contradicts (3.8)).

Hence, since E is reflexive, we can extract from (u_n) a subsequence (u_{n_k}) (still denoted (u_n)) such that (u_n) converges weakly to some $u_0 \in E$, since J is w.l.s.c, we have

$$J(u_0) \leq \liminf_{n\to+\infty} J(u_n) = -\infty.$$

This is a contradiction which implies that J is bounded from below.

Now, we show that J achieves its infimum at some w. Let $(u_n)_{n\in\mathbb{N}} \subset E$ a minimizing sequence of J; i.e. $\lim J(u_n) = \inf_{u\in E} J(u) = m$. As above $(u_n)_{n\in\mathbb{N}}$ is bounded in E, then there exist a weakly convergent subsequence $u_{n_k} \rightharpoonup w$ in E. Since J is w.l.s.c, we have

$$m = \inf_{u\in E} J(u) \leq J(w) \leq \liminf_{n\to+\infty} J(u_n) = \lim_{n\to+\infty} J(u_n) = \inf_{u\in E} J(u) = m,$$

we deduce that $J(w) = m = \inf_{u\in E} J(u)$ and this is what we want to prove.

the last result we have already demonstrated (see Theorem 3.1). ■

Combining Theorem 3.2 and Theorem 3.3 , we obtain the following corollary.

Corollary 3.1 *Let E be a reflexive Banach space and let $J : E \to \mathbb{R}$ be a functional. If J satisfies:*

(i) Cœrcive,

(ii) Continuous and convex.

Then, J is bounded from below on E and achieves its infimum at some $u_0 \in E$. If moreover,

(iii) J is Gateaux differentiable (Fréchet differentiable) at u_0,

then $J'_G(u_0) = 0$ ($J'(u_0) = 0$; i.e. J admits a critical point at u_0).

The following result gives a sufficient condition for the uniqueness of the minimum for a convex functional.

Proposition 3.2 *Let $J : E \to \mathbb{R}$ be a strictly convex functional, then J admits at most minimum point.*

Proof. Suppose that J has two minimum points $u_1, u_2 \in E$, $u_1 \neq u_2$ ($J(u_1) = J(u_1) = \min_{u\in E} J(u)$), the convexity implies that

$$\begin{aligned} J\Big(\frac{1}{2}u_1 + (1-\frac{1}{2})u_2\Big) &< \frac{1}{2}J(u_1) + \frac{1}{2}J(u_2) \\ &= \frac{1}{2}\min_{u\in E} J(u) + \frac{1}{2}\min_{u\in E} J(u) \\ &= \min_{u\in E} J(u) \end{aligned}$$

we set $v = \frac{1}{2}u_1 + \frac{1}{2}u_2 \in E$, we can deduce that

$$J(v) < \min_{u\in E} J(u)$$

which is contradiction. Consequently, we have $u_1 = u_2$. ■

The following results are immediately clear.

Corollary 3.2 *Let $J : E \to \mathbb{R}$ be a strictly convex functional and differentiable, admits at most one critical point.*

Corollary 3.3 *Let $J : E \to \mathbb{R}$ be a functional such that*

- *strictly convex,*

- *differentiable,*
- *cœrcive.*

Then, J has a unique critical point.

We often need sufficient conditions to prove convexity, the following is one of the important criteria for determining it.

Proposition 3.3 *Let E be a Banach space and let $J : E \to \mathbb{R}$ be a differentiable functional. Assume that for all $u, v \in E$,*

$$\langle J'(u) - J'(v), u - v\rangle \geq 0.$$

Then J is convex. If

$$\langle J'(u) - J'(v), u - v\rangle > 0,$$

for any $u \neq v$, then J is strictly convex.

Proof. Let $u, v \in E$ fixed and define a function $\varphi : \mathbb{R} \to \mathbb{R}$ as:

$$\varphi(t) = J(u + t(v - u)).$$

The function φ is differentiable and we have:

$$\varphi'(t) = \langle J'(u + t(v - u)), v - u\rangle.$$

Fix now $s < t$, we have:

$$\begin{aligned}
\varphi'(t) - \varphi'(s) &= \langle J'(u + t(v - u)) - J'(u + s(v - u)), v - u\rangle \\
&= \frac{1}{t - s}\langle J'(u + t(v - u)) - J'(u + s(v - u)), (u + t(v - u)) - (u + s(v - u))\rangle \\
&\geq 0.
\end{aligned}$$

Hence φ' is a nondecreasing function, so φ is convex, and in particular

$$\varphi(t) = \varphi(1t + 0(1 - t)) \leq t\varphi(1) + (1 - t)\varphi(0),$$

which means that

$$J(tv + (1 - t)u) \leq tJ(v) + (1 - t)J(u).$$

If in the assumption the strict inequality holds, then we get that φ' is strictly increasing, and hence φ is strictly convex, implying the same for J. ■

3.3.2 Applications

Let us now apply the abstract results of the previous sections to some concrete differential problems.

Example 1

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. We consider the following linear problem

$$\begin{cases} -\Delta u + q(x)u = h(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Where $q \in L^\infty(\Omega)$ and $h \in L^2(\Omega)$. Recall that $u \in H_0^1(\Omega)$ is a weak solution of Problem 3.9 if and only if it satisfies Definition 3.3

Theorem 3.4 *Assume that $q \in L^\infty(\Omega)$ and $q(x) \geq 0$ for a.e. $x \in \Omega$. Then for any $h \in L^2(\Omega)$, Problem 3.9 has a unique a weak solution $u \in H_0^1(\Omega)$.*

Proof.

- Let us consider the associated energy functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined as in (3.2). It is okay to rewrite it again.

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\int_\Omega q(x)|u|^2\, dx - \int_\Omega h(x)u\, dx.$$

We have previously verified that this functional is differentiable and

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \nabla v\, dx + \int_\Omega q(x)uv\, dx - \int_\Omega h(x)v\, dx.$$

Thus a critical point of J is a weak solution of Problem 3.9. Let $u, v \in H_0^1(\Omega)$, $u \neq v$,

$$\begin{aligned} \langle J'(u) - J'(v), u - v\rangle &= \langle J'(u), u - v\rangle - \langle J'(v), u - v\rangle \\ &= \Big[\int_\Omega \nabla u(\nabla u - \nabla v)\, dx + \int_\Omega q(x)u(u-v)\, dx - \int_\Omega h(u-v)\, dx\Big] \\ &\quad - \Big[\int_\Omega \nabla v(\nabla u - \nabla v)\, dx + \int_\Omega q(x)v(u-v)\, dx - \int_\Omega h(u-v)\, dx\Big] \\ &= \int_\Omega |\nabla u - \nabla v|^2\, dx + \int_\Omega q(x)(u-v)^2\, dx \\ &\geq \int_\Omega |\nabla u - \nabla v|^2\, dx = \|u - v\|^2 \\ &> 0, \quad \forall u \neq v. \end{aligned}$$

So, J is strictly convex thanks to Proposition 3.3.

- We show that J is cœrcive. We have

$$\frac{1}{2}\int_\Omega |\nabla u|^2\, dx = \frac{1}{2}\|u\|_{H_0^1}^2.$$

As $q(x) \geq 0$ a.e. $x \in \Omega$,

$$\frac{1}{2}\int_\Omega q(x)|u|^2\, dx \geq 0.$$

Using Cauchy Schwarz and Poincaré inequalities,

$$\int_\Omega h(x)u\, dx \leq \|h\|_{L^2}\|u\|_{L^2} \leq C_\Omega \|h\|_{L^2}\|u\|_{H_0^1}.$$

We deduce that,

$$J(u) \geq \frac{1}{2}\|u\|^2_{H^1_0} - C_\Omega \|h\|_{L^2}\|u\|_{H^1_0}. \tag{3.10}$$

Knowing that

$$\lim_{\|u\|_{H^1_0}\to+\infty} \frac{1}{2}\|u\|^2_{H^1_0} - C_\Omega \|h\|_{L^2}\|u\|_{H^1_0} = +\infty.$$

then,

$$\lim_{\|u\|_{H^1_0}\to+\infty} J(u) = +\infty.$$

- Applying Proposition 3.3, we conclude that the functional J has a global minimum which is its only critical point. Therefore Problem (3.9) has a unique solution.

■

Example 2

We consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u + q(x)u = f(u) + h(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Let us consider the following conditions:

(H_0) $h \in L^2(\Omega)$, $q \in L^\infty(\Omega)$ and $q(x) \geq 0$ for a.e. $\in \Omega$.

(H_1) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|f(t)| \leq a + b|t|^{2^*-1}, \quad \forall t \in \mathbb{R}. \tag{3.12}$$

(H_2)

$$f(t)t \leq 0 \quad \forall t, s \in \mathbb{R}, \tag{3.13}$$

$$(f(t) - f(s))(t - s) \leq 0, \quad \forall t, s \in \mathbb{R}. \tag{3.14}$$

We say that $u \in H^1_0(\Omega)$ is a weak solution of Problem 3.11 if and only if,

$$\int_\Omega \nabla u \nabla v \, dx + \int_\Omega q(x)uv \, dx - \int_\Omega h(x)v \, dx - \int_\Omega f(u)v \, dx = 0, \tag{3.15}$$

for any $v \in H^1_0(\Omega)$.

All terms in (3.15) are well defined. In particular, the third term, which is the nonlinear term, is well defined thanks to the hypothesis (H_1) (we leave it to the reader to verify that).

The following theorem gives us the existence result.

Theorem 3.5 *Under hypotheses (H_0), (H_1) and (H_2), Problem has a unique solution solution in $H^1_0(\Omega)$.*

Proof. We first start with the definition of the energy functional associated to Problem 3.11. Let $I : H_0^1(\Omega) \to \mathbb{R}$ defined by,

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\int_\Omega q(x)|u|^2\, dx - \int_\Omega F(u)\, dx - \int_\Omega h(x)u\, dx, \tag{3.16}$$

where $F(t) = \int_0^t f(s)\, ds$.

The differentiability of the nonlinear term $\int_\Omega F(u)\, dx$ is discussed in (2.8), depending of course on the growth condition (3.12). The differentiability of the rest of the terms are clear. Furthermore, we have

$$\langle I'(u), v\rangle = \int_\Omega \nabla u \nabla v\, dx + \int_\Omega q(x)uv\, dx - \int_\Omega f(u)v\, dx - \int_\Omega h(x)v\, dx.$$

for any $u, v \in H_0^1(\Omega)$. Thus, Critical points of I are weak solutions of Problem (3.11).

Now, we show that I is strictly convex; let $u, v \in H_0^1(\Omega)$ with $u \neq v$, we have:

$$\begin{aligned}\langle I'(u) - I'(v), u - v\rangle &= \langle I'(u), u - v\rangle - \langle I'(v), u - v\rangle \\ &= \int_\Omega |\nabla u - \nabla v|^2\, dx + \int_\Omega q(x)(u - v)^2\, dx - \int_\Omega (f(u) - f(v))(u - v)\, dx.\end{aligned}$$

It is easy to see that by (H_0) and (3.14), we have

$$\langle I'(u) - I'(v), u - v\rangle \geq \int_\Omega |\nabla u - \nabla v|^2\, dx = \|u - v\|_{H_0^1} > 0, \quad \forall u \neq v.$$

this shows that I is strictly convex.

We pass to prove the cœrcivity of I. On the other hand, Condition 3.13 states that $F(t) \leq 0$ for any $t \in \mathbb{R}$ (we leave it to the reader to verify), which implies that,

$$\int_\Omega F(u)\, dx \leq 0. \tag{3.17}$$

Hence, by the same techniques in (3.10) and using (3.17), we get,

$$I(u) \geq \frac{1}{2}\|u\|_{H_0^1}^2 - C_\Omega \|h\|_{L^2}\|u\|_{H_0^1},$$

this implies that $I(u) \to +\infty$ as $\|u\|_{H_0^1} \to +\infty$, so that I is cœrcive.

Since the functional I is coercive, strictly convex and continuous, it has a unique global minimum point, which is a critical point. Consequently, Problem (3.11) has a unique weak solution in $H_0^1(\Omega)$. ∎

3.4 Minimization Techniques for Problems involving a nonlinear term satisfies the growth condition

We consider the same Problem (3.11) with the following hypotheses

(h_1) $q(x) \geq 0$ a.e. $x \in \Omega$.

(h_2) $h \in L^2(\Omega)$.

(h_3) $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded function.

Theorem 3.6 *Assume that $(h_1) - (h_3)$ are holds, then Problem (3.11) has at least a weak solution in $H_0^1(\Omega)$.*

Proof. We consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ as defined in (3.16), It is clear that it is well defined under the assumptions $(h_1) - (h_3)$ and differentiable. In particular, the associated function to the nonlinear term is bounded, so it fulfills a special case of growth condition (3.12). In addition, we have

$$\langle I'(u), v\rangle = \int_\Omega \nabla u \nabla v \, dx + \int_\Omega q(x)uv \, dx - \int_\Omega f(u)v \, dx - \int_\Omega h(x)v \, dx.$$

Cœrcivity: Since f is bounded, there exists $M > 0$ such that $|f(t)| \leq M$ for any $t \in \mathbb{R}$. Thus, we have

$$|F(t)| \leq M|t|, \forall t \in \mathbb{R}. \tag{3.18}$$

By the continuous embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$, there exists $c_1 > 0$ such that

$$\|u\|_{L^1} \leq c_1 \|u\|_{H_0^1}.$$

So that

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|_{H_0^1}^2 - M\|u\|_{L^1} - C_\Omega \|h\|_{L^2}\|u\|_{H_0^1} \\ &\geq \frac{1}{2}\|u\|_{H_0^1}^2 - Mc_1\|u\|_{H_0^1} - C_\Omega \|h\|_{L^2}\|u\|_{H_0^1}. \end{aligned}$$

This implies that $I(u) \to +\infty$ as $\|u\| \to +\infty$. So I is Cœrcive.

W.l.s.c.: Let $(u_n)_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. We know that $\|\cdot\|_{H_0^1} : H_0^1(\Omega) \to \mathbb{R}$ is w.l.s.c. (see Example (3.1)). Then we have

$$\|u\|_{H_0^1} \leq \liminf_{n\to+\infty} \|u_n\|_{H_0^1}. \tag{3.19}$$

Recall that the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact as $1 \leq 2 < 2^*$. hence, we deduce that $u_n \to u$ in $L^2(\Omega)$ as $n \to +\infty$. This leads to,

$$\int_\Omega q(x)(u_n - u)^2 \, dx \leq \|q\|_\infty \|u_n - u\|_{L^2} \to 0,$$

as $n \to +\infty$. By a simple calculation, we get,

$$\begin{aligned} 0 = \lim_{n\to+\infty} \int_\Omega q(x)(u_n - u)^2 \, dx &= \lim_{n\to+\infty} \int_\Omega q(x)u_n^2 \, dx + \int_\Omega q(x)u^2 \, dx - 2\int_\Omega q(x)u_n u \, dx, \\ &= \lim_{n\to+\infty} \int_\Omega q(x)u_n^2 \, dx - \int_\Omega q(x)u^2 \, dx. \end{aligned}$$

So that

$$\lim_{n\to+\infty} \int_\Omega q(x)u_n^2 \, dx = \int_\Omega q(x)u^2 \, dx. \tag{3.20}$$

The Cauchy-Schwarz inequality ensures that we have

$$\left| \int_\Omega h(x)(u_n - u) \, dx \right| \leq \|h\|_{L^2}\|u_n - u\|_{L^2} \to 0,$$

as $n \to +\infty$. Then, we have,

$$\lim_{n\to+\infty} \int_\Omega h(x)u_n \, dx = \int_\Omega h(x)u \, dx. \tag{3.21}$$

Now, we show that

$$\lim_{n\to+\infty} \int_\Omega F(u_n) \, dx = \int_\Omega F(u) \, dx. \tag{3.22}$$

Since $u_n \to u$ in $L^2(\Omega)$, there exists a subsequence $(u_{n_{k_l}})$ (denoted (u_n)) such that $u_n(x) \to u(x)$ a.e. $x \in \Omega$. In addition, there exists $g \in L^2(\Omega)$ $(g \in L^1(\Omega))$ such that $|u_n(x)| \leq g(x)$ a.e. $x \in \Omega$. The continuity of F implies that

$$F(u_n) \to F(u_n) \quad \text{a.e. in } \Omega. \tag{3.23}$$

Using (3.18), we get

$$|F(u_n)| \leq M|u_n| \leq Mg \in L^1(\Omega). \tag{3.24}$$

From (3.23) and (3.24), Lebesgue's dominated convergence theorem gives that the equality (3.22) is verified.

According to (3.19),(3.20), (3.21) and (3.22) we can see that,

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_\Omega q(x)|u|^2 \, dx - \int_\Omega F(u) \, dx - \int_\Omega h(x)u \, dx \\
&\leq \lim_{n\to+\infty} \frac{1}{2}\int_\Omega |\nabla u_n|^2 \, dx + \frac{1}{2}\int_\Omega q(x)|u_n|^2 \, dx - \int_\Omega F(u_n) \, dx - \int_\Omega h(x)u_n \, dx \\
&= \liminf_{n\to+\infty} I(u_n).
\end{aligned}$$

Consequently, the functional I is w.l.s.c. In conclusion, as I is cœrcive, w.l.s.c. and differentiable, I admits at least a global minimum in $H_0^1(\Omega)$ which is a critical point. Then, it is a weak solution of Problem (3.11). ■

3.4.1 Sublinear case in the growth condition

We consider the growth condition as follows,

(h_4) f is continuous and

$$\exists \sigma \in (0,1) : |f(t)| \leq a + b|t|^\sigma, \quad \forall t \in \mathbb{R}. \tag{3.25}$$

Remark 3.3 *If f satisfies (h_4), then we have,*

$$\exists a_1, b_1 > 0 : |F(t)| \leq a_1 + b_1|t|^{\sigma+1}, \quad \forall t \in \mathbb{R}. \tag{3.26}$$

Let us try to check that out. By integrating both sides of the inequality (3.25), we get

$$|F(t)| \leq a|t| + \frac{b}{\sigma+1}|t|^{\sigma+1}, \quad \forall t \in \mathbb{R}.$$

For t large enough (for $|t| > 1$), we have $|t| \leq |t|^{\sigma+1}$, so that

$$\begin{aligned}
|F(t)| &\leq a|t|^{\sigma+1} + \frac{b}{\sigma+1}|t|^{\sigma+1} \\
&= \Big(a + \frac{b}{\sigma+1}\Big)|t|^{\sigma+1}.
\end{aligned} \tag{3.27}$$

If $|t| \leq 1$, we have

$$F(t) \leq a + \frac{b}{\sigma+1}|t|^{\sigma+1} \tag{3.28}$$

Relations (3.27) and (3.28) imply that, for any $t \in \mathbb{R}$ we have,

$$F(t) \leq a + \Big(a + \frac{b}{\sigma+1}\Big)|t|^{\sigma+1}$$

Just take $a_1 = a$ and $b_1 = a + \frac{b}{\sigma+1}$.

Based on hypothesis (h_4), we have the following result:

Theorem 3.7 *Suppose that assumptions (h_1), (h_2) and (h_4) are satisfied. Then, Problem (3.11) admits at least a weak solution in $H_0^1(\Omega)$.*

Proof. We consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ as defined in (3.16. Under the assumptions (h_1), (h_2) and (h_4) the functional I is well defined and differentiable. In particular, the associated function to the nonlinear term satisfies the growth condition (3.12). In addition, we have

$$\langle I'(u), v\rangle = \int_\Omega \nabla u \nabla v \, dx + \int_\Omega q(x)uv \, dx - \int_\Omega f(u)v \, dx - \int_\Omega h(x)v \, dx.$$

Cœrcivity: Using (3.26) we get,

$$\int_\Omega F(u)\, dx \leq a_1 \int_\Omega dx + b_1 \int_\Omega |u|^{\sigma+1}\, dx = a_1|\Omega| + b_1\|u\|^{\sigma+1}_{L^{\sigma+1}}$$

Thanks to the continuous embedding of $H_0^1(\Omega)$ in $L^{\sigma+1}(\Omega)$ (as $1 < \sigma+1 < 2 < 2^*$), there exists $c_2 > 0$ such that

$$a_1|\Omega| + b_1\|u\|^{\sigma+1}_{L^{\sigma+1}} \leq a_1|\Omega| + b_1c_2\|u\|^{\sigma+1}_{H_0^1}. \tag{3.29}$$

Hence, using (3.29) and with the same techniques we used before, we obtain,

$$I(u) \geq \frac{1}{2}\|u\|^2_{H_0^1} - a_1|\Omega| - b_1c_2\|u\|^{\sigma+1}_{H_0^1} - C_\Omega\|h\|_{L^2}\|u\|_{H_0^1}$$

Since $2 > \sigma+1 > 1$, we have that $I(u) \to +\infty$ as $\|u\| \to +\infty$. So I is Cœrcive.

W.l.s.c.: We can follow the same method as the proof of the *w.l.s.c.* part in the proof of Theorem 3.6, except for slight differences in the convergence $\lim_{n\to+\infty}\int_\Omega F(u_n)\, dx = \int_\Omega F(u)\, dx$, where when applying Lebesgue's dominated convergence theorem, precisely, in finding the dominated function of the sequence $(F(u_n))_{n\in\mathbb{N}}$, we need to use (3.26); we have

$$|F(u_n)| \leq a_1 + b_1|u_n|^2 \leq a_1 + b_1 g^2 \in L^1(\Omega),$$

where $g \in L^2(\Omega)$ is the function obtained by Lebesgue's inverse theorem (see the proof of Theorem 3.6). ■

3.4.2 Eigenvalue and eigenfunction of the Laplacian operator

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset.

Definition 3.7 *We say that $\lambda \in \mathbb{R}$ is an eigenvalue of the Laplacian operator $-\Delta$ under Dirichlet boundary conditions if there exists $\varphi \in H_0^1(\Omega) \setminus \{0\}$ such that*

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & \text{in } \Omega; \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

The function φ is called an eigenfunction associated to λ.

Remark 3.4 *Problem (3.30) is to be interpreted in the weak sense, namely φ solves it provided*

$$\int_\Omega \nabla\varphi\nabla v \, dx = \int_\Omega \varphi(x) v \, dx,$$

for all $v \in H_0^1(\Omega)$.

The basic properties of eigenvalues and eigenfunctions are summarized in the following theorem.

Theorem 3.8 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset. Then there exist a sequences $(\lambda_n)_{n\in\mathbb{N}}) \subset \mathbb{R}$ and $(\varphi_n)_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ such that, each λ_n is an eigenvalue of the operator Laplacian Δ and each φ_n is an eigenfunction corresponding to λ_n. Furthermore, we have,*

1. *$\varphi_n \to +\infty$ as $n \to +\infty$.*
2. *$(\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$.*
3. *For all $n \in \mathbb{N}$, $\varphi_n \in L^\infty(\Omega)$ and $\varphi_n(x) \neq 0$ a.e. in Ω.*

Remark 3.5 *Eigenvalues and eigenfunctions play a very relevant role in nonlinear elliptic problems. The most important eigenvalue is the smallest, λ_1, also called the first eigenvalue or the principal eigenvalue.*

The first eigenvalue has distinctive properties, the most important of which we discuss in the following theorem.

Theorem 3.9 *Let Ω be an open and bounded subset of $\mathbb{R}^N$. Define a functional $J : H_0^1(\Omega) \setminus \{0\} \to \mathbb{R}$ as*

$$J(u) = \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx}.$$

Then we have

1. *$\lambda_1 = \min_{u\in H_0^1(\Omega)\setminus\{0\}} J(u)$.*
2. *$\lambda_1 = J(w)$ if and only if v is a (weak) solution of*

$$\begin{cases} -\Delta w = \lambda_1 w, & \text{in } \Omega; \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.31}$$

3. *every non identically zero solution of (3.31) has constant sign in Ω (in particular, is a.e. different from zero in Ω);*
4. *the set of solutions of (3.31) has dimension one; one says that λ_1 is simple.*

3.4.3 Linear case in the growth condition

We return to the study of the nonlinear problem (3.11) where we consider the following hypotheses

(h_5) $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and there exist $a>0$ and $b\in(0,\lambda_1)$ such that

$$|f(t)|\leq a+b|t|,\quad \forall t\in\mathbb{R}$$

It is easy to see that (h_5) implies that,

$$|F(t)|\leq a|t|+\frac{b}{2}|t|^2,\quad \forall t\in\mathbb{R}. \tag{3.32}$$

Theorem 3.10 *Under the assumptions (h_1), (h_2) and (h_5) are verified. Problem (3.11) admits at least one solution.*

Proof. We consider the functional $I:H_0^1(\Omega)\to\mathbb{R}$ as defined in (3.16. Under the assumptions (h_1), (h_2) and (h_5) the functional I is well defined and differentiable. In particular, the associated function to the nonlinear term satisfies a particular case of the growth condition (3.12). In addition, I is differentiable and we have

$$\langle I'(u),v\rangle=\int_\Omega \nabla u\nabla v\,dx+\int_\Omega q(x)uv\,dx-\int_\Omega f(u)v\,dx-\int_\Omega h(x)v\,dx.$$

W.l.s.c.: We can follow exactly as in the preceding theorems.

Cœrcivity of I: We know that $\lambda_1=\min\frac{\|u\|^2_{H_0^1}}{\|u\|^2_{L^2}}$, this leads to

$$\|u\|_{L^2}\leq\frac{1}{\sqrt{\lambda_1}}\|u\|_{H_0^1}.$$

We use this inequality and (3.32) to estimate the functional I as follows:

$$\begin{aligned}
I(u) &\geq \frac{1}{2}\|u\|^2_{H_0^1}-a\|u\|_{L^1}-\frac{b}{2}\|u\|^2_{L^2}-\|h\|_{L^2}\|u\|_{L^2}\\
&\geq \frac{1}{2}\|u\|^2_{H_0^1}-ac_1\|u\|_{H_0^1}-\frac{b}{2\lambda_1}\|u\|^2_{H_0^1}-\frac{1}{\sqrt{\lambda_1}}\|h\|_{L^2}\|u\|_{H_0^1}\\
&= \frac{1}{2}(1-\frac{b}{\lambda_1})\|u\|^2-ac_1\|u\|_{H_0^1}-\frac{1}{\sqrt{\lambda_1}}\|h\|_{L^2}\|u\|_{H_0^1}.
\end{aligned}$$

Since $b\in(0,\lambda_1)$, we have $I(u)\to+\infty$ as $n\to+\infty$ which prove that I is cœrcive. ■

3.4.4 Existence of nontrivial solutions

We consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u+q(x)u=f(u), & \text{in } \Omega;\\ u=0, & \text{on } \partial\Omega.\end{cases} \tag{3.33}$$

We notice that if $f(0)=0$, then $u\equiv 0$ is a solution of Problem (3.33) which called the *trivial solution.*

Remark 3.6 *Without further assumptions, it may very well be that the trivial solution is the only solution; Indeed, if we suppose that $f(t)t \leq 0$ for any $t \in \mathbb{R}$, then any weak solution of (3.33) satisfies,*

$$\|u\|_{H_0^1}^2 + \int_\Omega q(x)u^2 \, dx = \int_\Omega f(u)u \, dx \leq 0,$$

which implies that $u \equiv 0$.

Theorem 3.11 *Let (h_1) holds. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies,*

$$f(0) = 0, \quad \limsup_{t \to \pm\infty} \frac{|f(t)|}{|t|} < \lambda_1.$$

Then, Problem (3.33) has at least at least one solution (which may be trivial). In addition, If f satisfies

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > \lambda_1 + \|q\|_\infty. \tag{3.34}$$

Then, Problem (3.33) has at least at least one nontrivial solution.

Remark 3.7 *By an elementary demonstration we can show that if a function $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the condition $\limsup_{t \to \pm\infty} \frac{|f(t)|}{|t|} < \lambda_1$, then f satisfies the hypothesis (h_5) (see Exercise ??).*

Proof. We consider the associated functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by,

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega q(x)|u|^2 \, dx - \int_\Omega F(u) \, dx,$$

Clearly, (h_1) and Remark 3.7 contribute to the fact that the function J is well defined and differentiable. The first part of Theorem (3.11) can be directly deduced using Remark (3.7) and applying Theorem 3.10 with $h(x) = 0$.

Now, based on condition (3.35) we prove that the solution found in the first part is nontrivial. Suppose that

$$\liminf_{t \to 0^+} \frac{f(t)}{t} = l, \tag{3.35}$$

this means that, for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $t \in [0, \delta]$, we have

$$l - \varepsilon \leq \frac{f(t)}{t} \leq l + \varepsilon.$$

Since $l > \lambda_1 + \|q\|_\infty$, we can choose $\varepsilon > 0$ small enough so that $B = l - \varepsilon > \lambda_1 + \|q\|_\infty$. Therefore,

$$f(t) \geq Bt, \quad \forall t \in [0, \delta],$$

so,

$$F(t) \geq \frac{B}{2} t^2, \quad \forall t \in [0, \delta]. \tag{3.36}$$

On the other hand, we notice that $J(0) = 0$, so if we check that the global minimum (critical point) of J which is the solution satisfies $J(u) < 0$, then we have shown that the solution is nontrivial. Indeed, we use a (*level argument*) as follows:

Let $\varphi_1 > 0$ be the first eigenfunction of $-\Delta$ associated to the first eigenvalue λ_1. We have $\varphi_1 \in L^\infty(\Omega) \cap H_0^1(\Omega)$. We choose $\varepsilon > 0$ small enough so that $\varepsilon\varphi_1 < \delta$. Hence, by (3.36) we have,

$$F(\varepsilon\varphi_1(x)) \geq \frac{1}{2}\beta\varepsilon^2\varphi_1^2(x), \tag{3.37}$$

a.e. $x \in \Omega$. From the definition of φ_1, we have,

$$\int_\Omega \nabla\varphi_1 \nabla v \, dx = \lambda_1 \int_\Omega \varphi_1 v \, dx,$$

for any $v \in H_0^1(\Omega)$. For $v = u$ we get

$$\int_\Omega |\nabla\varphi_1|^2 \, dx = \lambda_1 \int_\Omega \varphi_1^2 \, dx \tag{3.38}$$

Using (3.37) and (3.38), we obtain

$$\begin{aligned}
J(\varepsilon\varphi_1) &\leq \frac{1}{2}\int_\Omega |\nabla\varepsilon\varphi_1|^2 \, dx + \frac{1}{2}\int_\Omega q(x)|\varepsilon\varphi_1|^2 \, dx - \frac{B}{2}\int_\Omega |\varepsilon\varphi_1|^2 \, dx \\
&\leq \frac{1}{2}\varepsilon^2\|\varphi_1\|_{H_0^1}^2 + \frac{1}{2}\|q\|_{L^\infty}\|\varphi_1\|_{L^2}^2 - \frac{B\varepsilon^2}{2}\|\varphi_1\|_{L^2}^2 \\
&= \frac{1}{2}\varepsilon^2\lambda_1\|\varphi_1\|_{L^2}^2 + \frac{1}{2}\|q\|_{L^\infty}\|\varphi_1\|_{L^2}^2 - \frac{B\varepsilon^2}{2}\|\varphi_1\|_{L^2}^2 \\
&= \frac{\varepsilon^2}{2}(\lambda_1 + \|q\|_{L^\infty} - B)|\varphi_1\|_{L^2}^2.
\end{aligned}$$

Since $B > \lambda_1 + \|q\|_{L^\infty}$, we have $J(\varepsilon\varphi_1) < 0$. Hence,

$$J(u) = \min_{v \in H_0^1(\Omega)} J(v) \leq J(\varepsilon\varphi_1) < 0.$$

Since $J(0) = 0$, u cannot be the trivial solution. ■

3.5 Elliptic problem involving the p-Laplacian operator

3.5.1 Some important results

Let $p > 1$. The p-Laplacian is the second order non linear differential operator defined as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

In the case $p = 2$, the p-Laplacian is the usual Laplacian operator.

In order to study problems involving the p-Laplacian operator we need several basic concepts such as Sobolev spaces $W^{1,p}(\Omega)$ and their associated properties such as the Poincaré inequality, and continuous and compact embedding in Lebesgue spaces (Sobolev inequalities). For more details see the section 1.3.

We will also need two important results about the differentiability of two fundamental operators in nonlinear problems.

Proposition 3.4 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open set. For $p > 1$, define a functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ by:*

$$I(u) = \int_\Omega |\nabla u|^p \, dx$$

Then J is differentiable and

$$\langle I'(u), v\rangle = p \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx$$

Proof. The same steps of proof can be followed in Example 2.1 and Example 2.5. ∎ Here is a generalization of the result in Exercise 5-section 2.

Proposition 3.5 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open set. Let $1 < p < N$ and let $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. suppose that there exists $\sigma \in [1, p^*]$, $p^* = \frac{Np}{N-p}$ is be the critical Sobolev exponent, and there exist $a, b \in \mathbb{R}$ such that*

$$|f(t)| \leq a + b|t|^{\sigma-1}, \quad \forall t \in \mathbb{R}. \tag{3.39}$$

We set $F(t) = \int_0^t f(s) \, ds$ and we consider $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega F(u(x)) \, dx.$$

Then J is is differentiable and

$$\langle J'(u), v\rangle = \int_\Omega f(u(x))v(x) \, dx.$$

3.5.2 Application

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open bounded set and let $1 < p < N$. Let $h \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|f(t)| \leq a + b|t|^{\sigma-1}, \quad \forall t \in \mathbb{R}. \tag{3.40}$$

where $a, b > 0$ and $1 \leq \sigma < p$. We consider the following nonlinear problem

$$\begin{cases} -\Delta_p u = f(u) + h(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.41}$$

Definition 3.8 *We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of Problem 3.41 if and only if*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx - \int_\Omega f(u)v \, dx - \int_\Omega h(x)v \, dx = 0, \tag{3.42}$$

for any $v \in W_0^{1,p}(\Omega)$.

We can check that the quantities in (3.42) are well defined; Indeed, let $u, v \in W_0^{1,p}(\Omega)$. Firstly, by Hölder inequality we have

$$\begin{aligned} \left| \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx \right| &\leq \left(\int_\Omega \left(|\nabla u|^{p-1} \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla v|^p \, dx \right)^{\frac{1}{p}} \\ &= \|\nabla u\|_{L^p}^{p-1} \|\nabla v\|_{L^p} \\ &= \|u\|_{W_0^{1,p}}^{p-1} \|v\|_{W_0^{1,p}} < \infty. \end{aligned}$$

For the second term, using (3.40) we get

$$\begin{aligned}\left|\int_\Omega f(u)v\,dx\right| &\leq \int_\Omega |f(u)||v|\,dx\\ &\leq a\int_\Omega |v|\,dx + b\int_\Omega |u|^{\sigma-1}v\,dx.\end{aligned}$$

By Hölder inequality and Poincaré inequality we obtain

$$\int_\Omega |v|\,dx \leq |\Omega|^{\frac{1}{q}}\|v\|_{L^p} \leq |\Omega|^{\frac{1}{q}}C_\Omega\|v\|_{W_0^{1,p}} < \infty,$$

where C_Ω is the Poincaré constant. In addition, by Hölder inequality and using the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\sigma(\Omega)$, as $1 \leq \sigma < p < p^*$, there exist $c_1, c_2 > 0$ such that

$$\begin{aligned}\int_\Omega |u|^{\sigma-1}v\,dx &\leq \|u\|_{L^\sigma}^{\sigma-1}\|v\|_{L^\sigma}\\ &\leq c_1^{\sigma-1}\|u\|_{W_0^{1,p}}^{\sigma-1} + c_2\|v\|_{W_0^{1,p}} < \infty.\end{aligned}$$

So, $\int_\Omega f(u)v\,dx < \infty$. By applying Hölder and Poincaré inequalities, we can also easily check that $\int_\Omega h(x)v\,dx < \infty$. From the above, we conclude that all the terms appearing in the equality (3.42) are well defined.

Here is the existence theorem,

Theorem 3.12 *Assume that f is continuous function satisfies (3.40). For any $h \in L^q(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$, Problem 3.41 has at least a weak solution in $W_0^{1,p}(\Omega)$.*

Proof. Firstly, The functional whose critical points are the (weak) solutions of Problem (3.41) is $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(u)\,dx - \int_\Omega h(x)u\,dx,$$

where $F(t) = \int_0^t f(s)\,ds$. From relation (3.40) , we can deduce that

$$|F(t)| \leq a|t| + \frac{b}{q}|t|^q, \quad \forall t \in \mathbb{R}. \tag{3.43}$$

We have the following steps

Step 1. *Differentiability* Thanks to Proposition 3.4, Proposition 3.5 and depending on the growth condition (3.40) we conclude that the functional J is differentiable on $W_0^{1,p}(\Omega)$, and we have

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx - \int_\Omega f(u)v\,dx - \int_\Omega h(x)v\,dx.$$

Step 2. ; *Cœrcivity:* Let $u \in W_0^{1,p}(\Omega)$. Using the inequality (3.43) we have

$$\left|\int_\Omega F(u)\,dx\right| \leq a\int_\Omega |u|\,dx + \frac{b}{p}\int_\Omega |u|^\sigma\,dx = a\|u\|_{L^1} + \frac{b}{p}\|u\|_{L^\sigma}^\sigma.$$

Since the Sobolev space $W_0^{1,p}(\Omega)$ is continuously embedded in $L^1(\Omega)$ and $L^\sigma(\Omega)$ as $1 \leq \sigma < p < p^*$, there exist $c_1, c_2 > 0$ such that

$$\|u\|_{L^1} \leq c_1\|u\|_{W_0^{1,p}}, \quad \|u\|_{L^\sigma} \leq c_2\|u\|_{W_0^{1,p}}.$$

This leads to us having

$$\left|\int_\Omega F(u)\,dx\right| \le ac_1\|u\|_{W_0^{1,p}} + \frac{b}{p}c_2\|u\|^\sigma_{W_0^{1,p}}.$$

On the other hand, by Hölder and Poincaré inequalities, we obtain that

$$\left|\int_\Omega h(x)u\,dx\right| \le \|h\|_q\|u\|_{L^p} \le C_\Omega\|h\|_q\|u\|_{W_0^{1,p}}.$$

As a result of the above, we have

$$J(u) \ge \frac{1}{p}\|u\|^p_{W_0^{1,p}} - ac_1\|u\|_{W_0^{1,p}} - \frac{b}{p}c_2\|u\|^\sigma_{W_0^{1,p}} - C_\Omega\|h\|_q\|u\|_{W_0^{1,p}}.$$

Since $1 \le \sigma < p$ we have

$$\frac{1}{p}\|u\|^p_{W_0^{1,p}} - ac_1\|u\|_{W_0^{1,p}} - \frac{b}{p}c_2\|u\|^\sigma_{W_0^{1,p}} - C_\Omega\|h\|_q\|u\|_{W_0^{1,p}} \to +\infty,$$

as $\|u\|_{W_0^{1,p}} \to +\infty$, which implies that $\lim\limits_{\|u\|_{W_0^{1,p}}\to+\infty} J(u) = +\infty$. Therefore, J is cœrcive.

Step 3. *w.l.s.c.* Let $(u_n) \subset W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Since the norm in any Banach space is w.l.s.c., in particular, the norm $\|\cdot\|_{W_0^{1,p}}$ is w.l.s.c. and we have

$$\|u\|_{W_0^{1,p}} \le \liminf_{n\to+\infty}\|u_n\|_{W_0^{1,p}}. \tag{3.44}$$

Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact as $1 < p < p^*$, we have $u_n \to u$ in $L^p(\Omega)$. By Hölder inequality we obtain

$$\left|\int_\Omega h(x)(u_n - u)\,dx\right| \le \|h\|_{L^q}\|u_n - u\|_{L^p} \to 0,$$

as $n \to +\infty$. Therefore, we have

$$\lim_{n\to+\infty}\int_\Omega h(x)u_n\,dx = \int_\Omega h(x)u\,dx. \tag{3.45}$$

For the nonlinear term, we apply the Lebesgue's dominated convergence theorem. We know that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact as $1 < \sigma < p < p^*$, we have $u_n \to u$ in $L^\sigma(\Omega)$. Hence, there exists a subsequence (u_{n_k}) of (u_n) (still denoted (u_n)) such that

$$u_n(x) \to u(x) \quad a.e.\ x \in \Omega, \tag{3.46}$$

and there exists a function $g \in L^\sigma(\Omega)$ such that

$$|u_n(x)| \le g(x) \quad a.e.\ x \in \Omega. \tag{3.47}$$

Continuity of F and relation (3.46) leads to

$$F(u_n(x)) \to F(u(x)). \tag{3.48}$$

Using (3.43) and (3.47), we also get

$$|F(u_n)| \le a|u_n| + \frac{b}{q}|u_n|^\sigma \le ag + \frac{b}{q}g^\sigma \in L^1(\Omega). \tag{3.49}$$

From (3.48) and (3.49) we deduce that

$$\int_\Omega F(u_n)\,dx \to \int_\Omega F(u)\,dx \tag{3.50}$$

Combining (3.44), (3.45) and (3.50) we obtain

$$J(u) \le \liminf_{n\to+\infty} J(u_n).$$

Thus, we have proven that J is w.l.s.c.

Finally, since the functional J is differentiable, cœrcive and w.l.s.c., it has at lest global minimum which is a critical point. Consequently, Problem (3.41) has at least a weak solution in $W_0^{1,p}(\Omega)$.

■

3.6 Ekeland Variational Principle

3.6.1 Palais-Smale condition

To express the compactness of minimizing sequences, or more generally of sequences which converge to a point which we hope to show is a critical point, we often resort to the Palais-Smale condition (see [7, 9]).

Definition 3.9 (Palais-Smale Sequence) *Let E be a Banach space and let $J : E \to \mathbb{R}$ be a differentiable functional. Any sequence $\{u_n\}_n \subset E$ satisfies*

$$\exists C > 0 : |J(u_n)| \le C \;\; \forall n \in \mathbb{N} \quad \text{and} \quad \|J'(u_n)\|_{E'} \to 0 \quad \text{as } n \to +\infty, \tag{3.51}$$

is called a Palais-Smaile sequence for J (We write briefly: (P.S) sequence for J).

Let $c \in \mathbb{R}$, if

$$|J(u_n)| \to c \quad \text{and} \quad \|J'(u_n)\|_{E'} \to 0 \quad \text{as } n \to +\infty, \tag{3.52}$$

then $\{u_n\}$ is called a Palais-Smale sequence for J at level c ((We write briefly: $(P.S)_c$ sequence for J)). Then c is called the Palais-Smale level for J.

Example 3.2 *Consider the real function $f(x) = xe^{1-x}$,*

$$f'(x) = (1-x)e^{1-x} \Rightarrow f'(x) = 0 \Leftrightarrow x = 1.$$

Hence, f has only a critical point $x_0 = 1$ at level $c = f(1) = 1$. We note the following:

1. *The sequence $u_n = 1 - \frac{1}{n}$ satisfies:*

$$|f(u_n)| = \left(1 - \frac{1}{n}\right)e^{\frac{1}{n}} \le e$$

so $\{f(u_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. And

$$f'(u_n) = \frac{1}{n}e^{\frac{1}{n}} \to 0 \quad \text{as } n \to +\infty.$$

So $\{u_n\}$ is (P.S) Sequence.

Since $f(u_n) = (1 - \frac{1}{n})e^{\frac{1}{n}} \to 1$, $\{u_n\}$ is a Palais-Smale Sequence for f at level $c = 1$ (or $(P.S)_1$ sequence for f).

2. *If we take $v_n = n$, the sequence $\{v_n\}_{n\in\mathbb{N}}$ is a $(P.S)_0$ Sequence for f.*

Definition 3.10 (Palais-Smale condition) *Let E be Banach spaces and let $J : E \to \mathbb{R}$ be a differentiable functional.*

- *We say that J satisfies the Palais-Smale condition ((P.S) condition) if every (P.S) Sequence for J; i.e. it satisfies (3.51), has a (strongly) convergent subsequence in E.*
- *We say that J satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ ($(P.S)_c$ condition) if every $(P.S)_c$ Sequence for J; i.e. it satisfies (3.52), has a (strongly) convergent subsequence in E.*

Example 3.3 *In the previous example, f satisfies the condition of $(P.S)$ at level $c = 1$ but not at level $c = 0$.*

3.6.2 Ekeland's Lemma and consequences

In this section we present a very important result based on the concept of weakly lower semi-continuity and the boundedness from below of a functional defined on a complete metric space. This result is known as Ekeland's variational principle (see [6]), which has many applications in the field of mathematical analysis.

Lemma 3.1 ([5, 6]) *Let (E, d) be a complete metric space and $J : E \to \mathbb{R}$ be a lower semi-continuous function and bounded from below. we set $m = \inf_{u\in E} J(u)$. Then for all $\epsilon > 0$, there exists $u_\epsilon \in E$ such that*

$$\begin{cases} m \leq J(u_\epsilon) < m + \epsilon, \\ \forall u \in E : u \neq u_\epsilon, \\ J(u) - J(u_\epsilon) + \epsilon d(u, u_\epsilon) > 0. \end{cases}$$

Corollary 3.4 *Let E be a Banach space and $J \in \mathcal{C}^1(E, \mathbb{R})$. Assume that*

1. *J is bounded from below.*
2. *J satisfies the Palais-Smale Condition at level $m = \inf_{u\in E} J(u)$.*

Then J achieves its minimum c, therefore, J admits at least a critical point.

Proof.
We reformulate the demonstration as an exercise:

Exercice 3.13 1. *Using Ekeland Lemma, show that there exist a subsequence $(u_n) \subset E$ such that*

$$\begin{cases} m \leq J(u_n) < m + \frac{1}{n}, \\ \forall v \in E : J(v) + \frac{1}{n} \|v - u_n\| \geq J(u_n). \end{cases}$$

2. *Show that* $\|J'(u_n)\|_{E'} \leq \frac{1}{n}$ *and then* $J'(u_n) \to 0$ *in* E' *as* $n \to +\infty$.

3. *Deduce that there exist* $u_0 \in E$ *such that :*

$$\begin{cases} J(u_0) = m, \\ J'(u_0) = 0. \end{cases}$$

Solution:

1. Since $J \in \mathcal{C}^1(E, \mathbb{R})$ and bounded from below, we can apply Ekeland Lemma with $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}^*$, then there exist $u_n \in E$ such that:

$$\begin{cases} m \leq J(u_n) < m + \frac{1}{n}, \\ \forall v \in E : J(v) + \frac{1}{n} \|v - u_n\| \geq J(u_n). \end{cases}$$

2. Let $w \in E$ such that $\|w\|_E = 1$, we take $v = u_n + tw \in E$:

$$J(u_n + tw) + \frac{1}{n} \|w\|_E \geq J(u_n),$$

then,

$$\lim_{t \to 0} \frac{J(u_n + tw) - J(u_n)}{t} \geq \lim_{t \to 0} -\frac{1}{n}$$

then,

$$\langle J'(u_n, w) \rangle \geq -\frac{1}{n}$$

We can be replaced w by $-w$, we obtain

$$\langle J'(u_n, -w) \rangle \geq -\frac{1}{n}$$

then,

$$-\frac{1}{n} \leq \langle J'(u_n, w) \rangle \leq \frac{1}{n}, \text{ which means } |\langle J'(u_n, w) \rangle| \leq \frac{1}{n}.$$

then,

$$\sup_{\|w\|_E = 1} |\langle J'(u_n, w) \rangle| \leq \frac{1}{n}.$$

This means that,

$$\|J'(u_n)\|_{E'} \leq \frac{1}{n}, \quad \forall w \in E$$

we deduce that,

$$\lim_{n \to \infty} \|J'(u_n)\|_{E'} = 0$$

So,

$$J'(u_n) \to 0 \text{ in } E', \qquad \text{as } n \to +\infty.$$

3. We Know that

$$\begin{cases} J'(u_n) \to m, \\ J'(u_n) \to 0, \text{ in } E' \end{cases}$$

therefore, $\{u_n\}$ is a $(P.S)_m$ sequence. Since J satisfies the $(P.S)_m$ Condition, we deduce that there exist a subsequence $\{u_{n_k}\} \subset E$ and $u_0 \in E$ such that:

$$u_{n_k} \to u_0, \text{ in } E.$$

Since J and J' are continuous, we have

$$J(u_{n_k}) \to J(u_0), \text{ in } E.$$

By the uniqueness of limit we have $J(u_0) = m$ and $J'(u_0) = 0$. So, J achieves it is minimum and admit a critical point.

■

Application

We consider the following problem :

$$\begin{cases} -u'' + g(x,u) = f(x) & \text{in}(a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{3.53}$$

where $f \in L^2(a,b)$ and g is a caratheodory function (measurable to first variable and continuous to second variable), furthermore,

(g_1) $|g(x,s)| \leq b_0 + b_1 |s|^p$, $p > 1, b_0$ and $b_1 > 0$.

(g_2) $G(x,s) \geq 0$, $\forall s \in \mathbb{R}$, a.e $x \in (a,b)$, where

$$G(x,t) = \int_0^t g(x,s)\ ds$$

Definition 3.11 *We say that $u \in H_0^1(a,b)$ is a weak solution of Problem ((3.53)), if and only if*

$$\int_a^b u'v'\ dx + \int_a^b g(x,u)v\ dx - \int_a^b f(x)v\ dx = 0.$$

for any $v \in H_0^1(a,b)$.

Theorem 3.14 *Under the hypotheses (g_1) and (g_2), Problem (3.53) admits at least weak solution in $H_0^1(a,b)$.*

Proof. Let us consider the functional $J : H_0^1(a,b) \to \mathbb{R}$, given by

$$J(u) = \frac{1}{2}\int_a^b u'^2\ dx - \frac{1}{2}\int_a^b G(x,u)\ dx - \int_a^b f(x)u\ dx.$$

By (g_1), obviously that $J \in \mathcal{C}^1(H_0^1(a,b), \mathbb{R})$ and we have:

$$\int_a^b u'v'\ dx + \int_a^b g(x,u)v\ dx - \int_a^b f(x)v\ dx = 0.$$

Therefore, u is critical point of J if and only if u is a weak solution of Problem (3.53). Now, we will apply Corollary 3.4 which requires that we show that J is bounded from below and satisfies (P.S) condition. Indeed,

- J is bounded from below: using (g_2) and Cauchy-Schwarz inequality we get,

$$\begin{aligned} J(u) &= \frac{1}{2}\int_a^b |u'|^2 \ dx - \frac{1}{2}\int_a^b G(x,u)\ dx - \int_a^b f(x)u\ dx, \\ &\geq \frac{1}{2}\|u\|_{H_0^1}^2 - C_\Omega \|f\|_{L^2}\|u\|_{H_0^1}, \end{aligned} \tag{3.54}$$

then J is bounded from below.

- J satisfies the (P.S) condition: Let $\{u_n\} \subset H_0^1(a,b)$ be a (P.S) sequence i.e,

$$J(u_n)\text{is bounded in}\mathbb{R},\text{and}J'(u_n) \to 0in \left(H_0^1(a,b)\right)^*,$$

as $n \to \infty$.

Step 1: we show that $\{u_n\}$ is bounded in $H_0^1(a,b)$; from (3.54)

$$J(u_n) \geq \|u_n\|_{H_0^1}^2 - C_\Omega \|f\|_{L^2}\|u_n\|_{H_0^1}, \tag{3.55}$$

the convergence of the sequence $(J(u_n))$ implies that there exists $c > 0$ such that $|J(u_n)| \leq c$ for any $n \in \mathbb{N}$, so by (3.55) we deduce that

$$\|u_n\|_{H_0^1}^2 - C_\Omega \|f\|_{L^2}\|u_n\|_{H_0^1} \leq c,$$

If we suppose that (u_n) is unbounded, then there exists a subsequence (u_{n_k}) such that

$$\lim_{nk \to +\infty} \|u_{n_k}\|_{H_0^1} = +\infty, \tag{3.56}$$

i.e,

$$\lim_{n_k \to +\infty} \|u_{n_k}\|_{H_0^1}^2 - C_\Omega \|f\|_{L^2}\|u_{n_k}\|_{H_0^1} = +\infty,$$

we have a contradiction with (3.56), then $\{u_n\}$ bounded in $H_0^1(a,b)$.

Step 2: Since (u_n) is bounded in $H_0^1(a,b)$, which is reflexive space,so there exist a subsequence denoted by $\{u_n\}$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad H_0^1(a,b) \tag{3.57}$$

Thanks to the embedding $H_0^1(a,b) \hookrightarrow\hookrightarrow C([a,b])$ is compact, we deduce that

$$u_n \to u \quad \text{in } C[a,b], \tag{3.58}$$

which means that $\sup_{x\in[a,b]} |u_n - u| \to 0$ as $n \to +\infty$. This implies that there exists $M > 0$ such that

$$|u_n(x)| \leq M, \quad \forall x \in [a,b]. \tag{3.59}$$

Step 3: We have

$$\langle J'(u_n) - J'(u), u_n - u\rangle \to 0, \text{ as } n \to +\infty, \tag{3.60}$$

Indeed, since $u_n \rightharpoonup u$ weakly in $H_0^1(a,b)$ and $J'(u_n) \to 0$ in $(H_0^1(a,b))^*$, we obtain

$$\langle J'(u_n), u_n - u\rangle \to 0, \text{ as } n \to +\infty,$$

Next, since $u_n \rightharpoonup u$ weakly in $H_0^1(a,b)$ and $J'(u) \in (H_0^1(a,b))'$, we get

$$\langle J'(u), u_n - u\rangle \to 0, \text{ as } n \to +\infty.$$

On the other hand, we have

$$\begin{aligned}\langle J'(u_n) - J'(u), u_n - u\rangle &= \langle J'(u_n), u_n - u\rangle - \langle J'(u), u_n - u\rangle \\ &= \|u_n - u\|_{H_0^1}^2 + \int_a^b (g(x,u_n) - g(x,u))(u_n - u)\ dx. \quad (3.61)\end{aligned}$$

But,

$$\int_a^b (g(x,u_n) - g(x,u))\,|u_n - u|\ dx \to 0, \tag{3.62}$$

as $n \to +\infty$. Indeed, By a simple estimating we have

$$\begin{aligned}\left|\int_a^b (g(x,u_n) - g(x,u))(u_n - u)\ dx\right| &\leq \int_a^b |g(x,u_n) - g(x,u)|\,|u_n - u|\ dx \\ &\leq \sup_{x\in[a,b]} |u_n - u| \int_a^b |g(x,u_n) - g(x,u)|\ dx\end{aligned}$$

Based on the continuity of $\mathbb{R} \ni t \mapsto g(x,t)$, the hypothesis (g_2) and (3.59) and applying the dominated convergence theorem we obtain

$$\int_a^b |g(x,u_n) - g(x,u)|\ dx \to 0, \tag{3.63}$$

as $n \to +\infty$. Consequently, from (3.58) and (3.63) we deduce that (3.62) is established.

Combining (3.60), (3.61) and (3.62) we deduce that

$$\|u_n - u\|_{H_0^1}^2 \to 0,$$

as $n \to +\infty$. Hence, J satisfies the (P.S) condition.

As conclusion, we have $J \in \mathcal{C}^1(\Omega, \mathbb{R})$, bounded from below and J satisfies the (P.S) condition, then J admits at least critical point which leads to the existence at least of a weak solution of Problem (3.53) in $H_0^1(a,b)$. ■

3.7 Exercises

Exercise 1:

Given a continuous function $h : [0,1] \to \mathbb{R}$, consider the boundary value problem

$$\begin{cases} -u'' + u^3 = h(t), & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

and the functional

$$\varphi(u) = \int_0^1 \left[\frac{1}{2}|u'|^2 + \frac{1}{4}u^4 - hu\right] dt$$

defined on the Sobolev space $H_0^1(0,1)$.