Example 2.8 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open set. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfies the condition (2.3). We set $F(t) = \int_0^t f(s)\, ds$ and we consider $J : H_0^1(\Omega) \to \mathbb{R}$ defined by*

$$J(u) = \int_\Omega F(u(x))\, dx.$$

Then J is is Fréchet differentiable and

$$\langle J'(u), v\rangle = \int_\Omega f(u(x))v(x)\, dx.$$

For details see Exercise 5-section2.

2.4 Exercises

Exercise 1:

Let E be a Banach space and let

$$a : E \times E \to \mathbb{R}$$

be a continuous bilinear form. Denote by $J : E \to \mathbb{R}$ the associated quadratic form given by

$$J(u) = a(u, u).$$

1. Show that J is differentiable on E and

$$J'(u)v = a(u, v) + a(v, u), \quad \text{or} \quad J'(u) = a(u, \cdot) + a(\cdot, u)$$

for all $u, v \in X$.

2. Suppose that a is symmetric, calculate J'.

3. Assume that $(E, \langle\cdot,\cdot\rangle)$ a Hilbert space and $J(u) = \langle u, u\rangle = \|u\|^2$. Deduce that the square of the norm is everywhere differentiable and calculate J'.

Exercise 2:

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Show that the functional $J(u) = \|u\|$ is differentiable at every $u \neq 0$ with

$$J'(u) = \langle \frac{u}{\|u\|}, \cdot\rangle,$$

and J is not differentiable at zero.

Exercise 3:(Properties of differentiable functionals)

Assume that I and J are differentiable at $u \in U \subseteq E$. Then the following properties hold:

1. If a and b are real numbers, $aI + bJ$ is differentiable at u and

$$(aI + bJ)'(u) = aI'(u) + bJ'(u);$$

2. The product IJ is differentiable at u and

$$(IJ)'(u) = J(u)I'(u) + I(u)J'(u);$$

3. If $\gamma : \mathbb{R} \to U$ is differentiable at t_0 and $u = \gamma(t_0)$, then the composition $\eta : \mathbb{R} \to \mathbb{R}$ defined by $\eta(t) = I(\gamma(t))$ is differentiable at t_0 and

$$\eta'(t_0) = I'(u)\gamma'(t_0);$$

4. If $A \subseteq \mathbb{R}$ is an open set, $f : A \to \mathbb{R}$ is differentiable at $I(u) \in A$, then the composition $K(u) = f(I(u))$ is defined in an open neighborhood V of u, is differentiable at u and

$$K'(u) = f'(I(u))I'(u).$$

Exercise 4:

Consider functions $a_{ij} \in L^\infty(\Omega)$ for $i, j \in \{1, \dots, N\}$, and define the $N \times N$ matrix $A(x) = (a_{ij}(x))_{ij}$. Take a function $q \in L^\infty(\Omega)$ and define a map $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as

$$a(u, v) = \int_\Omega (A(x)\nabla u(x)) \cdot \nabla v(x)\, dx + \int_\Omega q(x)u(x)v(x)\, dx.$$

1. Verify that a is a continuous bilinear form and that if $A(x)$ is a symmetric matrix, then a is also symmetric.

2. We define

$$J(u) = a(u, u) = \int_\Omega (A(x)\nabla u(x)) \cdot \nabla u(x)\, dx + \int_\Omega q(x)u^2(x)\, dx.$$

   Show that J is differentiable on $H^1(\Omega)$ and we have

$$\langle J'(u), v\rangle = 2a(u, v).$$

3. Define $L : H^1(\Omega) \setminus \{0\}$ as

$$L(u) = \frac{J(u)}{\int_\Omega u^2\, dx}.$$

   Show that L is differentiable and

$$\langle L'(u), v\rangle = \frac{1}{\int_\Omega u^2\, dx}\Big(\int_\Omega (A(x)\nabla u) \cdot \nabla v\, dx + \int_\Omega q(x)uv\, dx - L(u) \int_\Omega uv\, dx\Big).$$

Exercise 5: Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set with a regular boundary. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose there exist $a, b > 0$ such that:

$$|f(t)| \leq a + b|t|^{2^*-1},$$

for all $t \in \mathbb{R}$. Recall that $2^* = \frac{2N}{N-2}$ is the critical exponent for the embedding of $H^1(\Omega)$ into $L^p(\Omega)$. We define:

$$F(t) = \int_0^t f(s)\, ds.$$

We want to study the differentiability properties of integral functionals such as $\int_\Omega F(u)\, dx$; we consider:

$$J(u) = \int_\Omega F(u)\, dx.$$

1. (a) Verify that:
$$\lim_{t\to 0}\frac{F(u+tv)-F(u)}{t}=f(u)v.$$
   (b) For fixed $t\in\mathbb{R}$, using the mean value theorem, show that there exists $c>0$ such that
$$\left|\frac{F(u+tv)-F(u)}{t}\right|\leq c(|v|+|u|^{2^*-1}v+|v|^{2^*}).$$
   (c) Verify that $|v|+|u|^{2^*-1}v+|v|^{2^*}\in L^1(\Omega)$.
   (d) Deduce that $\lim_{t\to 0}\int_\Omega \frac{F(u+tv)-F(u)}{t}\,dx = \int_\Omega f(u)v\,dx$, and hence J is Gâteaux differentiable.

2. Show that the application $J'_G : H^1(\Omega)\to \left(H^1(\Omega)\right)'$ is continuous; for this:
   (a) Take a sequence $(u_n)\subset H^1(\Omega)$ that converges strongly to $u\in H^1(\Omega)$. Verify that there exists a subsequence, also denoted by (u_n), such that:
   - $u_n\to u$ in $L^{2^*}(\Omega)$.
   - $u_n(x)\to u(x)$ almost everywhere x in Ω.
   - There exists $w\in L^{2^*}(\Omega)$ such that $|u_n(x)|\leq w(x)$ almost everywhere in Ω and for all $n\in\mathbb{N}$.

   (b) Deduce that $\lim_{n\to+\infty}|f(u_n(x))-f(u(x))|=0$ almost everywhere $x\in\Omega$, then show that:
$$|f(u_n)-f(u)|^{\frac{2^*}{2^*-1}}\leq C(1+|w|^{2^*}+|u|^{2^*})\in L^1(\Omega).$$

3. Show that:
$$\left|\langle J'_G(u_n)-J'_G(u),v\rangle\right|\leq\left(\int_\Omega |f(u_n)-f(u)|^{\frac{2^*}{2^*-1}}\right)^{\frac{2^*-1}{2^*}}\left(\int_\Omega |v|^{2^*}\right)^{\frac{1}{2^*}}.$$

4. Verify that $\|J'_G(u_n)-J'_G(u)\|_{(H^1(\Omega))'}\to 0$. What can be deduced?