Chapter 2

Differentiability of Functionals and Critical Points

In what follows, we present the two principal definitions of differentiability and their main properties, we start with the directional derivative.

2.1 Gâteaux differentiability

Definition 2.1 *Let E be a Banach space, $\Omega \subset E$ an open set, and let $I : \Omega \longrightarrow \mathbb{R}$ be a functional, we say that I is Gâteau differentiable (G-differentiable) at $u \in \Omega$, if there exists $A \in E'$ (A linear and continuous), denoted by $I'_G(u)$ such that, for all $v \in E$,*

$$\lim_{t \to 0} \frac{I(u+tv) - I(u)}{t} = \langle A, v \rangle, \tag{2.1}$$

If I is Gâteaux differentiable at u, there exists only one linear functional $A \in E'$ satisfying (2.1). It is called the Gâteaux differential of I at u and is denoted by $I'_G(u)$.

Definition 2.2 *If the functional I is Gâteaux differentiable at every u of an open set $U \subset E$, we say that I is Gâteaux differentiable on U. The map*

$$\begin{array}{rcl} I'_G : E & \to & E' \\ u & \mapsto & I'_G(u) \end{array}$$

is called the Gâteaux derivative of I.

Example 2.1 *Let $1 < p < \infty$, the functional*

$$\begin{array}{rcl} I : L^p(\Omega) & \to & \mathbb{R} \\ u & \mapsto & I(u) = \int_\Omega |u|^p \, dx \end{array}$$

is Gâteaux differentiable and we have

$$\langle I'_G(u), v \rangle = p \int_\Omega |u|^{p-2} uv dx.$$

Indeed; let $v \in L^p(\Omega)$, *we have*

$$\lim_{t\to 0}\frac{I(u+tv)-I(u)}{t}=\lim_{t\to 0}\frac{\int_\Omega |u+tv|^p\,dx-\int_\Omega |u|^p\,dx}{t}.$$

We define

$$\begin{aligned} g_{u,v}:[0,t] &\to \mathbb{R}\\ s &= g_{u.v}(s)=|u+sv|^p. \end{aligned}$$

since $g_{u,v}$ *is continuous on* $[0,t]$ *and differentiable on* $]0,t[$, *then according to the mean value theorem, there exists a real* $c_t \in]0,t[$ *such that*

$$g_{u,v}(t)-g_{u,v}(0)=g'_{u,v}(c_t)t$$

or,

$$|u+tv|^p-|u|^p=p\,|u+c_tv|^{p-2}\,(u+c_tv)vt,$$

when $t\to 0$, *we have* $c_t\to 0$, *so*

$$\begin{aligned}\lim_{t\to 0}\frac{|u+tv|^p-|u|^p}{t} &= \lim_{c_t\longrightarrow 0} p|u+c_tv|^{p-2}(u+c_tv)v\\ &= p|u|^{p-2}uv. \end{aligned}\tag{2.2}$$

Let us recall by the following preliminary inequalities,

$$\begin{aligned}(a+b)^s &\le 2^{s-1}(a^s+b^s), \quad \forall s\ge 1.\\ (a+b)^s &\le a^s+b^s, \quad \forall 0\le s\le 1.\end{aligned}$$

Using (2.3), we get

$$\begin{aligned}\Big||u+c_tv|^{p-2}(u+c_tv)v\Big| &\le |u+c_tv|^{p-1}|v|\\ &\le (|u|+c_t|v|)^{p-1}|v|\\ &\le \begin{cases} 2^{p-2}(|u|^{p-1}|v|+c_t^{p-1}|v|^p), & \text{if } p\ge 2;\\ |u|^{p-1}|v|+c_t^{p-1}|v|^p, & \text{if } 1<p<2.\end{cases}\\ &\le C(|u|^{p-1}|v|+|v|^p),\end{aligned}$$

where $C=\max\{1,2^{p-2}\}$. *Now, we verify that* $(|u|^{p-1}|v|+|v|^p)\in L^1(\Omega)$; *that is, by Hölder inequality we get*

$$\int_\Omega |u|^{p-1}|v|\,dx \le \|u\|_{L^p}^{p-1}\|v\|_{L^p}<\infty$$

then $|u|^{p-1}|v|+|v|^p\in L^1(\Omega)$. *Using this and (2.2), applying Dominated Convergence Theorem, we get*

$$\lim_{t\to 0}\frac{\int_\Omega |u+tv|^p\,dx-\int_\Omega |u|^p\,dx}{t}=p\int_\Omega |u|^{p-2}uvdx.$$

Now, we define

$$\begin{aligned} A:L^p &\to \mathbb{R}\\ v &\mapsto A(v)=p\int_\Omega |u|^{p-2}uvdx.\end{aligned}$$

we prove that $A \in (L^p(\Omega))'$, *i.e.* A *is linear and continuous.*
A *is linear, that is, let* $v_1, v_2 \in L^p(\Omega)$ *and let* $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} A(\alpha v_1 + \beta v_2) &= p\int_\Omega |u|^{p-2}u(\alpha v_1 + \beta v_2)dx \\ &= p[\int_\Omega |u|^{p-2}u\alpha v_1 dx + \int_\Omega |u|^{p-2}u\beta v_2 dx] \\ &= \alpha p\int_\Omega |u|^{p-2}uv_1 dx + \beta p\int_\Omega |u|^{p-2}uv_2 dx \\ &= \alpha A(v_1) + \beta A(v_2). \end{aligned}$$

then A *is linear. For continuity of* A, *let* $u, v \in L^p(\Omega)$

$$\begin{aligned} |A(v)| = |p\int_\Omega |u|^{p-2}uvdx| &\leq p\int_\Omega |u|^{p-1}|v|dx \\ &\leq p\Big(\int_\Omega |u|^{(p-1)p'}dx\Big)^{\frac{1}{p'}}\Big(\int_\Omega |v|^p dx\Big)^{\frac{1}{p}} \\ &\leq p\Big(\int_\Omega |u|^{(p-1)(\frac{p}{p-1})}dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |v|^p dx\Big)^{\frac{1}{p}} \\ &= \|u\|_{L^p}^{p-1}\|v\|_{L^p}. \end{aligned}$$

Then A *is a continuous, so the functional* I *is G-differentiable and* $\langle I'_G(u), v\rangle = p\int_\Omega |u|^{p-2}uvdx$.

Example 2.2 *In the previous example 2.1, if we set* $p = 2$, *the functional*

$$\begin{aligned} I : L^2(\Omega) &\to \mathbb{R} \\ u &\mapsto I(u) = \int_\Omega |u|^2\, dx = \|u\|_{L^2}^2 \end{aligned}$$

is Gâteaux differentiable and we have

$$\langle I'_G(u), v\rangle = 2\int_\Omega uv\, dx.$$

Example 2.3 *The functional defined by*

$$\begin{aligned} J : W_0^{1,p}(\Omega) &\to \mathbb{R} \\ u &\mapsto J(u) = \int_\Omega |\nabla u|^p\, dx = \|u\|_{W_0^{1,p}}^2 \end{aligned}$$

is Gâteaux differentiable and we have

$$\langle J'_G(u), v\rangle = p\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\, dx.$$

In particular case $p = 2$, *the functional defined by*

$$\begin{aligned} J : H_0^1(\Omega) &\to \mathbb{R} \\ u &\mapsto J(u) = \int_\Omega |\nabla u|^2\, dx = \|u\|_{H_0^1}^2 \end{aligned}$$

is Gâteaux differentiable and we have

$$\langle J'_G(u), v\rangle = 2\int_\Omega \nabla u\nabla v\, dx.$$

Example 2.4 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open set. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. suppose that there exists $\sigma \in [1, 2^*]$, $2^* = \frac{2N}{N-2}$ is be the critical Sobolev exponent, and there exist $a, b \in \mathbb{R}$ such that*

$$|f(t)| \leq a + b|t|^{\sigma-1}, \quad \forall t \in \mathbb{R}. \tag{2.3}$$

We set $F(t) = \int_0^t f(s)\, ds$ and we consider $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega F(u(x))\, dx.$$

Then J is is Gâteaux differentiable and

$$\langle J'_G(u), v\rangle = \int_\Omega f(u(x))v(x)\, dx.$$

To facilitate understanding the steps of justifying Example 2.4, see Exercise ??.

2.2 Fréchet differentiability

Definition 2.3 *Let E be a Banach space, $\Omega \subset E$ an open set and let $I : \Omega \longrightarrow \mathbb{R}$ be a functional, we say that I is Fréchet differentiable at $u \in \Omega$, if there exists $A_u \in E'$ such that*

$$\lim_{\|v\| \to 0} \frac{I(u+v) - I(v) - A_u v}{\|v\|} = 0 \tag{2.4}$$

Thus, for a Fréchet differentiable functional I, we have

$$I(u+v) - I(u) = A_u(v) + o(\|v\|),$$

where $\lim_{\|v\|\to 0} \frac{o(\|v\|)}{\|v\|} = 0$.

If a functional I is differentiable at u, then A_u is a unique.

Definition 2.4 *Let $I : \Omega \to \mathbb{R}$ be differentiable at $u \in \Omega$. The unique element A_u such that (2.4) holds is called the Fréchet differential of I at u, and is denoted by $I'(u)$. Then we can write*

$$I(u+v) - I(u) = \langle I'(u), v\rangle + o(\|v\|)$$

as $\|v\| \to 0$.

Definition 2.5 *Let $\Omega \subset E$ be an open set.*

1. *If the functional I is differentiable at every $u \in \Omega$, we say that I is differentiable on Ω.*
2. *The map $I' : \Omega \to E'$ that sends $u \in \Omega$ to $I'(u) \in E'$ is called the Fréchet derivative of I. Note that I' is in general a nonlinear map.*
3. *If the derivative I' is continuous from Ω to E' we say that I is of class C^1 on Ω and we write $I \in C^1(\Omega)$.*

Remark 2.1

- *Notice that $I'(u)$ is defined on E, even if I is defined only in Ω.*

- *If I is Fréchet differentiable at $u \in \Omega$ then I is continuous at u.*
- *If I is Fréchet differentiable at $u \in \Omega$ then I is Gâteaux differentiable at $u \in \Omega$ and $I'(u) = I'_G(u)$.*

The converse of the third point of Remark 2.1 is not always true but we have the following result:

Proposition 2.1 *Suppose that $\Omega \subseteq E$ is on open set, such that I G-differentiable in Ω and that $I'_G : \Omega \in E \to E'$ is continuous, then I is also Fréchet differentiable at u, and we have $I'_G = I'(u)$.*

The proof of Proposition 2.1 is given in [11].

Remark 2.2 *The importance of Proposition 4.22 reside in the fact that calculate the Gâteau derivative and then to prove that it is continuous is often technically easier to directly prove the Fréchet differentiability. Also we deduce that the functional is of class C^1.*

2.3 Basic examples

Example 2.5 *We prove that the functional $J : L^p(\Omega) \to \mathbb{R}$ $(p > 1)$ defined by*

$$u \to J(u) = \int_\Omega |u|^p dx.$$

is a Fréchet differentiable on $L^p(\Omega)$. Indeed; we already prove that J is G-differentiable, with

$$\langle J'_G(u), v\rangle = p \int_\Omega |u|^{p-1} uv \, dx, \quad \forall u, v \in L^p(\Omega). \tag{2.5}$$

it remains to prove that

$$\begin{array}{rcl} J'_G : L^p(\Omega) & \to & (L^p(\Omega))' \\ u & \mapsto & J'_G(u) \end{array}$$

where $J'_G(u)$ is defined as in (2.5).

Let $(u_n) \subset L^p(\Omega)$ such that

$$u_n \to u \ \text{ in } L^p(\Omega),$$

we prove that $J'_G(u_n) \to J'_G(u)$ in $(L^p(\Omega))'$. By definition we have,

$$\|J'_G(u_n) - J'_G(u)\|_{(L^p(\Omega))'} = \sup_{\|v\|_{L^p}=1} |\langle J'_G(u_n) - J'_G(u), v\rangle|$$

let $v \in L^p(\Omega)$, such that $\|v\|_{L^p} = 1$, we have

$$\begin{array}{rcl} |\,\langle J'_G(u_n) - J'_G(u), v\rangle\,| & = & |\,\langle J'_G(u_n), v\rangle - \langle J'_G(u), v\rangle\,| \\ & = & p|\displaystyle\int_\Omega |u_n|^{p-2} u_n v dx - \int_\Omega |u|^{p-2} uv \, dx| \\ & \leq & p|\displaystyle\int_\Omega |u_n|^{p-2} u_n v dx - \int_\Omega |u|^{p-2} uv dx| \\ & \leq & p[\displaystyle\int_\Omega ||u_n|^{p-2} u_n - |u|^{p-2} u|^{\frac{p}{p-1}} dx](\int_\Omega |v|^p dx)^{\frac{1}{p}} \\ & \leq & p\displaystyle\int_\Omega ||u_n|^{p-2} u_n - |u|^{p-2} u|^{\frac{p}{p-1}} dx. \end{array}$$

We set $w_n = |u_n|^{p-2}u_n$, we know that $u_n \to u$ in $L^p(\Omega)$, there is a subsequence u_{nk} (denoted u_n), such that $u_n(x) \to u(x)$ a.e. in $]0,1[$, so we have

$$w_n(x) \to |u|^{p-2}u = w(x) \quad a.e.\ in \Omega. \tag{2.6}$$

Moreover there exists $g \in L^p$ such that

$$|u_n(x)| \le g(x)\ a.e.\ in\ \Omega.$$

Thus,

$$|w_n(x)| = |u_n(x)|^{p-1} \le |g(x)|^{p-1} \quad \in L^{\frac{p}{p-1}}(\Omega). \tag{2.7}$$

According (2.6) and (2.7), by the dominated convergence theorem we have $w_n \to w$ in $L^{\frac{p}{p-1}}(\Omega)$; that is

$$\int_0^1 ||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}} dx \to 0,$$

as $n \to +\infty$. Consequently

$$|\langle J'_G(u_n) - J'_G(u), v\rangle| \to 0.$$

as $n \to +\infty$, which means that $J'_G(u_n) \to J'_G(u)$ in $\Big(L^p(0,1)\Big)'$, so J'_G is continuous functional. In conclusion, the functional J is Fréchet differentiable and it is of class C^1.

Example 2.6 *In Example 2.5, we take the particular case $p = 2$, the functional*

$$\begin{aligned} I : L^2(\Omega) &\to \mathbb{R} \\ u &\mapsto I(u) = \int_\Omega |u|^2\, dx = \|u\|^2_{L^2} \end{aligned}$$

is Fré differentiable and we have

$$\langle I'(u), v\rangle = 2\int_\Omega uv\, dx.$$

We note that this example can be justified directly using Definition 2.4 without going through the steps followed in justifying Example 2.5.

Example 2.7 *The functional defined by*

$$\begin{aligned} J : W_0^{1,p}(\Omega) &\to \mathbb{R} \\ u &\mapsto J(u) = \int_\Omega |\nabla u|^p\, dx = \|u\|^2_{W_0^{1,p}} \end{aligned}$$

is Fréchet differentiable and we have

$$\langle J'(u), v\rangle = p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx.$$

In particular case $p = 2$, the functional defined by

$$\begin{aligned} J : H_0^1(\Omega) &\to \mathbb{R} \\ u &\mapsto J(u) = \int_\Omega |\nabla u|^2\, dx = \|u\|^2_{H_0^1} \end{aligned}$$

is Fréchet differentiable and we have

$$\langle J'(u), v\rangle = 2\int_\Omega \nabla u \nabla v\, dx.$$

Based on Example 2.3 and using the same steps in justifying example 2.5, the result of the first part example can be verified. Concerning the second part we can use Definition 2.4 directly.