1.4 Exercices

Exercise 1:

Prove that if X is reflexive, then:

1. Boundedness of sequences implies weak compactness: Any bounded sequence $\{x_n\} \subset X$ has a weakly convergent subsequence.

2. Dual space property: X^* is also reflexive.

3. Finite-dimensional case: If X is reflexive, then any closed subspace of X is also reflexive.

Exercise 2:

For $1 < p < +\infty$, let $f \in L^p[0,1]$ and $(f_n)_{n\in\mathbb{N}} \subset L^p[0,1]$. Prove that $f_n \rightharpoonup f$ weakly if and only if:

$$\int_0^x f_n(t)\, dt \to \int_0^x f(t)\, dt, \quad \forall x \in [0,1],$$

and there exists $M > 0$ such that $\int_0^1 |f_n(t)|^p\, dt \leq M$, $\forall n \in \mathbb{N}$.

Exercise 3:

Let $(X, \|\cdot\|)$ be a Banach space, and let (x_n) be a sequence in X. Show that if $x_n \to x$ in the norm topology (i.e., $\|x_n - x\| \to 0$), then x_n also converges weakly to x.

Exercise 4:

Let $C([a,b])$ denote the Banach space of continuous functions on the interval $[a,b]$ with the supremum norm, i.e., for $f \in C([a,b])$, we define

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|.$$

Let $T : C([a,b]) \to \mathbb{R}$ be the functional defined by

$$T(f) = \int_a^b |f(x)|\, dx.$$

1. Prove that T is weakly lower semi-continuous on $C([a,b])$, i.e., if $f_n \to f$ weakly in $C([a,b])$, then
$$\liminf_{n\to\infty} T(f_n) \geq T(f).$$

2. Show that T is **not** strongly continuous on $C([a,b])$. That is, find a sequence $f_n \to f$ in the supremum norm on $C([a,b])$, such that
$$\lim_{n\to\infty} T(f_n) \neq T(f).$$

Exercise 5:(Brezis-Lieb Lemma) Let (f_n) be a sequence of measurable functions on a measure space (X,μ) such that:

1. $f_n \to f$ almost everywhere as $n \to \infty$,

2. $\sup_n \int_X |f_n|^p\, d\mu < \infty$ for some $1 \leq p < \infty$.

Show that:

$$\lim_{n\to\infty} \left(\int_X |f_n|^p\, d\mu - \int_X |f_n - f|^p\, d\mu \right) = \int_X |f|^p\, d\mu.$$