Chapter 1

Basic concepts in functional analysis

1.1 Reflexive space

1.1.1 Dual space

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two Banach spaces.

Definition 1.1 *Let $A : E \to F$ be a linear operator. One says that A is bounded (or continuous) if there is a constant $c \geq 0$ such that:*

$$\|Au\|_F \leq c\|u\|_E, \quad \forall u \in E.$$

Remark 1.1

- *We denote by $\mathcal{L}(E, F)$ the space of continuous (bounded) linear operators from E into F equipped with the norm*

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{x \in E, \|x\|_E \leq 1} \|T(x)\|_F$$

- *If $E = F$, we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.*
- *Let E be a vector space over $\mathbb{R}$. We recall that **a functional** is a function defined on E, or on some subspace of E, with values in $\mathbb{R}$. We denote by E' the **dual space** of E, that is, the space of all continuous linear functionals on E; the dual norm on E (the norm in E') is defined by*

$$\|f\|_{E'} = \sup_{x \in E, \|x\|_E \leq 1} |f(x)| = \sup_{x \in E, \|x\|_E \leq 1} f(x).$$

- *When there is no confusion we shall also write $\|f\|$ instead of $\|f\|_{E'}$.*
- *Given $f \in E'$ and $x \in E$ we shall often write $\langle f, x\rangle$ instead of $f(x)$; we say that $\langle \cdot, \cdot \rangle$ is the scalar product for the duality E', E. So, we have the following inequality*

$$\forall f \in E', \forall x \in E : |\langle f, x\rangle| \leq \|f\|_{E'} \|x\|_E. \tag{1.1}$$

- *It is well known that E' is a Banach space, i.e., E' is complete (even if E is not); this follows from the fact that $\mathbb{R}$ is complete.*

- *If E is a Hilbert space, then, according to Riesz's theorem (see [4]), the spaces E and its dual E' can be identified, we write*

$$E' = E. \tag{1.2}$$

The equality (6.1) does not mean equal elements, but rather in the sense that there is a canonical isometry from E onto E'.

Example 1.1 *Let $1 < p < \infty$, we have $\left(L^p(\Omega)\right)' = L^q(\Omega)$ where $\frac{1}{p}+\frac{1}{q} = 1$. $\left(L^1(\Omega)\right)' = L^\infty(\Omega)$.*

1.1.2 Some notions of convergence

Definition 1.2 (Strong convergence) *Convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ in a normed vector space $(E, \|\cdot\|_E)$ to an element $x \in E$, defined in the following way:*

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow \|x_n - x\|_E \leq \varepsilon.$$

Or,

$$\lim_{n\to+\infty} \|x_n - x\|_E = 0.$$

We denote by $x_n \to +x$ as $n \to +\infty$.

Definition 1.3 (Weak convergence) *We say that the sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $x \in E$ if and only if*

$$\langle f, x_n\rangle \to \langle f, x\rangle, \;\; \forall f \in E'. \tag{1.3}$$

for $n \to +\infty$. This weak convergence is written as $x_n \rightharpoonup x$,

Remark 1.2

- *Note that the convergence in (1.3) is convergence for a sequence of real numbers.*
- *If $x_n \rightharpoonup x$, then $(x_n)_{n\in\mathbb{N}}$ is bounded in E; i.e.*

$$\exists M > 0 : \quad \|x_n\| \leq M.$$

Example 1.2

1. *Let $\{e_n\}_{n\in\mathbb{N}}$ be an Hilbertian base for a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then, for each $x \in H$ can be decomposed as*

$$x = \sum_{n=1}^{\infty} \langle e_n, x\rangle e_n$$

Or

$$\|x - \sum_{n=1}^{m} \langle e_m, x\rangle e_m\| \to 0,$$

as $m \to +\infty$. From this it follows that $\langle e_n, x\rangle \to 0 = \langle 0, x\rangle$ for any $x \in H' = H$, this means that $e_n \rightharpoonup 0$ in H.

On the other hand, we have $\|e_n - 0\| = \|e_n\| = 1$, so $\{e_n\}$ does not converge. We deduce that $\{e_n\}_n$ is weakly convergent but it is not strongly convergent.

2. *Let $\Omega \subset \mathbb{R}^N$ an open set, let $1 < p < \infty$. A sequence $u_n \rightharpoonup u$ in $L^p(\Omega)$ means that*

$$\int_\Omega u_n v \, dx \to \int_\Omega uv \, dx, \quad \forall \in v \in L^q(\Omega),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of functions defined by*

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{n}}, & \text{if } x \in [n, 2n]; \\ 0, & \text{if not.} \end{cases}$$

We can verify that $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to 0 in $L^2([0; +\infty[)$ but does not converge strongly in $L^2([0; +\infty[)$.

Definition 1.4 (Bidual space) *Let $(E, \|\cdot\|)$ be a Banach space and let E' be the dual space. The bidual E'' is the dual of E' with norm*

$$\|f\|_{E''} = \sup_{g\in E', \|g\|_{E'}\leq 1} |\langle f, g\rangle|, \quad \forall f \in E''.$$

There is a canonical injection $i : E \to E''$ defined as follows:

$$\begin{array}{rcl} i : E & \to & E'' \\ x & \mapsto & ix, \end{array}$$

where ix is defined as:

$$\begin{array}{rcl} ix : E' & \to & \mathbb{R} \\ f & \mapsto & \langle ix, f\rangle_{E'',E'} = \langle f, x\rangle_{E',E}, \end{array}$$

It is clear that i is linear and that i is an isometry (see Corollary 1.4. in [**?**]), that is,

$$\|ix\|_{E''} = \|x\|_E.$$

Definition 1.5 (Reflexive space) *The space E is said to be reflexive if the canonical injection $i : E \to E''$ is surjective, i.e. $i(E) = E''$.*

When E is reflexive, E'' is usually identified with E; $(E'' = E)$.

Example 1.3 1. *finite dimensional spaces are reflexive (since $\dim E = \dim E' = \dim E''$).*

2. *L^p spaces are reflexive for $1 < p < \infty$. L^1 and L^∞ are not reflexive.*

3. *Hilbert spaces are reflexive.*

The following result shows an important compactness property in reflexive spaces (see [4])

Theorem 1.1 *Assume that E is a reflexive Banach space and let (x_n) be a bounded sequence in E. Then there exists a subsequence (x_{n_k}) that converges weakly to some x in E.*

1.2 Operators on Banach spaces

1.2.1 Continuity, convexity and coercivity of operators

In the following, we state some definitions concerning continuity properties.

Definition 1.6 *The operator $T : E \to F$ is called continuous at the point $x_0 \in E$ if for any sequence $(x_n) \subset E$ we have*

$$x_n \to x_0 \ \text{ in } \ E \ \ \Rightarrow \ \ T(x_n) \to T(x_0) \ \text{ in } F$$

Or,

$$\|x_n - x_0\|_E \to 0 \Rightarrow \|T(x_n) - T(x_0)\|_F \to 0$$

This definition is equivalent to:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in E : \|x - x_0\|_E \leq \delta \Rightarrow \|T(x) - T(x_0)\|_F \leq \varepsilon.$$

Example 1.4 *Let $(E, \|\cdot\|)$ be a Banach space. We consider $N : E \to \mathbb{R}$; $N(x) = \|x\|$. N is a continuous functional. Indeed, for any $(x_n)_n \subset E$ such that $x_n \to x$ in E. Since*

$$\Big|N(x_n) - N(x)\Big| = \Big|\|x_n\| - \|x\|\Big| \leq \|x_n - x\|,$$

and $\|x_n - x\| \to 0$, then we have $N(x_n) \to N(x)$ in $\mathbb{R}$.

There are some other definitions that are related to the concept of weak convergence.

Definition 1.7 (Weakly continuous) *Let $T : E \to F$ be an operator. T is called weakly continuous at $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have*

$$x_n \rightharpoonup x_0 \quad \text{in } \ E \Rightarrow T(x_n) \rightharpoonup T(x_0).$$

Remark 1.3 *For functionals defined on E, the concepts of convergence and weak convergence coincide. So the functional $T :\to \mathbb{R}$ is called weakly continuous if for every sequence $(x_n) \subset E$, we have*

$$x_n \rightharpoonup x_0 \quad \text{in } \ E \Rightarrow T(x_n) \to T(x_0) \quad \text{in} \mathbb{R}.$$

Note that the concept of weak continuity is stronger than the concept of continuity because the convergence implies the weak convergence.

Definition 1.8 (Compact operator) *An operator $T : E \to F$ is called* compact *if it is continuous and $T(B)$ is relatively compact set in F (i.e. $\overline{T(B)}$ is compact set in F); here $B = \{x \in E : \|x\|_E < 1\}$.*

There is an equivalent definition using sequences.

Definition 1.9 *An operator $T : E \to F$ is called* compact *if it is continuous and if every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset E$ has a subsequence (x_{n_k}) for which $\Big(T(x_{n_k})\Big)$ converges.*

Remark 1.4 *If T is a linear operator then the requirement that T is continuous on E, which then equivalent that T is bounded operator, is superfluous.*

The compact operators defined on reflexive spaces have an important property that appears in the following theorem.

Theorem 1.2 *Let E be a reflexive Banach space and F a Banach space. If $T : E \to F$ is linear and compact. Then we have*

$$x_n \rightharpoonup x_0 \;\; in \;\; E \Rightarrow T(x_n) \to T(x_0) \;\; in \; F$$

Proof. See Proof of Theorem 1.2.11 in [11]. ■

In the search for the global minimum, which we will discuss in Chapter 3, a related concept is convexity and coercivity.

Definition 1.10 1. *We say that a part K of E is convex if*

$$\forall x, y \in K, \quad \forall \theta \in [0,1], \quad \theta x + (1-\theta)y \in K.$$

2. *When K is convex and $J : K \longrightarrow \mathbb{R}$ is a functional. We say that J is convex if:*

$$\forall x, y \in K, \quad \forall \theta \in [0,1], \quad J(\theta x + (1-\theta)y) \leq \theta J(x) + (1-\theta)J(y).$$

We say that J is strictly convex if:

$$\forall x, y \in K, \quad avec \quad x \neq y \quad \forall \theta \in]0,1[, \quad J(\theta x + (1-\theta)y) < \theta J(x) + (1-\theta)J(y).$$

3. *We say that J is (strictly) concave if $-J$ is (strictly) convex.*

Definition 1.11 (Coercivity) *A functional $J : E \to \mathbb{R}$ on a Banach space E is called coercive if, for every sequence $(u_n)_{n\in\mathbb{N}} \subset E$,*

$$\lim_{\|u_n\| \to +\infty} J(u_n) = +\infty.$$

1.2.2 Lower and Upper Semi-continuity

The following concepts are often found in minimization theory of functionals.

Definition 1.12 1. *A functional $J : E \to \mathbb{R}$ is called* lower semi-continuous *at a point $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have:*

$$x_n \to x_0 \Rightarrow f(x_0) \leq \liminf_{n\to+\infty} f(x_n).$$

2. *$J : E \to\to \mathbb{R}$ is called* weakly lower semi-continuous ***(w.l.s.c)*** *at a point $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have:*

$$x_n \rightharpoonup x_0 \Rightarrow f(x_0) \leq \liminf_{n\to+\infty} f(x_n).$$

3. *$J : E \to \mathbb{R}$ is called* upper semi-continuous *at a point $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have:*
$$x_n \to x_0 \Rightarrow f(x_0) \geq \limsup_{n\to+\infty} f(x_n).$$

4. *$J : E \to \mathbb{R}$ is called* weakly upper semi-continuous ***(w.u.s.c)*** *at a point $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have:*
$$x_n \rightharpoonup x_0 \Rightarrow f(x_0) \geq \limsup_{n\to+\infty} f(x_n).$$

Example 1.5 *Let $(H, \langle \cdot, \cdot \rangle$ be a Hilbert space. The functional $N : E \ni x \mapsto N(x) = \|x\|$, where $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. We have N is w.l.s.c. Indeed; let $(x_n) \subset H$ and $x_0 \in H$ such that $x_n \rightharpoonup x_0$, then*
$$\langle x_n, y\rangle \to \langle x, y\rangle, \forall y \in H' = H,$$
in particular, taking $y = x_0 \in H$, we get
$$\langle x_n, x_0\rangle \to \langle x_0, x_0\rangle = \|x_0\|^2. \tag{1.4}$$
By a direct calculation, we get
$$\begin{aligned} \|x_n - x_0\|^2 &= \langle x_n - x_0, x_n - x_0\rangle \\ &= \|x_n\|^2 - 2\langle x_n, x_0\rangle + \|x_0\|^2. \end{aligned}$$
Since $\|x_n - x_0\|^2 \geq 0$, we deduce that
$$\|x_n\|^2 \geq 2\langle x_n, x_0\rangle - \|x_0\|^2. \tag{1.5}$$
Passing to the limit in (1.5), and using (1.4), we get
$$\liminf_{n\to+\infty} \|x_n\|^2 \geq 2\|x_0\|^2 - \|x_0\|^2 = \|x_0\|^2.$$
which implies that N is w.l.s.c.

1.3 Functional spaces

In this section, we briefly mention some spaces and their most important properties and theories, which we will need in the following lessons. For more details see e.g [1].

1.3.1 L^p Spaces

Definition 1.13 *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we set*
$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N ; f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$
with
$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, dx \right]^{\frac{1}{p}}.$$

Definition 1.14 *We set*

$$L^{\infty}(\Omega)=\Big\{f:\Omega\to\mathbb{R}^N; f \textit{is measurable and } |f(x)|\leq c \;\; a.e.on \;\; \Omega\Big\},$$

with

$$\|f\|_{L^{\infty}}=\|f\|_{\infty}=inf\left\{c:|f(x)|\leq c \;\; a.e.on \;\; \Omega\right\},$$

with c is a constant.

If $f\in L^{\infty}(\Omega)$ then we have $|f(x)|\leq\|f\|_{\infty}$ a.e. in $x\in\Omega$.

Theorem 1.3 ([4]) *$(L^p(\Omega),\|\cdot\|_{L^p})$ is a Banach space for any $1\leq p\leq\infty$, and reflexive for $1<p<\infty$, and separable for $1\leq p<\infty$.*

Notation 1.4 *Let $1\leq p\leq\infty$, we denote by p' the conjugate exponent*

$$\frac{1}{p}+\frac{1}{p'}=1.$$

Theorem 1.5 (Hölder's inequality) *Assume that $f\in L^p$ and $g\in L^p$ with $1\leq p\leq\infty$, then $fg\in L^1$ and*

$$\left|\int fg\,dx\right|\leq\|f\|_{L^p}\|g\|_{L^{p'}}.$$

1.3.2 Some convergence criteria

Theorem 1.6 *(Lebesgue's Dominated Convergence Theorem) Let (f_n) be a sequence of functions of L^1. We suppose that*

1. *$f_n(x)\longrightarrow f(x)$ a.e. $x\in\Omega$,*
2. *there exists a function $g\in L^1$ such that for every n, $|f_n(x)|\leq g(x)$ a.e.in $x\in\Omega$.*

We have $f\in L^1(\Omega)$ and $\|f_n-f\|_{L^1}\longrightarrow 0$.

It is often useful to know a partial converse of the dominated convergence theorem.

Theorem 1.7 *Let (f_n) be a sequence in L^p and let $f\in L^p$ be such that $\|f_n-f\|_{L^p}\longrightarrow 0$. Then, there exist a subsequence (f_{n_k}) and a function $h\in L^p$ such that*

1. *$f_{n_k}(x)\longrightarrow f(x)$ a.e.on Ω,*
2. *$|f_{n_k}(x)|\leq h(x)$ $\forall k$, a.e.in Ω.*

Remark 1.5 *It should be noted that in Theorem 1.7 we only assert the existence of a subsequence converging almost everywhere, and we can give examples of sequences converge in $L^p(\Omega)$ which do not converge almost everywhere.*

Lemma 1.1 (Fatou Lemma) *Let (f_n) be a sequence of functions in L^1 that satisfy*

1. *for all $n\in\mathbb{N}$, $f_n\geq 0$ a.e.*

2. $\sup_n \int f_n < \infty$.

For almost $x \in \Omega$ *we set* $f(x) = \liminf_{n\to+\infty} f_n(x)$. *Then* $f \in L^1$ *and*

$$\int f \leq \liminf_{n\to+\infty} \int f_n.$$

Lemma 1.2 (Brezis-Lieb) *Let* $1 \leq p < \infty$ *and let* (f_n) *be a bounded sequence in* $L^p(\Omega)$ *converges a.e. to* f. *Then* $f \in L^p(\Omega)$ *and*

$$\|f\|^p_{L^p} = \lim_{n\to+\infty}(\|f_n\|^p_{L^p} + \|f_n - f\|^p_{L^p}).$$

An immediate corollary of this result is the following:

Corollary 1.1 *Let* $1 \leq p < \infty$ *and let* $f \in L^p(\Omega)$ *and* (f_n) *be a sequence in* $L^p(\Omega)$. *Assume that*

$$f_n(x) \to f(x) \quad a.e. x \in \Omega \quad and \quad \lim_{n\to+\infty} \|f_n\|_{L^p} = \|f\|_{L^p}.$$

Then

$$\lim_{n\to+\infty} \|f_n - f\|_{L^p} = 0.$$

Now here is a link between convergence almost everywhere and weak convergence in L^p spaces.*

Lemma 1.3 *Let* $1 < p < +\infty$ *and let* (f_n) *is a bounded sequence in* $L^p(\Omega)$ *converging a.e. to* f. *Then* $f_n \rightharpoonup f$ *weakly in* $L^p(\Omega)$.

1.3.3 $W^{1.p}(\Omega)$ Spaces

Let $\Omega \in \mathbb{R}^N$ be an open set, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.15 *We denote by* $\mathcal{D}(\Omega)$ *the set of function of class* $\mathcal{C}^\infty(\Omega)$ *with support compact include in* Ω. *The Sobolev space* $W^{1.p}(\Omega)$ *is defined by*

$$W^{1.p}(\Omega) = \left\{u \in L^p(\Omega), \exists g_i \in L^p(\Omega) \text{such that:} \int_\Omega u\frac{\partial \varphi}{\partial x_i}dx = -\int_\Omega g_i \varphi dx, \forall \varphi \in \mathcal{D}(\Omega), \forall i = 1, 2, ..., N\right\}.$$

We set

$$H^1(\Omega) = W^{1.2}(\Omega).$$

For $u \in W^{1.p}(\Omega)$ *we set* $\frac{\partial u}{\partial x_i} = g_i$, *and we write*

$$\nabla u = grad\ u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}).$$

The space $W^{1.p}$ *is equipped with the norm* $\|u\|_{W^{1.P}} = \|u\|_{L^p} + \|\nabla u\|_{L^P}$, *or sometimes with the equivalent norm* $\|u\|_{W^{1.p}} = (\|u\|^p_{L^p} + \|\nabla u\|^p_{L^p})^{\frac{1}{p}}$ *if* $(1 \leq p < \infty)$.
The space $H^1(\Omega)$ *is equipped with the scalar product*

$$\int_\Omega uv dx + \int_\Omega \nabla u \nabla v dx.$$

The associated norm $\|u\|_{H^1} = (\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2})^{\frac{1}{2}}$ *is equivalent to the* $W^{1.2}$ *norm.*

Proposition 1.1 *[4].*$W^{1.p}(\Omega)$ *is a Banach space for* $1 \leq p \leq \infty$, *reflexive for* $1 < p < \infty$, *and separable for* $1 \leq p < \infty$.
In particular $H^1(\Omega)$ *is reflexive, separable and Hilbert space.*

1.3.4 Sobolev Embeddings

Let $(E, \|.\|_E), (F, \|.\|_F)$ Banach spaces

Notation 1.8

1. E is injected continuously into F, means that the canonical injection $j : E \to F$ is continuous i.e, $\exists c > 0, \forall x \in E : \|x\|_F \leq c\|x\|_E$, and we denote by $E \hookrightarrow F$.

2. E is injected in compact into F means that the canonical injection $j : E \to F$ is compact i.e for all sequence bounded u_n in E we can extract subsequence u_{nk} convergent in F, and we denote by $E \hookrightarrow_c F$.
   If $1 \leq p < \infty$, the Sobolev exponent of p defined by $p^* = \frac{Np}{N-p}$ or $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

Theorem 1.9 *Let $1 \leq p \leq \infty$, we suppose that Ω is on open set of class $\mathcal{C}^1$ a bounded frontier or else that $\Omega = \mathbb{R}^N_+$. We have:*

1. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*] \quad$ *if* $\; p < N$.

2. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, \infty[\quad$ *if* $\; p = N$.

3. $W^{1.p}(\Omega) \hookrightarrow L^\infty(\Omega) \quad$ *if* $\; p > N$.

Theorem 1.10 (Rellich-Kondrachon) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1. Then we have the following compact embeddings:*

1. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1.p^*) \quad$ *if* $\; p < N$.

2. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1, \infty) \quad$ *if* $\; p = N$.

3. $W^{1.p}(\Omega) \subset \mathcal{C}\left(\overline{\Omega}\right) \quad$ *if* $\; p > N$.

1.3.5 $W_0^{1.p}(\Omega)$ Space

Definition 1.16 *.Let $1 \leq p \leq \infty$, $W_0^{1.p}$ means the closing of $\mathcal{D}(\Omega)$ in $W^{1.p}$, we notice*

$$\begin{aligned} W_0^{1.p}(\Omega) &= \overline{\mathcal{D}(\Omega)}^{W^{1.p}} \\ &= \left\{u \in W^{1.p}(\Omega) : u = 0 \; sur \; \partial\Omega\right\}, \end{aligned}$$

and

$$H_0^1(\Omega) = W_0^{1.2}(\Omega).$$

The space $W_0^{1.p}(\Omega)$ provided with norm induced by $W^{1.p}$, H_0^1 is a Hilbert space for the scalar product of H^1 we put $\|u\|_{H_0^1} = \|u\|$.

Remark 1.6 *When $\Omega = \mathbb{R}^N$, we know that $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1.p}(\mathbb{R}^N)$, and there for*

$$W_0^{1.p}(\mathbb{R}^N) = W^{1.p}(\mathbb{R}^N).$$

Proposition 1.2 (Poincaré's inequality) *Let $\Omega \subset \mathbb{R}^N$ on open set, Then there exists a constant $c > 0$ such that*

$$\|u\|_{W^{1.p}(\Omega)} \leq c\|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1.p}(\Omega).$$

In other words, on $W_0^{1.p}$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ is a norm equivalent to the $W^{1.p}$ norm.