M. I. Faculty
Department of Maths.
PDE Master I, (S2)

Introduction to Evolution Problems

Final Exam

Instructions: Clarity and logical structure will be graded. Box your final results.

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Exercise 1 (Functional Spaces)

Let $\Omega = (0, 1)$, $T > 0$, and let $X = H_0^1(\Omega)$.

1. Define the space

$$L^p(0, T; X), \qquad \text{for } 1 \le p < \infty \text{ and } p = \infty.$$

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Give the associated norms of X and $L^p(0, T; X)$ in each case.

2. Prove that

$$L^\infty(0, T; H_0^1(\Omega)) \subset L^2(0, T; L^2(\Omega)).$$

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Is the converse inclusion true? Justify. 1 1

3. For $\alpha, \beta \in \mathbb{R}$, determine necessary and sufficient conditions such that 1 1

$$t^\alpha x^\beta \in L^5(0, T; L^7(\Omega)).$$

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Exercise 2 (Heat Equation) Consider the initial-boundary value problem

$$\begin{cases} u_t - u_{xx} + b(x)u = f(x, t), & (x, t) \in (0, 1) \times (0, T), \\ u(0, t) = u(1, t) = 0, & t \in (0, T), \\ u(x, 0) = \sin(5\pi x), & x \in (0, 1), \end{cases}$$

where

$$b \in C([0, 1]), \qquad f \in L^2(0, T; L^2(0, 1)).$$

1. Derive the weak formulation in $H_0^1(0, 1)$. 2
2. Assume that $b(x) \ge C_0$ and define the bilinear form

$$B(u, v) = \int_0^1 u_x v_x, dx + \int_0^1 b(x)uv, dx.$$

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Determine a lower bound for C_0 to insure that B is coercive on $H_0^1(0, 1)$, i.e.

$$B(u, v) \ge c \int_0^1 u_x^2 dx. \quad 1$$

for some constant $c > 0$.

3. Using the test function $v = u(t)$, establish the energy identity

$$\frac{1}{2}\frac{d}{dt}|u(t)|_{L^2(0,1)}^2 + B(u(t), u(t)) = \int_0^1 f(x, t)u(x, t), dx.$$

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4. Assume that $b(x) \ge 0$ and $f = 1$. Using Poincaré inequality, prove an estimate of the form

$$|u(t)|_{L^2(0,1)}^2 \le C_2 e^{-\gamma t} + C_2. \quad 2$$

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Determine explicitly C_1, C_2 and the rate γ.

5. Assume now that

$$f = 0, \quad \text{and} \quad b(x) = \alpha > 0$$

where α is a constant.

(a) Solve the problem explicitly using separation of variables. 2

(b) Find the optimal decay rate of $E(t)$ using the series of the solution. 2

Useful facts: $ab \le (a^2 + b^2)/2$, the Poincaré's constant on $(0, 1)$ is $\frac{1}{\pi}$ and $\|u\|_{H_0^1(0,1)} = \left(\int_0^1 |u_x|^2\, dx\right)^{1/2}$.

Complete Solution

Evolution Problems – Master 1

Exercise 1 (Functional Spaces)

Let $\Omega = (0,1)$, $T > 0$, and $X = H_0^1(\Omega)$.

1. Definition of $L^p(0,T;X)$

Case $1 \le p < \infty$. We define

$$L^p(0,T;X) = \left\{ u : (0,T) \to X \text{ measurable } : \int_0^T \|u(t)\|_X^p \, dt < \infty \right\}.$$

The norm is

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p}.$$

Case $p = \infty$.

$$L^\infty(0,T;X) = \left\{ u : (0,T) \to X : \operatorname*{ess}_{t \in (0,T)} \|u(t)\|_X < \infty \right\}.$$

The norm is

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess}_{t \in (0,T)} \|u(t)\|_X.$$

2. Inclusion

We prove that

$$L^\infty(0,T;H_0^1(\Omega)) \subset L^2(0,T;L^2(\Omega)).$$

Let $u \in L^\infty(0,T;H_0^1(\Omega))$. Then there exists $M > 0$ such that

$$\|u(t)\|_{H_0^1(\Omega)} \le M \quad \text{for a.e. } t \in (0,T).$$

Using the Poincaré inequality,

$$\|u(t)\|_{L^2(\Omega)} \le \frac{1}{\pi} \|u_x(t)\|_{L^2(\Omega)} \le \frac{M}{\pi}.$$

Therefore

$$\int_0^T \|u(t)\|_{L^2(\Omega)}^2 \, dt \le \int_0^T \frac{M^2}{\pi^2} \, dt = \frac{TM^2}{\pi^2} < \infty.$$

Hence $u \in L^2(0,T;L^2(\Omega))$. Thus

$$\boxed{L^\infty(0,T;H_0^1(\Omega)) \subset L^2(0,T;L^2(\Omega)).}$$

Converse inclusion. The converse is false. Take $u(x,t) = 1$. Then

$$\|u(t)\|^2_{L^2(\Omega)} = \int_0^1 1\,dx = 1,$$

so

$$\int_0^T \|u(t)\|^2_{L^2(\Omega)}\,dt = T < \infty.$$

Hence $u \in L^2(0,T;L^2(\Omega))$. However, $u(0,t) = u(1,t) = 1 \neq 0$, therefore $u \notin H^1_0(\Omega)$. Consequently,

$$u \notin L^\infty(0,T;H^1_0(\Omega)).$$

3. Condition for $t^\alpha x^\beta \in L^5(0,T;L^7(\Omega))$

We compute

$$\|t^\alpha x^\beta\|^7_{L^7(\Omega)} = \int_0^1 t^{7\alpha} x^{7\beta}\,dx = t^{7\alpha} \int_0^1 x^{7\beta}\,dx.$$

The integral

$$\int_0^1 x^{7\beta}\,dx$$

is finite if and only if $7\beta > -1$. Hence

$$\|t^\alpha x^\beta\|_{L^7(\Omega)} = Ct^\alpha, \quad \text{where} \;\; C < \infty.$$

Now,

$$\int_0^T \|t^\alpha x^\beta\|^5_{L^7(\Omega)}\,dt = C\int_0^T t^{5\alpha}\,dt.$$

This integral is finite if and only if $5\alpha > -1$.

Therefore the necessary and sufficient conditions are

$$\boxed{\alpha > -\frac{1}{5}, \qquad \beta > -\frac{1}{7}.}$$

Exercise 2 (Heat Equation)

We consider

$$u_t - u_{xx} + b(x)u = f(x,t), \qquad (x,t) \in (0,1)\times(0,T), \tag{1}$$
$$u(0,t) = u(1,t) = 0, \tag{2}$$
$$u(x,0) = 1 - \cos(\pi x). \tag{3}$$

1. Weak formulation

Let $v \in H^1_0(0,1)$. Multiply the equation by v and integrate over $(0,1)$:

$$\langle u_t, v\rangle - \int_0^1 u_{xx} v\,dx + \int_0^1 b(x)uv\,dx = \int_0^1 fv\,dx.$$

Recall that $u_t(t) \in H^{-1}(0,1)$ and $\langle u_t, v\rangle$ is the dualty pair between $H^{-1}(0,1)$ and $H_0^1(0,1)$. Integrating by parts,

$$-\int_0^1 u_{xx} v\, dx = \int_0^1 u_x v_x\, dx,$$

because $v(0) = v(1) = 0$. Hence the weak formulation is: Find

$$u \in L^2(0,T; H_0^1(0,1)),$$

such that for all $v \in H_0^1(0,1)$,

$$\boxed{\langle u_t, v\rangle + \int_0^1 u_x v_x\, dx + \int_0^1 b(x) u v\, dx = \int_0^1 f v\, dx.}$$

with $u(x,0) = \sin(5\pi x)$.

2. Coercivity

We compute

$$B(u,u) = \int_0^1 |u_x|^2\, dx + \int_0^1 b(x) u^2\, dx.$$

Assume

$$b(x) \geq C_0.$$

Then

$$B(u,u) \geq \|u_x\|^2_{L^2(0,1)} + C_0 \|u\|^2_{L^2(0,1)}.$$

- Clearly, if $C_0 \geq 0$, then
$$B(u,u) \geq \|u_x\|^2_{L^2(0,1)}.$$
and B is coercive in this case.
- Since we are looking for a lower bound for C_0 we will assume that $C_0 \leq 0$. By the Poincaré inequality,
$$\|u\|^2_{L^2(0,1)} \leq \frac{1}{\pi^2}\|u_x\|^2_{L^2(0,1)} \Rightarrow C_0\|u\|^2_{L^2(0,1)} \geq \frac{C_0}{\pi^2}\|u_x\|^2_{L^2(0,1)}$$
Thus
$$B(u,u) \geq \left(1 + \frac{C_0}{\pi^2}\right)\|u_x\|^2_{L^2(0,1)}.$$

Therefore coercivity holds if and only if $1 + \frac{C_0}{\pi^2} > 0$. Hence the minimal condition is

$$\boxed{C_0 > -\pi^2.}$$

3. Energy identity

Take the test function $v = u(t)$. Then

$$\int_0^1 u_t u\, dx + B(u,u) = \int_0^1 f u\, dx.$$

Since

$$\langle u_t, v\rangle = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,1)},$$

we obtain

$$\boxed{\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,1)} + B(u(t),u(t)) = \int_0^1 f(x,t) u(x,t)\, dx.}$$

4. Case $b(x) \geq 0$ and $f = 1$

We have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(0,1)} + \|u_x\|^2_{L^2(0,1)} \leq \int_0^1 u\,dx.$$

Using Cauchy–Schwarz,

$$\int_0^1 u\,dx = \int_0^1 1 \times u\,dx \leq \left(\int_0^1 1^2\right)^{1/2}\left(\int_0^1 u^2\right)^{1/2} = \|u\|_{L^2(0,1)}.$$

Using Young's inequality $ab \leq \left(a^2 + b^2\right)/2$,

$$\|u\|_{L^2(0,1)} = 1 \times \|u\|_{L^2(0,1)} \leq \frac{1}{2}\|u\|^2_{L^2(0,1)} + \frac{1}{2}.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(0,1)} + \|u_x\|^2_{L^2(0,1)} \leq \frac{1}{2}\|u\|^2_{L^2(0,1)} + \frac{1}{2}.$$

By the Poincaré inequality $\|u_x\|^2_{L^2(0,1)} \geq \pi^2\|u\|^2_{L^2(0,1)}$, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(0,1)} + \pi^2\|u\|^2_{L^2(0,1)} \leq \frac{1}{2}\|u\|^2_{L^2(0,1)} + \frac{1}{2},$$

Denote $y(s) = \|u(s)\|^2_{L^2(0,1)}$, therefore

$$y'(s) + \left(2\pi^2 - 1\right)y(s) \leq 1.$$

Set $\gamma = 2\pi^2 - 1$ and multiply by $e^{\gamma s}$,

$$\left(y'(s) + \gamma y(s)\right)e^{\gamma s} = \left(y(s)e^{\gamma s}\right)' \leq 1.$$

In tegrating on $(0, t)$, we obtain

$$y(t)e^{\gamma t} - y(0) \leq t \Rightarrow y(t) \leq \left(y(0) + T\right)e^{-\gamma t}.$$

Compute

$$y(0) = \|u_0\|^2_{L^2(0,1)} = \int_0^1 (\sin(5\pi x))^2\,dx = \frac{1}{2}.$$

Therefore

$$\boxed{\|u(t)\|^2_{L^2(0,1)} \leq C_1 e^{-\gamma t} + C_2,}$$

with $C_1 = \frac{1}{2} + T$ and $C_2 = 0$.

5. Case $f = 0$ and $b(x) = \alpha > 0$

We solve

$$u_t - u_{xx} + \alpha u = 0.$$

(a) Separation of variables. Seek solutions of the form $u(x,t) = X(x)T(t)$. Substituting into the equation,

$$XT' - X''T + \alpha XT = 0 \Rightarrow \frac{T'}{T} + \alpha = \frac{X''}{X} = -\lambda.$$

Hence

$$X'' + \lambda X = 0, \quad X(0) = X(1) = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = n^2\pi^2, \qquad X_n(x) = \sin(n\pi x).$$

The time equation becomes

$$T_n' + (\lambda_n + \alpha)T_n = 0 \Rightarrow T_n(t) = e^{-(n^2\pi^2+\alpha)t}.$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n^2\pi^2+\alpha)t} \sin(n\pi x),$$

where

$$a_5 = 2\int_0^1 \sin^2(5\pi x)\,dx = 1 \ \text{ and } \ a_n = 2\int_0^1 \sin(5\pi x)\sin\left(n\pi x\right)\,dx = 0 \ \text{ if } n \neq 0.$$

Thus

$$\boxed{u(x,t) = e^{-(25\pi^2+\alpha)t}\sin(5\pi x).}$$

(b) Asymptotic behavior. We have

$$|u(x,t)| = \left|e^{-(25\pi^2+\alpha)t}\sin(\pi x)\right| \leq e^{-(25\pi^2+\alpha)t}, \qquad t > 0.$$

Therefore

$$\boxed{u(x,t) \to 0 \quad \text{exponentially as } t \to \infty.}$$

The decay rate is

$$\boxed{25\pi^2 + \alpha.}$$

For the energy

$$E(t) = \frac{1}{2}\|u(t)\|^2_{L^2(0,1)},$$

we compute

$$\|u(t)\|^2_{L^2(0,1)} = e^{-2(25\pi^2+\alpha)t}\int_0^1 \sin^2(5\pi x)\,dx = E(0)e^{-(50\pi^2+2\alpha)t}.$$

Hence the energy decay rate is

$$\boxed{50\pi^2 + 2\alpha.}$$