University of M'Sila
M. I. Faculty
Department of Maths.
PDE Master I, (S2)

Introduction to Evolution Problems 2024/2025

Final Exam

Instructions: Clarity and logical structure will be graded. Box your final results.

Exercise 1 ($2+2+2=6pts$)**.** Let $\Omega=(0,1)$, $T>0$, and $X=L^2(\Omega)$.

1. Define $L^p(0,T;X)$ for $1\leq p\leq\infty$. What is the norm for $p=\infty$?
2. Prove or disprove with justification:
$$L^4(0,T;H^1(\Omega))\subset L^2(0,T;L^2(\Omega)).$$
3. For $\alpha,\beta\in\mathbb{R}$, find necessary and sufficient conditions on α,β such that:
$$t^\alpha x^\beta\in L^4(0,T;H^1(\Omega)).$$

Exercise 2 ($2+2+[2+3]+[2+2+1]=14pts$)**.**
Consider the initial-boundary value problem:

$$\begin{cases} u_t-u_{xx}-a(x)u=0, & (x,t)\in(0,1)\times(0,T),\\ u(0,t)=u(1,t)=0, & t\in(0,T),\\ u(x,0)=\sin(2\pi x)+x(1-x), & x\in(0,1),\end{cases}\tag{1}$$

where $a(x)\in L^\infty(0,1)$.

1. Derive the weak formulation in $H_0^1(0,1)$.
2. Identify the conditions on $a(x)$ that guarantee coercivity of the bilinear form
$$B[u,v]=\int_0^1 u_xv_x\,dx-\int_0^1 a(x)uv\,dx$$
3. *Energy Estimates method:*
   (a) Define the energy functional:
   $$E(t)=\frac{1}{2}\int_0^1 u^2(x,t)\,dx.$$
   Prove the estimate:
   $$\frac{dE}{dt}+\gamma\int_0^1 u_x^2\,dx\leq 0$$
   and determine the constants γ explicitly in terms of $\|a\|_\infty$ and the Poincaré's constant.
   (b) Under what condition on $a(x)$ can you guarantee exponential decay of the energy? Find the optimal decay rate.
4. *Separation of variables method:* Assume $a(x)\equiv\eta>0$ constant.
   (a) Solve the problem explicitly using separation of variables.
   (b) Find the optimal decay rate of $E(t)$ using the series of the solution.
   (c) Compare the decay rate with one obtained from the energy method.

Note: The Poincaré's constant$=\frac{1}{\pi}$.

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Exercise 1 (Function Spaces)- $2+2+2=6pts$

1. **Definition:** For $1 \leq p < \infty$, $L^p(0,T;X)$ consists of all measurable functions $u:(0,T) \to X$ with

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} < \infty$$

For $p=\infty$, the norm is

$$\|u\|_{L^\infty(0,T;X)} := \text{ess sup}_{t\in(0,T)}\|u(t)\|_X$$

2. **Embedding:** The embedding $L^4(0,T;H^1(\Omega)) \subset L^2(0,T;L^2(\Omega))$ holds true.

Justification: By the Sobolev embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and Cauchy-Schwarz inequality in time:

$$\begin{aligned}\int_0^T \|u(t)\|_{L^2}^2 dt &\leq \left(\int_0^T 1dt\right)^{1/2}\left(\int_0^T \|u(t)\|_{L^2}^4 dt\right)^{1/2} \leq T^{1/2}\left(\int_0^T \|u(t)\|_{L^2}^4 dt\right)^{1/2} \\ &\leq T^{1/2}\left(\int_0^T \left(\|u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2\right)^{4/2} dt\right)^{1/2} = CT^{1/2}\|u\|_{L^4(0,T;H^1)}^2\end{aligned}$$

3. **Membership:** For $t^\alpha x^\beta \in L^4(0,T;H^1(0,1))$, we require:

1. $L^4(0,T;L^2(0,1))$ *condition:*

$$\int_0^T \left\|t^\alpha x^\beta\right\|_{L^2}^4 dt = \int_0^T \left(\int_0^1 t^{2\alpha}x^{2\beta}dx\right)^2 dt = \int_0^T t^{4\alpha}dt\left(\int_0^1 x^{2\beta}dx\right)^2 < \infty$$

- Space integral: $\int_0^1 x^{2\beta}dx$ converges when $2\beta > -1 \Rightarrow \beta > -\frac{1}{2}$.
- Time integral: $\int_0^T t^{4\alpha}dt$ converges when $4\alpha > -1 \Rightarrow \alpha > -\frac{1}{4}$.

2. $L^4(0,T;H^1(0,1))$ *condition (derivative):*

$$\int_0^T \left\|\beta t^\alpha x^{\beta-1}\right\|_{L^2}^4 dt = \int_0^T \left(\int_0^1 \beta^2 t^{2\alpha}x^{2(\beta-1)}dx\right)^2 dt = \beta^4\int_0^T t^{4\alpha}dt\left(\int_0^1 x^{2(\beta-1)}dx\right)^2 < \infty$$

- Space integral:$\int_0^1 x^{2(\beta-1)}dx$ converges when $2(\beta-1) > -1 \Rightarrow \beta > \frac{1}{2}$.

Final Conditions:

$$\alpha > -\frac{1}{4} \quad \text{and} \quad \beta > \frac{1}{2}$$

Exercise 2 (Heat Equation)- $2+2+[2+3]+[2+2+1]=14pts$

1. Weak Formulation

To derive the weak formulation:

- Multiply the PDE by a test function $v \in H_0^1(0,1)$:

$$u_t v - u_{xx}v - a(x)uv = 0$$

- Integrate over $(0,1)$:

$$\int_0^1 u_t v\, dx - \int_0^1 u_{xx}v\, dx - \int_0^1 a(x)uv\, dx = 0$$

- Integrate by parts the second term:

$$\int_0^1 u_t v\, dx - u_x v]_{x=0}^{x=1} + \int_0^1 u_x v_x\, dx - \int_0^1 a(x) u v\, dx = 0$$

- The weak formulation is: Find $u \in L^2(0,T;H_0^1(0,1))$ with $u_t \in L^2(0,T;H^{-1}(0,1))$ such that :

$$\begin{cases} \langle u_t, v\rangle + B[u,v] = (f,v)_{L^2}, & \forall v \in H_0^1(0,1) \\ u(0) = \sin(2\pi x) + x(1-x), & x \in (0,1). \end{cases}$$

- **Coercivity**: If $\|a\|_\infty < \pi^2$, then:

$$B[u,u] \geq \int_0^1 u_x^2\, dx - \|a\|_\infty \int_0^1 u^2\, dx \geq \left(1 - \frac{\|a\|_\infty}{\pi^2}\right) \|u\|_{H_0^1}^2$$

  by Poincaré's inequality $\int_0^1 u^2 dx \leq \frac{1}{\pi^2}\int_0^1 u_x^2 dx$.

2. Energy Estimates

(a) Energy Inequality

Define $E(t) = \frac{1}{2}\int_0^1 u^2 dx$. Then:

$$\begin{aligned} \frac{dE}{dt} &= \int_0^1 u u_t dx = \int_0^1 u(u_{xx} + a(x)u)dx \\ &= -\int_0^1 u_x^2 dx + \int_0^1 a(x)u^2 dx \leq -\int_0^1 u_x^2 dx + \|a\|_\infty \int_0^1 u^2 dx \leq -\left(1 - \frac{\|a\|_\infty}{\pi^2}\right)\int_0^1 u_x^2 dx \end{aligned}$$

where: $\gamma = 1 - \frac{\|a\|_\infty}{\pi^2}$.

(b) Exponential Decay

When $\|a\|_\infty < \pi^2$:

$$\frac{dE}{dt} + 2(\pi^2 - \|a\|_\infty)E(t) \leq 0$$

The inequality takes the form:

$$\frac{dE}{dt} \leq -2(\pi^2 - \|a\|_\infty)E(t)$$

Let $\lambda := 2(\pi^2 - \|a\|_\infty)$. Multiply both sides by the integrating factor $e^{\lambda t}$:

$$e^{\lambda t}\frac{dE}{dt} + \lambda e^{\lambda t}E(t) = \frac{d}{dt}\left(e^{\lambda t}E(t)\right) \leq 0$$

Integrate from 0 to t:

$$e^{\lambda t}E(t) \leq E(0)$$

Thus:

$$E(t) \leq E(0)e^{-\lambda t} = E(0)e^{-2(\pi^2 - \|a\|_\infty)t}$$

Thus we have exponential decay with rate $\lambda := 2\left(\pi^2 - \|a\|_\infty\right)$.

3. Separation of variables

(a) Exact Solution

For $a(x) \equiv \eta$ constant, solve via separation of variables:
Assume $u(x,t) = X(x)T(t)$:

$$\frac{T'}{T} = \frac{X''}{X} + \eta = -\lambda$$

Then,

$$T' + \lambda T = 0 \Longrightarrow T = Ce^{-\lambda t}$$

Eigenvalue problem:

$$X'' + (\eta + \lambda)X = 0 \quad \text{with} \quad X(0) = X(1) = 0.$$

Solution:

$$X_n(x) = \sin(n\pi x), \quad \lambda_n = n^2\pi^2 - \eta$$

General solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n^2\pi^2-\eta)t} \sin(n\pi x)$$

where coefficients c_n come from the initial condition:

$$c_n = 2\int_0^1 [\sin(2\pi x) + x(1-x)]\sin(n\pi x)dx = 2\underbrace{\int_0^1 \sin(2\pi x)\sin(n\pi x)\,dx}_{I_1} + 2\underbrace{\int_0^1 x(1-x)\sin(n\pi x)\,dx}_{I_2}$$

Using the orthogonality of sine functions:

$$I_1 = \int_0^1 \sin(2\pi x)\sin(n\pi x)\,dx = \begin{cases} \frac{1}{2} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

Calculation of I_2: Integrate by parts twice:

$$\begin{aligned} I_2 = \int_0^1 x(1-x)\sin(n\pi x)\,dx &= \left[-\frac{x(1-x)\cos(n\pi x)}{n\pi}\right]_0^1 + \frac{1}{n\pi}\int_0^1 (1-2x)\cos(n\pi x)\,dx \\ &= 0 + \frac{1}{n\pi}\left[\frac{(1-2x)\sin(n\pi x)}{n\pi}\right]_0^1 + \frac{2}{(n\pi)^2}\int_0^1 \sin(n\pi x)\,dx, \end{aligned}$$

hence

$$I_2 = \frac{2}{(n\pi)^2}\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{2}{(n\pi)^3}\left[1-(-1)^n\right]$$

Combining both results:

$$c_n = \begin{cases} 1 + \frac{4}{(2\pi)^3}(1-(-1)^2) = 1 & \text{for } n = 2 \\ 0 + \frac{4}{(n\pi)^3}\left[1-(-1)^n\right] & \text{for } n \neq 2 \end{cases} \implies c_n = \begin{cases} 1 & \text{if } n = 2 \\ \frac{8}{(n\pi)^3} & \text{if } n \text{ odd} \\ 0 & \text{if } n > 2 \text{ even} \end{cases}$$

The full solution is:

$$u(x,t) = e^{-(4\pi^2-\eta)t}\sin(2\pi x) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{(n\pi)^3} e^{-(n^2\pi^2-\eta)t}\sin(n\pi x).$$

(b) Decay Rate

Step 1: Parseval's Identity: Using the orthonormality of $\{\sqrt{2}\sin(n\pi x)\}_{n=1}^{\infty}$:

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 u^2(x,t)\,dx &= \frac{1}{2}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left|\frac{8}{(n\pi)^3}\right|^2 e^{-2(n^2\pi^2-\eta)t} \\ &\leq \left(\frac{1}{2}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left|\frac{8}{(n\pi)^3}\right|^2 e^{-2(n^2-1)\pi^2 t}\right) e^{-2(\pi^2-\eta)t} \leq \left(\frac{1}{2}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left|\frac{8}{(n\pi)^3}\right|^2\right) e^{-2(\pi^2-\eta)t} \end{aligned}$$

hence

$$E(t) \leq E(0)e^{-2(\pi^2-\eta)t}.$$

(c) Comparison

The previous energy method result

$$E(t) \leq Ce^{-2(\pi^2-\eta)t}$$

The decay rates from both methods coincideexactly when $a(x) \equiv \eta$.
