University of M'Sila
M. I. Faculty
Department of Maths.
PDE Master I, (S2)

Introduction to Evolution Problems 2023/2024

Final Exam

Clarity of reasoning and quality of writing will be considered in grading. Final results should be presented within a box. The Constants α, A, C_1, C_2, and C_3 need to be determined in Exercises 2 and 3.

Exercise 1. Let $T > 0$ and X be a Banach space with norm $\|\ .\ \|_X$ and $1 \leq p, q, r, s < +\infty$.
i) Recall the definition of the space $L^p(0, T; X)$, for $1 \leq p < +\infty$.
ii) Give a sufficient condition on p, q, r, and s to have the continuous embedding

$$L^p(0, T; L^q(\Omega)) \subset L^r(0, T; L^s(\Omega)).$$

Exercise 2. Let $T > 0$. Consider the following heat equation

$$\begin{cases} y_t - y_{xx} = 0, & \text{for } (x,t) \in]0,1[\times]0,T[, \\ y(0,t) = y(1,t) = 0, & \text{for } t \in]0,T[, \\ y(x,0) = \sin(3\pi x) + \sin(5\pi x), & \text{for } x \in]0,1[, \end{cases} \qquad (*)$$

1. Give the weak formulation of $(*)$.
2. Find the exact solution by separation of variables.
3. Show that

$$|y(t,x)| \leq Ae^{-\alpha t} \text{ for } t > 0.$$

Exercise 3. Let $T > 0$ and $\gamma \geq 0$. Consider the following wave equation

$$\begin{cases} u_{tt} - u_{xx} + \gamma x u = 0, & x \in (0,1), t \in (0,T), \\ u(0,t) = 0 \ \text{ and } \ u_x(1,t) = 0, & t \in (0,T), \\ u(x,0) = 0, \quad u_t(x,0) = x \sin \pi x & x \in (0,1), \end{cases} \qquad (**)$$

and we define

$$E_\gamma(t) = \frac{1}{2} \int_0^1 |u_t(t)|^2 + |u_x(t)|^2 + \gamma x |u(t)|^2 \, dx, \quad t \geq 0,$$

1. Give the variational formulation of Problem $(**)$.
2. Let $\gamma = 0$.
   (a) Compute the initial energy $E_0(0)$.
   (b) What is the sign of $\frac{d}{dt} E_0(t)$?
   (c) Can we deduce the uniqueness of the solution? Explain.
3. Let $\gamma > 0$.
   (a) Show that we have $C_1 \, E_0(t) \leq E_\gamma(t) \leq C_2 \, E_0(t), \quad \forall t \in [0,T]$
   (b) Prove that $E_0(t) \leq C_3, \quad \forall t \in [0,T]$.

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Evolution problems [Master EDP1. 2023/2024]

Exercice 1 i) $L^p(0,T;X) = \left\{ u : [0,T] \to X \text{ mesurable and } \int_0^T \|u(t)\|_X^p \, dt < +\infty \right\}$.

ii) To have $L^p(0,T;L^q(\Omega)) \subset L^r(0,T;L^s(\Omega))$ it suffices to have $\boxed{r < p}$ and $\boxed{s < q}$. The proof uses Hölder's inequality.

Exercice 2 1) $\langle y_t(t), u \rangle + \int_0^1 y_x(t) u_x \, dx = 0, \quad \forall u \in H_0^1(0,1)$.

2) Exact solution $y(x,t) = \sin(3x)\, e^{-9\pi^2 t} + \sin(5x)\, e^{-25\pi^2 t}$.

3) $|y(x,t)| \leq 2 e^{-9\pi^2 t}$. [we can even show $|y(x,t)| \leq e^{-9\pi^2 t}$]

Exercice 3 (1) $\langle u_{tt}(t), u \rangle + \int_0^1 u_x(t) . u \, dx + \gamma \int_0^1 x\, u(t) u \, dx = 0$

where $u \in V = \{ \varphi \in H^1(0,1),\ \varphi(0) = 0 \}$.

(2) Take $\gamma = 0$. a) $E_0(0) = \frac{1}{2} \int_0^1 |x \sin(\pi x)|^2 dx = \frac{1}{6} - \frac{1}{4\pi^2}$. (Integration by parts)

b) $\frac{d}{dt} E_0(t) = 0$, c) yes we can, see the course.

(3) Take $\gamma > 0$. a) since $0 < x < 1$, then we have $E_0(t) \leq E_\gamma(t)$.

we have also
$$E_\gamma(t) = E(0) + \frac{\gamma}{2} \int_0^1 |u(t)|^2 dx$$
$$\leq E(0) + \frac{\gamma C_p^2}{2} \int_0^1 |u_x(t)|^2 dx \quad \text{(by Poincaré)}.$$
$$\leq \frac{1}{2} \int_0^1 |u_t(t)|^2 + \frac{1+\gamma C_p^2}{2} \int_0^1 |u_x(t)|^2 dx$$
$$\leq (1 + \gamma C_p^2) E_0(t)$$

thus $E_0(t) \leq E_\gamma(t) \leq (1+\gamma C_p^2) E_0(t)$. $\quad (C_p = 1/\pi)$.

b) $E_0(t) = \frac{1}{6} - \frac{1}{4\pi^2}$ and $E_\gamma(t) \leq \left[1 + \gamma\left(\frac{1}{\pi^2}\right)\right]\left(\frac{1}{6} - \frac{1}{4\pi^2}\right)$.

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