The 1-D Wave Equation

1 1-D Wave Equation : Physical derivation

We consider a string of length l with ends fixed, and rest state coinciding with x-axis. The string is plucked into oscillation. Let $u(x,t)$ be the position of the string at time t.

Assumptions:

1. Small oscillations, i.e. the displacement $u(x,t)$ is small compared to the length l.

   (a) Points move vertically. In general, we don't know that points on the string move vertically. By assuming the oscillations are small, we assume the points move vertically.

   (b) Slope of tangent to the string is small everywhere, i.e. $|u_x(x,t)| \ll 1$, so stretching of the string is negligible

   (c) arc length $\alpha(t) = \int_0^l \sqrt{1+u_x^2}dx \simeq l$.

2. String is perfectly flexible (it bends). This implies the tension is in the tangent direction and the horizontal tension is constant, or else there would be a preferred direction of motion for the string.

Consider an element of the string between x and $x+\Delta x$. Let $T(x,t)$ be tension and $\theta(x,t)$ be the angle wrt the horizontal x-axis. Note that

$$\tan\theta(x,t) = \text{slope of tangent at } (x,t) \text{ in } ux\text{-plane} = \frac{\partial u}{\partial x}(x,t). \tag{1}$$

Short version of Matthew J. Hancock's course notes.

Newton's Second Law ($F = ma$) states that

$$F = (\rho \Delta x) \frac{\partial^2 u}{\partial t^2} \tag{2}$$

where ρ is the linear density of the string (ML^{-1}) and Δx is the length of the segment. The force comes from the tension in the string only - we ignore any external forces such as gravity. The horizontal tension is constant, and hence it is the vertical tension that moves the string vertically (obvious).

Balancing the forces in the horizontal direction gives

$$T(x + \Delta x, t) \cos\theta (x + \Delta x, t) = T(x, t) \cos\theta (x, t) = \tau = const \tag{3}$$

where τ is the constant horizontal tension. Balancing the forces in the vertical direction yields

$$\begin{aligned} F &= T(x + \Delta x, t) \sin\theta (x + \Delta x, t) - T(x, t) \sin\theta (x, t) \\ &= T(x + \Delta x, t) \cos\theta (x + \Delta x, t) \tan\theta (x + \Delta x, t) - T(x, t) \cos\theta (x, t) \tan\theta (x, t) \end{aligned}$$

Substituting (3) and (1) yields

$$\begin{aligned} F &= \tau \left(\tan\theta (x + \Delta x, t) - \tan\theta (x, t) \right) \\ &= \tau \left(\frac{\partial u}{\partial x} (x + \Delta x, t) - \frac{\partial u}{\partial x} (x, t) \right). \end{aligned} \tag{4}$$

Substituting F from (2) into Eq. (4) and dividing by Δx gives

$$\rho \frac{\partial^2 u}{\partial t^2} (\xi, t) = \tau \frac{\frac{\partial u}{\partial x} (x + \Delta x, t) - \frac{\partial u}{\partial x} (x, t)}{\Delta x}$$

for $\xi \in [x, x + \Delta x]$. Letting $\Delta x \to 0$ gives the <u>1-D Wave Equation</u>

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{\tau}{\rho} > 0. \tag{5}$$

Note that c has units $[c] = \left[\frac{\text{Force}}{\text{Density}} \right]^{1/2} = LT^{-1}$ of speed.

1.1 Boundary conditions

In order to guarantee that Eq. (5) has a unique solution, we need initial and boundary conditions on the displacement $u(x, t)$. There are now 2 initial conditions and 2 boundary conditions.

E.g. The string is fixed at both ends, $u(0,t) = u(l,t) = 0$, $t > 0$ (homogeneous type I BCs)

E.g. The string is connected to frictionless cylinders of mass m that move vertically on tracks at $x = 0, l$. Performing a force balance at either $x = 0$ or $x = 1$ gives

$$T \sin\theta = mg \tag{6}$$

In other words, the vertical tension in the string balances the mass of the cylinder. But $\tau = T\cos\theta = const$ and $\tan\theta = u_x$, so that (6) becomes

$$\tau u_x = T\cos\theta\tan\theta = mg$$

Rearranging yields

$$u_x = \frac{mg}{\tau}, \qquad x = 0, 1$$

These are Type II BCs. If the string is really tight and the cylinders are very light, then $mg/\tau \ll 1$ and we approximate $u_x \approx 0$ at $x = 0, 1$, and the BCs become Type II homogeneous BCs.

1.2 Initial conditions

We need to specify both the initial position of the string and the initial velocity: $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, $0 < x < l$. The idea is the same as finding the trajectory of a falling body; we need to know both the initial position and initial velocity of the particle to compute its trajectory. We can also see this mathematically.

The Taylor series of $u(x,t)$ about $t = 0$ is

$$u(x,t) = u(x,0) + u_t(x,0)\,t + u_{tt}(x,0)\,\frac{t^2}{2} + \frac{\partial^3 u}{\partial t^3}(x,0)\,\frac{t^3}{3!} + \cdots$$

From the initial conditions, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ and the PDE gives

$$\begin{aligned} u_{tt}(x,0) &= c^2 u_{xx}(x,0) = c^2 f''(x), \\ \frac{\partial^3 u}{\partial t^3}(x,0) &= c^2 u_{txx}(x,0) = c^2 g''(x). \end{aligned}$$

Higher order terms can be found similarly. Therefore, the two initial conditions for $u(x,0)$ and $u_t(x,0)$ are sufficient to determine $u(x,t)$ near $t = 0$.

To summarize, the dimensional basic 1-D Wave Problem with Type I BCs (fixed ends) is

$$\text{PDE}: \qquad u_{tt} = c^2 u_{xx}, \qquad 0 < x < l \tag{7}$$

$$\text{BC}: \qquad u(0,t) = 0 = u(l,t), \qquad t > 0, \tag{8}$$

$$\text{IC}: \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 < x < l \tag{9}$$

1.3 Non-dimensionalization

We now scale the basic 1-D Wave Problem. The characteristic quantities are length L_* and time T_*. Common sense suggests choosing $L_* = l$, the length of the string. We introducing the non-dimensional variables

$$\hat{x} = \frac{x}{L_*}, \qquad \hat{t} = \frac{t}{T_*}, \qquad \hat{u}\left(\hat{x}, \hat{t}\right) = \frac{u\left(x,t\right)}{L_*}, \qquad \hat{f}\left(\hat{x}\right) = \frac{f\left(x\right)}{L_*}, \qquad \hat{g}\left(\hat{x}\right) = \frac{T_* g\left(x\right)}{L_*}.$$

From the chain rule,

$$\frac{\partial u}{\partial x} = L_* \frac{\partial \hat{u}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} = \frac{\partial \hat{u}}{\partial \hat{x}}, \qquad \frac{\partial u}{\partial t} = L_* \frac{\partial \hat{u}}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial t} = \frac{L_*}{T_*} \frac{\partial \hat{u}}{\partial \hat{t}}$$

and similarly for higher derivatives. Substituting the dimensionless variables into 1-D Wave Equation (7) gives

$$\hat{u}_{\hat{t}\hat{t}} = \frac{T_*^2 c^2}{L_*^2} \hat{u}_{\hat{x}\hat{x}}$$

This suggests choosing $T_* = L_*/c = l/c$, so that

$$\hat{u}_{\hat{t}\hat{t}} = \hat{u}_{\hat{x}\hat{x}}, \qquad 0 < \hat{x} < 1, \qquad \hat{t} > 0. \tag{10}$$

The BCs (8) become

$$\hat{u}\left(0, \hat{t}\right) = 0 = \hat{u}\left(1, \hat{t}\right), \qquad \hat{t} > 0. \tag{11}$$

The ICs (9) become

$$\hat{u}\left(\hat{x}, 0\right) = \hat{f}\left(\hat{x}\right), \qquad \hat{u}_{\hat{t}}\left(\hat{x}, 0\right) = \hat{g}\left(\hat{x}\right), \qquad 0 < \hat{x} < 1. \tag{12}$$

1.3.1 Dimensionless 1-D Wave Problem with fixed ends

Dropping hats, the dimensionless 1-D Wave Problem is, from (10) – (12),

$$\text{PDE}: \qquad u_{tt} = u_{xx}, \qquad 0 < x < 1 \tag{13}$$

$$\text{BC}: \qquad u\left(0,t\right) = 0 = u\left(1,t\right), \qquad t > 0, \tag{14}$$

$$\text{IC}: \qquad u\left(x,0\right) = f\left(x\right), \qquad u_t\left(x,0\right) = g\left(x\right), \qquad 0 < x < 1 \tag{15}$$

2 Separation of variables solution

Substituting $u\left(x,t\right) = X\left(x\right) T\left(t\right)$ into the PDE (13) and dividing by $X\left(x\right) T\left(t\right)$ gives

$$\frac{T''\left(t\right)}{T\left(t\right)} = \frac{X''\left(x\right)}{X\left(x\right)} = -\lambda \tag{16}$$

where λ is a constant. The negative sign is for convention. The BCs (14) become

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \\ u(1,t) &= X(1)T(t) = 0 \end{aligned}$$

which implies

$$X(0) = X(1) = 0 \tag{17}$$

Thus, the problem for $X(x)$ is the same Sturm-Liouville Boundary Value Problem as for the Heat Equation,

$$X''(x) + \lambda X(x) = 0; \qquad X(0) = X(1) = 0. \tag{18}$$

Recall that the eigenvalues and eigenfunctions of (18) are

$$\lambda_n = (n\pi)^2, \qquad X_n(x) = b_n \sin(n\pi x), \qquad n = 1, 2, 3, ...$$

The function $T(t)$ satisfies

$$T'' + \lambda T = 0$$

and hence each eigenvalue λ_n corresponds to a solution $T_n(t)$

$$T_n(t) = \alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t).$$

Thus, a solution to the PDE and BCs is

$$u_n(x,t) = (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

where we have absorbed the constant b_n into α_n, β_n.

In general, the individual $u_n(x,t)$'s will not satisfy the ICs. Thus we sum infinitely many of them, using the principle of superposition,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

Imposing the ICs gives

$$\begin{aligned} f(x) &= u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x) \\ g(x) &= u_t(x,0) = \sum_{n=1}^{\infty} n\pi\beta_n \sin(n\pi x) \end{aligned}$$

Therefore, the coefficients α_n and β_n are given by

$$\begin{aligned} \alpha_n &= 2\int_0^1 f(x) \sin(n\pi x)\, dx \\ \beta_n &= \frac{2}{n\pi}\int_0^1 g(x) \sin(n\pi x)\, dx \end{aligned}$$

Note: The convergence of this series is harder to show, because we don't have decaying exponentials $e^{-n^2\pi^2 t}$ in the sum terms (more later).

Note: Given BCs and an IC, the wave equation has a unique solution (Myint-U & Debnath §6.3).

3 Interpretation - Normal modes of vibration

The terms

$$u_n(x,t) = (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

for $n = 1, 2, 3, ...$ are called the normal modes of vibration. The solution $u(x,t)$ is a superposition of the normal modes $u_n(x,t)$. In physical variables, the normal modes are

$$u_n'(x',t') = \left(\alpha_n \cos\left(\frac{n\pi c}{l}t'\right) + \beta_n \sin\left(\frac{n\pi c}{l}t'\right)\right) \sin\left(\frac{n\pi x'}{l}\right).$$

3.1 Frequency and Period

A function $f(t)$ is periodic if for some real number T,

$$f(t+T) = f(t)$$

for all t. If t measures time (either physical or dimensionless), we define the period of a function as the smallest number T such that $f(t+T) = f(t)$. We say that $f(t)$ has period T or, equivalently, that $f(t)$ is T-periodic. The period T has the same dimensions as t. If t is dimensionless then so is T; if t has the dimensions of time, so does T. Note that each normal mode $u_n(x,t)$ has period $2/n$, and in physical variables, $u_n'(x',t')$ has period $2l/(nc)$.

The analog to period for spatial coordinates is the wavelength. The wavelength of a function $g(x)$ (x is a scaled or physical spatial coordinate) is defined as the smallest L such that $g(x+L) = g(x)$. We say $g(x)$ is L-periodic. Again, the wavelength L has the same dimensions as x. If x is dimensionless then so is L; if x has the dimensions of length, so does L.

The frequency f is defined as $f = 1/T = 1/\text{period}$. In physical variables, each normal mode $u_n'(x',t')$ has frequency

$$f_n = \frac{\omega_n}{2\pi} = \frac{nc}{2l} = \frac{n}{2l}\sqrt{\frac{\tau}{\rho}}$$

with dimensions 1/time (e.g. Hz $=$ cycle/sec $=$ sec^{-1}).

The angular frequency is defined as $\omega = 2\pi f$ where f is the frequency. Each mode has angular frequency $\omega_n = 2\pi f_n = n\pi c/l$. In terms of the frequency, we can write the normal mode as

$$\begin{aligned} u_n'(x', t') &= \left(\alpha_n \cos\left(\frac{\pi n c}{l}t'\right) + \beta_n \sin\left(\frac{n\pi c}{l}t'\right)\right)\sin\left(\frac{n\pi x'}{l}\right) \\ &= \left(\alpha_n \cos(2\pi f_n t') + \beta_n \sin(2\pi f_n t')\right)\sin\left(\frac{n\pi x'}{l}\right) \\ &= \left(\alpha_n \cos(\omega_n t') + \beta_n \sin(\omega_n t')\right)\sin\left(\frac{n\pi x'}{l}\right) \end{aligned}$$

The first harmonic is the normal mode of lowest frequency, $u_1(x,t)$ or in physical variables, $u_1'(x', t')$.

The fundamental frequency is f_1, i.e. the frequency of the first harmonic. In dimensionless variables, $f_1 = 1/2$. In physical variables, $f_1' = c/(2l) = \left(\sqrt{\tau/\rho}\right)/2l$.

Note that $f_n = n f_1$, in other words, the frequencies of higher harmonics are just integer multiples of the fundamental frequency f_1.

Examples. Check for yourself that $\sin(\pi x)$ is 2-periodic i.e. has period 2, $\sin(x)$ is 2π-periodic, $\sin(n\pi t)$ is $2/n$-periodic, $\sin(n\pi c t/l)$ has period $2l/(nc)$, $\sin(n\pi x/l)$ has period $2l/n$. Note that if a function has period T, then $f(t + mT) = f(t)$ for all $m = 1, 2, 3, ...$ In particular,

$$\sin(n\pi x), \cos(n\pi x), \sin(n\pi t), \cos(n\pi t)$$

are all $2/n$-periodic, and hence

$$\begin{aligned} \sin\left(n\pi\left(x + \frac{2m}{n}\right)\right) &= \sin n\pi x, \qquad \cos\left(n\pi\left(x + \frac{2m}{n}\right)\right) = \cos n\pi x \\ \sin\left(n\pi\left(t + \frac{2m}{n}\right)\right) &= \sin n\pi t, \qquad \cos\left(n\pi\left(t + \frac{2m}{n}\right)\right) = \cos n\pi t \end{aligned}$$

for all $m, n = 1, 2, 3, ..$ Therefore, the dimensionless solution $u(x,t)$ of the wave equation has time period 2 ($u(x, t+2) = u(x,t)$) since

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)\right)\sin(n\pi x)$$

and for each normal mode, $u_n(x,t) = u_n(x, t+2)$ (check for yourself). Thus, the period of $u(x,t)$ is the same as that for the first harmonic $u_1(x,t)$. In physical variables, the period of $u'(x', t')$ is $T_1' = 2l/c = 2l/\sqrt{\tau/\rho}$ and the frequency is $f_1' = 1/T_1' = c/(2l)$.

3.2 A note on dimensions

I've tried to denote dimensional quantities with primes or specifically comment whether or not we are working with physical x, t (a length and time) or with dimensionless x, t (dimensions 1). However, in general you should always ask the question, "What are the dimensions?" The quantity c/l has dimensions of 1/time since I have defined c, l to be a speed and a length. The argument of any mathematical function like cos, sin, exp, etc. must be dimensionless. The cosine of 1 m or 2 s does not make sense. Thus, the quantity t in $\cos(n\pi t)$ is dimensionless. Note that radians are a dimensionless measure of angle. Consider the quantity t in $\cos(\omega t)$ or $\cos(2\pi f t)$ or $\cos\left(\frac{\pi c t}{l}\right)$, where you are given that ω and f have dimensions of 1/time, c is a speed, and l is a length. Then t must have dimensions of time, so that the arguments ωt, $2\pi f t$, or $\pi c t/l$ of cosine are dimensionless.

3.3 Amplitude

Note that $u_n(x,t)$ can be written

$$u_n(x,t) = \gamma_n \sin(n\pi x)\sin(n\pi t + \psi_n) \tag{19}$$

where $\gamma_n = \sqrt{\alpha_n^2 + \beta_n^2}$, $\psi_n = \arctan\left(\frac{\alpha_n}{\beta_n}\right)$. At each point x, the n'th mode vibrates sinusoidally according to

$$u_n(x,t) = A_n(x)\sin(n\pi t + \psi_n)$$

where

$$A_n(x) = \gamma_n \sin(n\pi x).$$

The mode $u_n(x,t)$ vibrates sinusoidally in time between the two limits $\pm A_n(x)$. We call $A(x)$ the time amplitude of the mode $u_n(x,t)$, since this sets the bounds on the oscillations in time. Locations where $A_n(x) = 0$ are called nodes and locations where $|A_n(x)| = \gamma_n$ are called antinodes.

4 Conservative system and energy

Recall the solution to the Heat Problem with a homogeneous PDE (i.e. no sources, sinks) Homogenous Type I BCs ($u = 0$ at $x = 0, 1$) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

and as $t \to \infty$, $u(x,t) \to 0$. Thus, the rod looses heat energy and with it the memory of the initial state, or initial condition.

In contrast, the solution to the wave equation with Homogeneous Type I BCs (fixed ends, $u = 0$ at $x = 0, 1$),

$$u(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

The oscillations do not decay, since we ignored gravity and resistive forces. Once plucked, the string vibrates/oscillates forever, with period $2l/c$ in physical coordinates. We call this type of system conservative: energy is conserved. In addition, the system comes back to its initial condition periodically - i.e. it maintains a memory of its initial state.

We define the energy of the system as

$$\text{Total energy } E'(t') = PE'(t') + KE'(t')$$

where primes denote physical or dimensional variables. The local kinetic energy is

$$KE'(x',t') = \frac{1}{2}mv^2 = \frac{1}{2}(\rho\Delta x)\left(\frac{\partial u}{\partial t}\right)^2$$

and the local potential energy is

$$\begin{aligned} PE'(x',t') &= \text{work done getting to displacement } u \\ &= \text{Force } \times \text{ change in length of string} \end{aligned}$$

Making a displacement Δu of a segment of string from x to $x+\Delta x$ results in a change in the string segment length, initially of length Δx, of

$$\Delta l = \sqrt{(\Delta x)^2 + (\Delta u)^2} - \Delta x = \Delta x\left(\sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} - 1\right)$$

Recall the binomial expansion $\sqrt{1+a^2} = 1 + \frac{1}{2}a^2 + O(a^4)$. Then

$$\Delta l = \Delta x\left(1 + \frac{1}{2}\left(\frac{\Delta u}{\Delta x}\right)^2 - 1 + O\left(\left(\frac{\Delta u}{\Delta x}\right)^4\right)\right) \approx \frac{\Delta x}{2}\left(\frac{\Delta u}{\Delta x}\right)^2$$

and hence the potential energy is

$$PE'(x',t') = \tau\Delta l \approx \frac{\tau}{2}\left(\frac{\Delta u}{\Delta x}\right)^2 \Delta x \approx \frac{\tau}{2}\left(\frac{\partial u}{\partial x}\right)^2 \Delta x$$

as $\Delta x \to 0$. Thus, the total energy is, in dimensional form,

$$E'(t') = \frac{1}{2}\int_0^l \left(\rho\left(\frac{\partial u}{\partial t'}\right)^2 + \tau\left(\frac{\partial u}{\partial x'}\right)^2\right) dx' = \frac{\rho}{2}\int_0^l \left(u_{t'}^2 + c^2 u_{x'}^2\right) dx' \qquad (20)$$

where we use the shorthand notation $u_{t'}^2 = (\partial u/\partial t')^2$. Substituting $c^2 = \tau/\rho$ and the dimensionless variables into (20) gives the dimensionless total energy

$$E(t) = \frac{E'(t')}{\tau l} = \frac{1}{2}\int_0^1 \left(u_t^2 + u_x^2\right) dx \qquad (21)$$

4.1 Conservation of energy for a single mode

The energy of a given normal mode $u_n(x,t)$ is found by substituting the form (19) into Eq. (21) for the energy,

$$\begin{aligned} E_n(t) &= \frac{1}{2}\int_0^1 \left(\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right) dx \\ &= \frac{(n\pi\gamma_n)^2}{2}\int_0^1 \left(\sin^2(n\pi x)\cos^2(n\pi t + \psi_n) + \cos^2(n\pi x)\sin^2(n\pi t + \psi_n)\right) dx \\ &= \frac{(n\pi\gamma_n)^2}{2}\left(\cos^2(n\pi t + \psi_n)\int_0^1 \sin^2(n\pi x)\, dx \right. \\ &\qquad \left. + \sin^2(n\pi t + \psi_n)\int_0^1 \cos^2(n\pi x)\, dx\right) \end{aligned}$$

Note that

$$\sin^2(n\pi x) = \frac{1}{2} - \frac{1}{2}\cos(2n\pi x), \qquad \cos^2(n\pi x) = \frac{1}{2} + \frac{1}{2}\cos(2n\pi x)$$

so that

$$\int_0^1 \sin^2(n\pi x)\, dx = \frac{1}{2}, \qquad \int_0^1 \cos^2(n\pi x)\, dx = \frac{1}{2}$$

Substituting these integrals into the energy $E_n(t)$ gives

$$E_n(t) = \frac{(n\pi\gamma_n)^2}{4}\left(\cos^2(n\pi t + \psi_n) + \sin^2(n\pi t + \psi_n)\right) = \frac{(n\pi\gamma_n)^2}{4}.$$

Thus the total energy (kinetic + potential across the rod) of each mode is constant!

An alternate method to show $E(t) = const$ employs the infinite series solution derived using the method of separation of variables:

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left(\left(\frac{\partial}{\partial t} \sum_{n=1} u_n \right)^2 + \left(\frac{\partial}{\partial x} \sum_{n=1} u_n \right)^2 \right) dx \\ &= \frac{1}{2}\int_0^1 \left(\sum_{n=1}\sum_{m=1} \frac{\partial u_n}{\partial t}\frac{\partial u_m}{\partial t} + \frac{\partial u_n}{\partial x}\frac{\partial u_m}{\partial x} \right) dx \\ &= \frac{1}{2}\sum_{n=1}\sum_{m=1}\int_0^1 \left(\frac{\partial u_n}{\partial t}\frac{\partial u_m}{\partial t} + \frac{\partial u_n}{\partial x}\frac{\partial u_m}{\partial x} \right) dx \end{aligned}$$

Substituting for the normal modes $u_n(x,t) = \gamma_n \sin(n\pi x)\sin(n\pi t + \psi_n)$ gives

$$\begin{aligned} E(t) &= \frac{1}{2}\sum_{n=1}\sum_{m=1} \gamma_n\gamma_m nm\pi^2 \cos(n\pi t + \psi_n)\cos(m\pi t + \psi_m) \\ &\qquad \times \int_0^1 \sin(n\pi x)\sin(m\pi x)\, dx \\ &+ \frac{1}{2}\sum_{n=1}\sum_{m=1} \gamma_n\gamma_m nm\pi^2 \sin(n\pi t + \psi_n)\sin(m\pi t + \psi_m) \\ &\qquad \times \int_0^1 \cos(n\pi x)\cos(m\pi x)\, dx \end{aligned}$$

Substituting the orthogonality relations gives

$$\begin{aligned} E(t) &= \frac{1}{2}\sum_{n=1}\sum_{m=1} \gamma_n\gamma_m nm\pi^2 \cos(n\pi t + \psi_n)\cos(m\pi t + \psi_m)\frac{\delta_{mn}}{2} \\ &+ \frac{1}{2}\sum_{n=1}\sum_{m=1} \gamma_n\gamma_m nm\pi^2 \sin(n\pi t + \psi_n)\sin(m\pi t + \psi_m)\frac{\delta_{mn}}{2} \\ &= \frac{1}{4}\sum_{n=1} (\gamma_n n\pi)^2 \left(\cos^2(n\pi t + \psi_n) + \sin^2(n\pi t + \psi_n)\right) \\ &= \frac{1}{4}\sum_{n=1} (\gamma_n n\pi)^2 \end{aligned}$$

which is constant for all t.

6 Waves on a finite string

We now consider D'Alembert's solution for a finite string. The dimensionless problem with homogeneous Type I BCs (ends of string fixed) is

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(0,t) &= 0 = u(1,t) \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned} \tag{43}$$

We found that this has solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

where

$$\begin{aligned} \alpha_n &= 2\int_0^1 f(x) \sin(n\pi x)\, dx \\ \beta_n &= \frac{2}{n\pi}\int_0^1 g(x) \sin(n\pi x)\, dx \end{aligned}$$

We wrote, equivalently, that

$$u(x,t) = P(x-t) + Q(x+t)$$

where

$$\begin{aligned} P(x-t) &= \frac{1}{2}\sum_{n=1}^{\infty} (\alpha_n \sin(n\pi(x-t)) - \beta_n \cos(n\pi(x-t))) \\ Q(x+t) &= \frac{1}{2}\sum_{n=1}^{\infty} (\alpha_n \sin(n\pi(x+t)) + \beta_n \cos(n\pi(x+t))) \end{aligned}$$