3

HEAT EQUATION

The heat equation and its variants occur in many diffusion phenomena, it is the simplest example of a parabolic equation which occur in many fields: mechanics, physics, chemistry, biology, optimal control, probability, finance, image processing etc. For simplicity we limit our selves to Cauchy-Dirichlet problem for this equation to introduce a possible weak formulation.

3.1 NOTATION AND WEAK FORMULATION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$ and consider the space-time cylinder

$$Q_T = \Omega \times]0, T[\,,$$

with a smooth lateral boundary

$$\Sigma = \Gamma \times]0, T[\,.$$

We shall consider the following problem. Given $\alpha > 0$, two functions $u^0 : \Omega \to \mathbb{R}$ and $f : Q \to \mathbb{R}$, find $u : (x, t) = Q \to \mathbb{R}$ solution of the problem

$$\begin{cases} u_t - \alpha \triangle u = f & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x, 0) = u^0(x), & \text{in } \Omega. \end{cases} \tag{3.1}$$

- Equation (3.1) is called the heat equation because it models the temperature distribution u in the domain Ω, at time t, in the presence of a heat source term f distributed on Ω.

- The constant α is the *thermal conductivity*.

- Initial condition: $u^0(x)$ is a given function.
- The (homogeneous) Dirichlet boundary condition, $u = 0$ on Σ, it could be replaced (among others) by the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ on Σ, (ν is the outward unit normal vector to Σ).
- Condition $u = 0$ on Σ corresponds to the assumption that the boundary is kept at zero temperature, condition $\frac{\partial u}{\partial \nu} = 0$ on Σ corresponds to the assumption that the heat flux across Σ is zero (insulated boundary). Other boundary conditions also arise occasionally.

Assuming this, here are two assertions that are intuitively clear (when $f = 0$).

- Say the initial and boundary temperatures are at most M . Then at any time in the future, the maximum temperature is at most M . This is called the maximum principle. We discuss it in the last section.
- Eventually, the temperature throughout the body tends to some "equilibrium temperature", $u(x,t) \to v(x)$, where $v(x)$ depends only on the boundary temperature, not on the initial temperature.

One test of the mathematical model is to prove these assertions from the data specified. It is convenient to consider $u(x,t)$ as a function of t with values in $V = L^2(\Omega)$

$$u : [0,T] \to V.$$

and for simplicity we write $u(t)$ instead of $u(x,t)$ and $\dot{u}(t)$ instead of u_t. Accordingly, we write $f(t)$ instead of $f(x,t)$. We consider the Hilbert triplet

$$V = H_0^1(\Omega),\ H = L^2(\Omega),\ V' = H^{-1}(\Omega).$$

We note $\|\cdot\|_1, |\cdot|$ and $\|\cdot\|_*$ the norms of V, H and V', respectively, and by $(\cdot,\cdot)_1$ and $(\cdot,\cdot)$ the scalar product of V and H. The duality product between V and V' is denoted by $\langle\cdot,\cdot\rangle_*$.

Remark 3.1 *The choice of V depends on the type of boundary condition we are dealing with. Other familiar choices are $V = H^1(\Omega)$ for the Neumann and*

$$V = H^1_{0,\Gamma_D}(\Omega) := \left\{ v \in H^1(\Omega) \text{ such that } v|_{\Gamma_D} = 0 \right\},\quad \Gamma_D \subset \Gamma,$$

in the case of mixed conditions.

Let us give the first (very) weak formulation.

Definition 3.1 *Assume that $u^0 \in H$ and $f \in L^2(0,T;H)$. We say that a function*

$$u \in L^2(0,T;V) \text{ with } \dot{u} \in L^2(0,T;V'),$$

is a weak solution of the initial boundary value Problem (3.1) if

$$\begin{cases} \int_0^T \langle \dot{u}(t), v(t)\rangle_* \,dt + \int_0^T (\nabla u(t), \nabla v(t))\,dt = \int_0^T (f(t), v(t))\,dt, & \forall v \in L^2(0,T;V), \\ u(0) = u^0. \end{cases} \tag{3.2}$$

Exercise 5 *Checck that if u, g and f are smooth functions and u is a weak solution, then u solves Problem (3.2) in a classical sense.*

For the numerical treatment of Problem (3.2) it is often more convenient an *equivalent* alternative formulation that we give below.

Definition 3.2 *Assume that $u^0 \in H$ and $f \in L^2(0,T;H)$. We say that a function*

$$u \in L^2(0,T;V) \text{ with } \dot{u} \in L^2(0,T;V'),$$

is a weak solution of the initial boundary value Problem (3.1) if

$$\begin{cases} \langle \dot{u}(t), v\rangle_* + \alpha(u(t), v)_1 = (f(t), v), & \forall v \in V \text{ and a.e. } t \in [0,T], \\ u(0) = u^0. \end{cases} \tag{3.3}$$

Remark 3.2 *Here the requirement $u(0) = u^0$ makes sense since by Theorem 2.12, $u \in C([0,T]; L^2(\Omega))$.*

Remark 3.3 *Since $\triangle u \in \mathscr{L}(H_0^1(\Omega), H^{-1}(\Omega))$ then*

$$\triangle u \in L^2(0,T;H^{-1}(\Omega)), \text{ for } u \in L^2(0,T;H_0^1(\Omega))$$

and thus

$$\dot{u} = \alpha\Delta u + f \in L^2(0,T;H^{-1}(\Omega)).$$

This explains the presence of the duality product in (3.3).

Remark 3.4 *The first equation in (3.3) holds in the distribution sense in $\mathscr{D}'(]0,T[)$. In fact, $\forall \varphi \in \mathscr{D}(]0,T[)$, by the Bochner's theorem, we have*

$$\begin{aligned} \int_0^T \langle \dot{u}, v\rangle \varphi dt &= \left\langle \int_0^T \dot{u}\varphi dt, v\right\rangle \\ &= -\left\langle \int_0^T u\varphi' dt, v\right\rangle = -\int_0^T \langle u, v\rangle \varphi' dt \\ &= -\int_0^T (u(t), v)\varphi'(t)dt = \int_0^T \frac{d}{dt}(u(t), v)\varphi(t)dt. \end{aligned}$$

Thus

$$\langle \dot{u}, v\rangle = \frac{d}{dt}(u(t), v) \text{ in } D'(0,T), \quad \forall v \in V. \tag{3.4}$$

As a consequence, equation (3.3) may be written in the form

$$\frac{d}{dt}(u(t), v) + \alpha(\nabla u(t), \nabla v) = (f(t), v),$$

in the sense of distributions in $D'(]0,T[)$, for all $v \in V$.

3.2 UNIQUENESS

Theorem 3.1 *There exists a unique weak solution of Problem (3.3).*

Proof. Assume that Problem (3.3) admit two solutions u_1, u_2 with the same data. Thus $w = u_1 - u_2$ satisfy

$$\begin{cases} w' - \alpha \triangle w = 0, & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,0) = 0. & \text{in } \Omega. \end{cases} \tag{3.5}$$

and the weak formulation is given by

$$\langle \dot{w}(t), v \rangle + \alpha (\nabla w(t), \nabla v) = 0, \qquad \forall v \in V. \tag{3.6}$$

Taking $v = w \in V,$ we get

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \alpha \|w(t)\|_1^2 = 0.$$

and integrating on $[0,t]$, it comes

$$|w(t)|^2 - 0 + 2\alpha \int_0^t \|w(t)\|_1^2 \, ds = 0.$$

which entails $w(t) = 0$ for all $t \in [0,T]$. This gives uniqueness of the weak solution. ■

We shall use Faedo-Galerkin method described in the precedent chapter.

3.3 THE APPROXIMATE PROBLEM

Since the space $V = H_0^1(\Omega)$ is separable,we can select a sequence of smooth functions $\{w_k\}_{k=1}^{\infty}$ constituting

$$\begin{cases} \text{an orthogonal basis in } V, \\ \text{an orthonormal basis in in } H. \end{cases}$$

In particular, we can write

$$u^0 = \sum_{k=1}^{\infty} \beta_k w_k,$$

where $\beta_k = (u^0, w_k)$ and such that the series converge in H. Now let V_m be the subspace of V such that

$$V_m = span\{w_1, w_2, ..., w_m\},$$

then

$$V_m \subset V_{m+1} \text{ and } \overline{\cup V_m} = V.$$

Let us fix m and define

$$u_m(t) = \sum_{k=1}^{m} c_k(t) w_k, \;\; u_m^0 = \sum_{k=1}^{m} \beta_k w_k,$$

where $c_k \in L^2(0,T)$ are unknown. We construct a sequence of function u_m solution to the approximate problem

$$\begin{cases} \text{find } u_m \text{ such that, } \forall s = 1, \cdots, m, \\ (\dot{u}(t), w_s) + \alpha(\nabla u_m(t), \nabla w_s) = (f(t), w_s), \; 0 \leq t \leq T, \\ u_m(0) = u_m \to u_0 \text{ in } V, \text{ as } m \to \infty \end{cases} \tag{3.7}$$

We call u_m a Galerkin approximation of the solution u.

Solution of the approximate problem

Lemma 3.1 $\forall m \geq 1$, *there exists a unique solution unique to Problem (3.7), satisfying*

$$u_m \in H^1(0,T;V_m).$$

Proof. The sequence $w_1, w_2, ..., w_m$ is orthonormal in H and we have

$$(\dot{u}_m(t), w_s) = \left(\sum_{k=1}^{m} c_k'(t) w_k, w_s\right) = c_s'(t), \text{ for } s = 1, \cdots, m.$$

On the other hand, since $w_1, w_2, ..., w_m$ is orthogonal in V then

$$(\sum_{k=1}^{m} c_k(t) \nabla w_k, , \nabla w_s) = (\nabla w_s, \nabla w_s) c_k(t) = |\nabla w_s|^2 c_s(t).$$

Setting

$$F_s(t) = (f(t), w_s), \;\; \mathbf{F}_m(t) = \begin{pmatrix} F_1(t) \\ \vdots \\ F_m(t) \end{pmatrix}, \;\; \mathbf{C}_m = \begin{pmatrix} c_1(t) \\ \vdots \\ c_m(t) \end{pmatrix}, \;\; \mathbf{A}_m = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix},$$

which are respectively in $L^2(0,T), L^2(0,T;\mathbb{R}^m), L^2(0,T;\mathbb{R}^m)$ and $\mathbb{R}^m$. Hence

$$c_k'(t) \nabla w_k, , \nabla w_s) = -\beta |\nabla w_s|^2 c_s(t) + F_s(t), \text{ for } s = 1, \cdots, m.$$

If we introduce the diagonal matrix

$$\mathbf{W}_m = diag\left\{|\nabla w_1|^2, \; |\nabla w_2|^2, ..., |\nabla w_m|^2\right\} : \mathbb{R}^m \to \mathbb{R}^m$$

then, Problem (3.7) is equivalent to the following system of m uncoupled linear ordinary differential equations, with constant coefficients

$$\begin{cases} \dot{\mathbf{C}}_m(t) = -\alpha \mathbf{W}_m \mathbf{C}_m(t) + \mathbf{F}_m(t), \\ \mathbf{C}_m(0) = \mathbf{A}_m. \end{cases} \tag{3.8}$$

Since $\mathbf{F}_m(t) \in L^2(0,T;\mathbb{R}^m)$, there exists a unique solution $\mathbf{C}_m \in H^1(0,T;\mathbb{R}^m)$. Then we can construct u_m by

$$u_m(t) = \sum_{k=1}^{m} c_k(t) w_k,$$

and we deduce that $u_m(t) \in H^1(0,T;V)$. ∎

3.4 A PRIORI ESTIMATES

Theorem 3.2 *For every $t \in [0,T]$, the following estimate holds:*

$$|u_m(t)|^2 + \alpha \int_0^t \|u_m(s)\|_1^2 \, ds \leq \left|u^0\right|^2 + \frac{C_p^2}{\alpha} \int_0^T |f(s)|^2 \, ds. \tag{3.9}$$

Proof. Since $u_m \in H^1(0,T;V)$, we can take $v = u_m(t)$ as a test function in (3.7), it comes

$$(\dot{u}_m(t), u_m(t)) + \alpha(\nabla u_m(t), \nabla u_m(t)) = (f, u_m(t)), \ \forall t \in [0,T], \tag{3.10}$$

Noting that

$$(\dot{u}_m(t), u_m(t)) = \frac{1}{2}\frac{d}{dt} |u_m(t)|^2 \ \text{ and } \ (\nabla u_m(t), \nabla u_m(t)) = |\nabla u_m(t)|^2 = \|u_m(t)\|_1^2, \tag{3.11}$$

then by the Poincaré's and Young's inequalities, we have

$$\begin{aligned} |(f(t), u_m(t))| &\leq |f(t)|\,|u_m(t)| \\ &\leq C_p\,|f(t)|\,\|u_m(t)\|_1 = \left(\frac{C_p}{\sqrt{\alpha}}\,|f(t)|\right)\left(\sqrt{\alpha}\,\|u_m(t)\|_1\right) \\ &\leq \frac{C_p^2}{2\alpha}\,|f(t)|^2 + \frac{\alpha}{2}\,\|u_m(t)\|_1^2. \end{aligned} \tag{3.12}$$

Considering (3.11) and (3.12) in (3.10) it comes that

$$\frac{d}{dt}\,|u_m(t)|^2 + \alpha\,\|u_m(t)\|_1^2 \leq \frac{C_p^2}{2\alpha}\,|f(t)|^2 + \frac{\alpha}{2}\,\|u_m(t)\|_1^2,$$

after integration on $[0,t]$, we obtain

$$\frac{1}{2}\,|u_m(t)|^2 + \alpha \int_0^t \|u_m(s)\|_1^2 \, ds \leq \frac{1}{2}\left|u_m^0\right|^2 + \frac{C_p^2}{2\alpha}\int_0^t |f(s)|^2 \, ds + \frac{\alpha}{2}\int_0^t \|u_m(s)\|_1^2 \, ds.$$

Since

$$\left|u_m^0\right|^2 = \sum_{k=1}^{m} |\beta_k|^2 \leq \sum_{k=1}^{+\infty} |\beta_k|^2 = \left|u^0\right|^2$$

then

$$\frac{1}{2}\,|u_m(t)|^2 + \frac{\alpha}{2}\int_0^t \|u_m(s)\|_1^2 \, ds \leq \frac{1}{2}\left|u^0\right|^2 + \frac{C_p^2}{2\alpha}\int_0^t |f(s)|^2 \, ds,$$

Which ends the proof. ∎

Now we estimate $\dot{u}_m$.

Theorem 3.3 *The following estimate holds:*

$$\int_0^T |\dot{u}_m(s)|_*^2 \, ds \leq 2\alpha \left|u^0\right|^2 + 4C_p^2 \int_0^T |f(s)|^2 \, ds, \tag{3.13}$$

where $C_p$is the Poincaré constant.

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Proof. Let $v \in V$, then we can write $v = w + z$, with $w \in V_m = span\{w_1, w_2, ..., w_m\}$, and $z \in V_m^{\perp}$, the ortogonolity for $(\ ,\ )_1$, then

$$(v, w)_1 = (w + z, w)_1 = (w, w)_1 + (z, w)_1 = \|w\|_1^2.$$

Thus

$$(v, w)_1 = \|w\|_1^2, \forall v \in V_m.$$

On the other hand, by the Cauchy-Schwarz inequality, we infer that

$$\|w\|_1^2 = (v, w) \le \|v\|_1 \|w\|_1.$$

Thus we have in particular

$$\|w\|_1 \le \|v\|_1. \tag{3.14}$$

Taking $w \in V_m$ as a test function in (3.7), we get

$$(\dot{u}_m(t), v) = (\dot{u}_m(t), w) = -\alpha(\nabla u_m(t), \nabla w) + (f(t), w). \tag{3.15}$$

Since

$$|(\nabla u_m(t), \nabla w)| = |(u_m(t), w)_1| \le \|u_m\|_1 \|w\|_1,$$

we infer, using the Cauchy-Schwarz and Poincaré inequalities,

$$\begin{aligned} |(\dot{u}_m(t), v)| &\le \alpha \|u_m(t)\|_1 \|w\|_1 + |(f(t)| |w(t)| \\ &\le (\alpha \|u_m(t)\|_1 + C_p |f(t)|) \|w\|_1, \end{aligned}$$

and using (3.14), we get

$$|\langle \dot{u}_m(t), v\rangle| = |(\dot{u}_m(t), v)| \le (\alpha \|u_m(t)\|_1 + C_p |f(t)|) \|v\|_1.$$

Then, by the definition of the dual norm in V', we may write

$$\|\dot{u}_m(t)\|_* = \sup_{v \in V} \frac{|\langle \dot{u}_m(t), v\rangle|}{\|v\|_1} \le \alpha \|u_m(t)\|_1 + C_p |f(t)|, \tag{3.16}$$

Squaring both sides and using inequality $(a+b)^2 \le 2(a^2 + b^2)$, we get

$$\|\dot{u}_m(t)\|_*^2 \le 2\alpha^2 \|u_m(t)\|_1^2 + 2C_p^2 |f(t)|^2 \tag{3.17}$$

then integrating over $(0, t)$ it follows that

$$\int_0^t \|\dot{u}_m(t)\|_*^2 dt \le 2\alpha^2 \int_0^t \|u_m(s)\|_1^2 ds + 2C_p^2 \int_0^t |f(s)|^2 ds,$$

Going back to (3.9) we get

$$\begin{aligned} \int_0^t \|\dot{u}_m(t)\|_*^2 dt &\le 2\alpha \left(\left|u^0\right|^2 + \frac{C_p^2}{\alpha} \int_0^T |f(s)|^2 ds \right) + 2C_p^2 \int_0^t |f(s)|^2 ds \\ &\le 2\alpha \left|u^0\right|^2 + 4C_p^2 \int_0^T |f(s)|^2 ds. \end{aligned}$$

This ends the proof. ∎

3.5 EXISTENCE

The two precedent theorem shows that

$$\begin{cases} u_m \text{ is bounded in } L^\infty(0,T;V) \subset L^2(0,T;V), \\ \dot{u}_m \text{ is bounded in } L^2(0,T;V'). \end{cases} \tag{3.18}$$

Due to the weak compactness of these spaces, we can extract a subsequence, which for simplicity we still denote by $\{u_m\}$, such that

$$\begin{cases} u_m \rightharpoonup u \ \text{ in } L^2(0,T;V), \\ \dot{u}_m \rightharpoonup \chi \ \text{ in } L^2(0,T;V'). \end{cases}, \ \text{as } m \to \infty. \tag{3.19}$$

Since $L^2(0,T;V)$ and $L^2(0,T;V')$ are continuously included $\mathscr{D}'(]0,T[\,;V')$ then

$$\begin{cases} u_m \rightharpoonup u \ \text{ in } \mathscr{D}'(]0,T[\,;V'), \\ \dot{u}_m \rightharpoonup \chi \ \text{ in } \mathscr{D}'(]0,T[\,;V'). \end{cases}, \ \text{as } m \to \infty$$

and due tothe continuity of derivation in $\mathscr{D}'(]0,T[\,;V')$ we have $\chi = \dot{u}$.

This u is a solution of Problem (3.3). Precisely

Theorem 3.4 *Let $f \in L^2(0,T;L^2(\Omega))$ and $u^0 \in L^2(\Omega)$. Then, u is a solution of Problem* (3.3).

Proof. The convergences (3.19) mean

$$\int_0^T (\nabla u_m(t), \nabla v(t))dt \to \int_0^T (\nabla u(t), \nabla v(t))dt, \quad \forall v \in L^2(0,T;V),$$
$$\int_0^T (\dot{u}_m(t), v(t))dt = \int_0^T \langle \dot{u}_m(t), v(t)\rangle\, dt \to \int_0^T \langle \dot{u}(t), v(t)\rangle\, dt, \quad \forall v \in L^2(0,T;V).$$

To pass to the limit as $m \to \infty$, the test functions have to be chosen in V_m. Fix $v \in L^2(0,T;V)$, we consider, the partial sum that converges to v;

$$v_N(t) = \sum_{k=1}^{N} b_k(t) w_k.$$

Let $m > N$, then $v_N \in L^2(0,T;V_m)$. Multiplying (3.7) by $b_k(t)$ and summing up, we get

$$(\dot{u}_m(t), v_N(t)) + \alpha(\nabla u_m(t), \nabla v_N(t)) = (f(t), v_N).$$

Integrating on $(0,T)$ it comes

$$\int_0^T (\dot{u}_m(t), v_N(t)) + \alpha(\nabla u_m(t), \nabla v_N(t))dt = \int_0^T (f(t), v_N)dt. \tag{3.20}$$

Passing to the limit as $m \to \infty$, we obtain

$$\int_0^T \langle \dot{u}(t), v_N(t)\rangle + \alpha(\nabla u(t), \nabla v_N(t))dt = \int_0^T (f(t), v_N)dt.$$

Letting $N \to \infty$, then $v_N \to v$ strongly in $L^2(0,T;V)$ and it follows that

$$\int_0^T \langle \dot{u}(t), v(t)\rangle + \alpha(\nabla u(t), \nabla v(t))dt = \int_0^T (f(t), v(t))dt, \tag{3.21}$$

which means

$$\langle \dot{u}(t), v(t)\rangle + \alpha(\nabla u(t), \nabla v(t)) = (f(t), v(t)), \text{ a.e. in } L^2(0,T).$$

for all $v \in V$ and a.e. $t \in [0,T]$. Therefore u satisfies (3.3). From Theorem 2.12, we know that $u \in C([0,T];H)$.

It remains to check that $u(t)$ satisfies the initial condition $u(0) = u^0$. Let $v \in C^1([0,T];V)$ with $v(T) = 0$. Integrating by parts , we obtain

$$\int_0^T (\dot{u}_m(t), v_N(t))dt \ = -(u_m^0(0), v_N(0)) - \int_0^T (u_m(t), \dot{v}_N(t))dt$$

then (3.20) becomes

$$\int_0^T -(u_m(t), \dot{v}_N(t)) + \alpha(\nabla u(t), \nabla v_N(t))dt = -(u_m^0, v(0)) + \int_0^T (f(t), v_N(t))dt.$$

and letting $m \to \infty$ then $N \to \infty$, we get

$$\int_0^T -(u(t), \dot{v}(t)) + \alpha(\nabla u(t), \nabla v_N(t))dt = -(u^0, v(0)) + \int_0^T (f(t), v_N(t))dt. \tag{3.22}$$

Thanks to (2.9), we can integrate by parts (3.21). Letting $m \to \infty$ then $N \to \infty$, we deduce

$$\int_0^T -\langle \dot{v}(t), u(t)\rangle + \alpha(\nabla u(t), \nabla v(t))dt = -(u(0), v(0)) + \int_0^T (f(t), v(t))dt. \tag{3.23}$$

Recalling that $\langle \dot{v}(t), u(t)\rangle = (u(t), \dot{v}(t))$, since $v(t) \in L^2(0,T;V)$, and comparing with (3.22) and (3.23), one find

$$(u(0), v(0)) = (u^0, v(0)), \quad \forall v \in C^2([0,T];V), \text{ with } v(T) = 0,$$

i.e.

$$(u(0), v(0)) = (u^0, v(0)), \quad \forall v(0) \in V.$$

Thus $u(0) = u^0$ and the existence of the solution follows. ∎

3.6 STABILITY AND REGULARITY

3.6.1 *Stability*

Theorem 3.5 *The solution of Problem (3.3) satisfies*

$$|u(t)|^2 + \alpha \int_0^t \|u(s)\|_1^2 ds \le \left|u^0\right|^2 + \frac{C_p^2}{\alpha}\int_0^T |f(s)|^2 ds, \text{ for every } t \in [0,T]$$

and

$$\int_0^T \|\dot{u}(s)\|_*^2 ds \le 2\alpha \left|u^0\right|^2 + 4C_p^2 \int_0^T |f(s)|^2 ds.$$

Proof. It suffices to let $m \to +\infty$ in (3.9) and (3.13) and use the fact that $\|x\| < \liminf \|x_n\|$, if $x_n \rightharpoonup x$ weakly ∎

3.6.2 *Regularity*

Theorem 3.6 *We assume that* Ω *is a* C^2 *– domain. Assume that* $u^0 \in H^1(\Omega)$ *and* $f \in L^2(0,T;L^2(\Omega))$ *Then*

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), \ \dot{u} \in L^2(0,T;L^2(\Omega)).$$

Moreover

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|u\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\dot{u}\|_{L^2(0,T;L^2(\Omega))} \leq C_\alpha \left(\left\|u^0\right\|_1^2 + \int_0^T |f(s)|^2 \, ds \right).$$

Remark 3.5 *This result allows us to prove smoothness of solutions under the assumption that the data of Problem (3.3) are smooth. A proof can be found in [2].*

3.7 THE MAXIMUM PRINCIPLE

3.7.1 *Positivity of the Data and Positivity of the Solution*

Let $u \in H^1(\Omega)$. Since $C^1(\Omega)$ is dense in $H^1(\Omega)$, $u \geq 0$ on $\partial\Omega$ if there exists a sequence $\{v_k\}_{k\geq 1} \subset C^1(\Omega)$ such that

$$v_k \to u \text{ in } H^1(\Omega) \quad \text{and } v_k \geq 0.$$

It is as if the trace of u on $\partial\Omega$ "inherits" the nonnegativity from the sequence $\{v_k\}_{k\geq 1}$. Since $v_k \geq 0$ on $\partial\Omega$ is equivalent to saying that the negative part $v_k^- = max\{-v_k, 0\}$ has zero trace on $\partial\Omega$, it then turns out that

$$u \geq 0 \text{ on } \partial\Omega \Leftrightarrow u^- \in H_0^1(\Omega).$$

Similarly,

$$u \leq 0 \text{ on } \partial\Omega \Leftrightarrow u^+ \in H_0^1(\Omega).$$

We shall establish the following result.

Theorem 3.7 *We assume that*

$$u^0(x) \geq 0 \quad a.e. \text{ in } \Omega, \tag{3.24}$$
$$f(x,t) \geq 0 \quad a.e. \text{ in } \Omega, \forall t \in [0,T], \tag{3.25}$$

then the solution of problem (3.3) satisfy, for all $t \in [0,T]$

$$u(x,t) \geq 0 \quad a.e. \text{ in } \Omega.$$

Proof. Let $v \in H_0^1(\Omega)$. We may write $v = v^+ - v^-$ where v^+ and v^- denote respectively the positive and negative parts of v. We can check that if that if $v \in H_0^1(\Omega)$ then v^+ and $v^- \in H_0^1(\Omega)$. We may therefore take $v = -u^-$ as a test function in (3.3). We note that

$$(\nabla v, \nabla v^-) = (\nabla v^-, \nabla v^-)$$

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so that we obtain

$$\frac{1}{2}\frac{d}{dt}\left|u^{-}(t)\right|^{2}+\alpha\left\|\nabla u^{-}(t)\right\|_{1}^{2}=-(f(t),u^{-}(t)), \tag{3.26}$$

from (3.25) the right hand side of (3.26) is < 0 so that (3.26) implies

$$\frac{1}{2}\frac{d}{dt}\left|u^{-}(t)\right|^{2}\leq 0.$$

Since, from (3.24), $u^{-}(0) = 0$, we deduce that $u^{-} = 0$ from which we have the conclusion of the theorem. ■

3.7.2 *A Weak Maximum Principle*

We shall show

Theorem 3.8 *We assume that $f = 0$ and that*

$$u^{0}\in L^{\infty}(\Omega),\quad M=\left\|u^{0}\right\|_{L^{\infty}(\Omega)}$$

Then, u, the solution of (3.3), satisfies

$$u\in L^{\infty}(Q_{T}),\forall T>0$$

and

$$\|u\|_{L^{\infty}(Q_{T})}\leq\left\|u^{0}\right\|_{L^{\infty}(\Omega)}.$$

Proof. We take $v = (u-M)^{+}$ as a test function in (3.3). This is permissible since$v = 0$ on Σ. Then (3.3) becomes

$$\left(\frac{\partial u}{\partial t},(u-M)^{+}\right)+\alpha(\nabla u,\nabla(u-M)^{+})=0,\quad a.e.t\in[0,T].$$

But

$$(\nabla u,\nabla(u-M)^{+})=(\nabla(u-M),\nabla(u-M)^{+})=(\nabla(u-M)^{+},\nabla(u-M)^{+})\geq 0,$$

therefore

$$\frac{d}{dt}\left|(u-M)^{+}\right|^{2}\leq 0.$$

Since $(u-M)^{+}(x,0)=(u^{0}-M)^{+}=0$ we deduce that $(u-M)^{+}=0$ therefore $u\leq M$. Changing u to $-u$, or taking $v=(u+M)^{-}$ we have the result. ■

Remark 3.6 *The maximum principle implies the uniqueness of the solution of (3.3). It suffices to take $u^{0}=0$ in the precedent theorem.*

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