The 1-D Heat Equation

1 The 1-D Heat Equation

1.1 Physical derivation

In a metal rod with non-uniform temperature, heat (thermal energy) is transferred from regions of higher temperature to regions of lower temperature. Three physical principles are used here.

1. Heat (or thermal) energy of a body with uniform properties:

$$\text{Heat energy } = cmu,$$

where m is the body mass, u is the temperature, c is the specific heat, units $[c] = L^2T^{-2}U^{-1}$ (basic units are M mass, L length, T time, U temperature). c is the energy required to raise a unit mass of the substance 1 unit in temperature.

2. Fourier's law of heat transfer: rate of heat transfer proportional to negative temperature gradient,

$$\frac{\text{Rate of heat transfer}}{\text{area}} = -K_0\frac{\partial u}{\partial x} \tag{1}$$

where K_0 is the thermal conductivity, units $[K_0] = MLT^{-3}U^{-1}$. In other words, heat is transferred from areas of high temp to low temp.

3. Conservation of energy.

Consider a uniform rod of length l with non-uniform temperature lying on the x-axis from $x = 0$ to $x = l$. By uniform rod, we mean the density ρ, specific heat c, thermal conductivity K_0, cross-sectional area A are ALL constant. Assume the sides

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Short version of Matthew J. Hancock's course notes.

of the rod are insulated and only the ends may be exposed. Also assume there is no heat source within the rod. Consider an arbitrary thin slice of the rod of width Δx between x and $x + \Delta x$. The slice is so thin that the temperature throughout the slice is $u(x,t)$. Thus,

$$\text{Heat energy of segment} = c \times \rho A \Delta x \times u = c\rho A \Delta x u(x,t).$$

By conservation of energy,

$$\begin{array}{l}\text{change of}\\ \text{heat energy of}\\ \text{segment in time } \Delta t\end{array} = \begin{array}{l}\text{heat in from}\\ \text{left boundary}\end{array} - \begin{array}{l}\text{heat out from}\\ \text{right boundary}\end{array}.$$

From Fourier's Law (1),

$$c\rho A \Delta x u(x, t+\Delta t) - c\rho A \Delta x u(x,t) = \Delta t A \left(-K_0 \frac{\partial u}{\partial x}\right)_x - \Delta t A \left(-K_0 \frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

Rearranging yields (recall ρ, c, A, K_0 are constant),

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{K_0}{c\rho}\left(\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right)$$

Taking the limit $\Delta t, \Delta x \to 0$ gives the Heat Equation,

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{2}$$

where

$$\kappa = \frac{K_0}{c\rho} \tag{3}$$

is called the thermal diffusivity, units $[\kappa] = L^2/T$. Since the slice was chosen arbitrarily, the Heat Equation (2) applies throughout the rod.

1.2 Initial condition and boundary conditions

To make use of the Heat Equation, we need more information:

1. Initial Condition (IC): in this case, the initial temperature distribution in the rod $u(x,0)$.

2. Boundary Conditions (BC): in this case, the temperature of the rod is affected by what happens at the ends, $x = 0, l$. What happens to the temperature at the end of the rod must be specified. In reality, the BCs can be complicated. Here we consider three simple cases for the boundary at $x = 0$.

(I) Temperature prescribed at a boundary. For $t > 0$,

$$u(0,t) = u_1(t).$$

(II) Insulated boundary. The heat flow can be prescribed at the boundaries,

$$-K_0 \frac{\partial u}{\partial x}(0,t) = \phi_1(t)$$

(III) Mixed condition: an equation involving $u(0,t)$, $\partial u/\partial x\,(0,t)$, etc.

Example 1. Consider a rod of length l with insulated sides is given an initial temperature distribution of $f(x)$ degree C, for $0 < x < l$. Find $u(x,t)$ at subsequent times $t > 0$ if end of rod are kept at 0^o C.

The Heat Eqn and corresponding IC and BCs are thus

$$\begin{aligned}
\text{PDE:} \quad & u_t = \kappa u_{xx}, && 0 < x < l, && (4)\\
\text{IC:} \quad & u(x,0) = f(x), && 0 < x < l, && (5)\\
\text{BC:} \quad & u(0,t) = u(L,t) = 0, && t > 0. && (6)
\end{aligned}$$

Physical intuition: we expect $u \to 0$ as $t \to \infty$.

1.3 Non-dimensionalization

Dimensional (or physical) terms in the PDE (2): k, l, x, t, u. Others could be introduced in IC and BCs. To make the solution more meaningful and simpler, we group as many physical constants together as possible. Let the characteristic length, time and temperature be L_*, T_* and U_*, respectively, with dimensions $[L_*] = L$, $[T_*] = T$, $[U_*] = U$. Introduce dimensionless variables via

$$\hat{x} = \frac{x}{L_*}, \qquad \hat{t} = \frac{t}{T_*}, \qquad \hat{u}\left(\hat{x},\hat{t}\right) = \frac{u(x,t)}{U_*}, \qquad \hat{f}(\hat{x}) = \frac{f(x)}{U_*}. \tag{7}$$

The variables $\hat{x}$, $\hat{t}$, $\hat{u}$ are dimensionless (i.e. no units, $[\hat{x}] = 1$). The sensible choice for the characteristic length is $L_* = l$, the length of the rod. While x is in the range $0 < x < l$, $\hat{x}$ is in the range $0 < \hat{x} < 1$.

The choice of dimensionless variables is an ART. Sometimes the statement of the problem gives hints: e.g. the length l of the rod (1 is nicer to deal with than l, an unspecified quantity). Often you have to solve the problem first, look at the solution, and try to simplify the notation.

From the chain rule,

$$
\begin{aligned}
u_t &= \frac{\partial u}{\partial t} = U_* \frac{\partial \hat{u}}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial t} = \frac{U_*}{T_*} \frac{\partial \hat{u}}{\partial \hat{t}}, \\
u_x &= \frac{\partial u}{\partial x} = U_* \frac{\partial \hat{u}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} = \frac{U_*}{L_*} \frac{\partial \hat{u}}{\partial \hat{x}} \\
u_{xx} &= \frac{U_*}{L_*^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}
\end{aligned}
$$

Substituting these into the Heat Eqn (4) gives

$$
u_t = \kappa u_{xx} \qquad \Rightarrow \qquad \frac{\partial \hat{u}}{\partial \hat{t}} = \frac{T_* \kappa}{L_*^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}
$$

To make the PDE simpler, we choose $T_* = L_*^2/\kappa = l^2/\kappa$, so that

$$
\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}, \qquad 0 < \hat{x} < 1, \qquad \hat{t} > 0.
$$

The characteristic (diffusive) time scale in the problem is $T_* = l^2/\kappa$. For different substances, this gives time scale over which diffusion takes place in the problem. The IC (5) and BC (6) must also be non-dimensionalized:

$$
\begin{aligned}
\text{IC:} \qquad & \hat{u}(\hat{x}, 0) = \hat{f}(\hat{x}), \qquad & 0 < \hat{x} < 1, \\
\text{BC:} \qquad & \hat{u}\left(0, \hat{t}\right) = \hat{u}\left(1, \hat{t}\right) = 0, \qquad & \hat{t} > 0.
\end{aligned}
$$

1.4 Dimensionless problem

Dropping hats, we have the dimensionless problem

$$
\begin{aligned}
\text{PDE:} \qquad & u_t = u_{xx}, \qquad & 0 < x < 1, \qquad & (8) \\
\text{IC:} \qquad & u(x, 0) = f(x), \qquad & 0 < x < 1, \qquad & (9) \\
\text{BC:} \qquad & u(0, t) = u(1, t) = 0, \qquad & t > 0, \qquad & (10)
\end{aligned}
$$

where x, t are dimensionless scalings of physical position and time.

2 Separation of variables

We look for a solution to the dimensionless Heat Equation (8) – (10) of the form

$$
u(x, t) = X(x)\, T(t) \tag{11}
$$

Take the relevant partial derivatives:

$$u_{xx} = X''(x)\,T(t)\,, \qquad u_t = X(x)\,T'(t)$$

where primes denote differentiation of a single-variable function. The PDE (8), $u_t = u_{xx}$, becomes

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

The left hand side (l.h.s.) depends only on t and the right hand side (r.h.s.) only depends on x. Hence if t varies and x is held fixed, the r.h.s. is constant, and hence T'/T must also be constant, which we set to $-\lambda$ by convention:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \qquad \lambda = \text{constant}. \tag{12}$$

The BCs become, for $t > 0$,

$$\begin{aligned} u(0,t) &= X(0)\,T(t) = 0 \\ u(1,t) &= X(1)\,T(t) = 0 \end{aligned}$$

Taking $T(t) = 0$ would give $u = 0$ for all time and space (called the trivial solution), from (11), which does not satisfy the IC unless $f(x) = 0$. If you are lucky and $f(x) = 0$, then $u = 0$ is the solution (this has to do with uniqueness of the solution, which we'll come back to). If $f(x)$ is not zero for all $0 < x < 1$, then $T(t)$ cannot be zero and hence the above equations are only satisfied if

$$X(0) = X(1) = 0. \tag{13}$$

2.1 Solving for $X(x)$

We obtain a boundary value problem for $X(x)$, from (12) and (13),

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < 1, \tag{14}$$

$$X(0) = X(1) = 0. \tag{15}$$

This is an example of a Sturm-Liouville problem (from your ODEs class).

There are 3 cases: $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$.

(i) $\lambda < 0$. Let $\lambda = -k^2 < 0$. Then the solution to (14) is

$$X = Ae^{kx} + Be^{-kx}$$

for integration constants A, B found from imposing the BCs (15),

$$X(0) = A + B = 0, \qquad X(1) = Ae^{k} + Be^{-k} = 0.$$

The first gives $A = -B$, the second then gives $A\left(e^{2k} - 1\right) = 0$, and since $|k| > 0$ we have $A = B = u = 0$, which is the trivial solution. Thus we discard the case $\lambda < 0$.

(ii) $\lambda = 0$. Then $X(x) = Ax + B$ and the BCs imply $0 = X(0) = B$, $0 = X(1) = A$, so that $A = B = u = 0$. We discard this case also.

(iii) $\lambda > 0$. In this case, (14) is the simple harmonic equation whose solution is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right). \tag{16}$$

The BCs imply $0 = X(0) = A$, and $B\sin\sqrt{\lambda} = 0$. We don't want $B = 0$, since that would give the trivial solution $u = 0$, so we must have

$$\sin\sqrt{\lambda} = 0. \tag{17}$$

Thus $\sqrt{\lambda} = n\pi$, for any nonzero integer n ($n = 1, 2, 3, ...$). We use subscripts to label the particular n-value. The values of λ are called the eigenvalues of the Sturm-Liouville problem (14),

$$\lambda_n = n^2\pi^2, \qquad n = 1, 2, 3, ...$$

and the corresponding solutions of (14) are called the eigenfunctions of the Sturm-Liouville problem (14),

$$X_n(x) = b_n \sin(n\pi x), \qquad n = 1, 2, 3, ... \tag{18}$$

We have assumed that $n > 0$, since $n < 0$ gives the same solution as $n > 0$.

2.2 Solving for $T(t)$

When solving for $X(x)$, we found that non-trivial solutions arose for $\lambda = n^2\pi^2$ for all nonzero integers n. The equation for $T(t)$ is thus, from (12),

$$T'(t) = -n^2\pi^2 T(t)$$

and, for n, the solution is

$$T_n = c_n e^{-n^2\pi^2 t}, \qquad n = 1, 2, 3, ... \tag{19}$$

where the c_n's are constants of integration.

2.3 Full solution $u(x,t)$

Putting things together, we have, from (11), (18) and (19),

$$u_n(x,t) = B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}, \qquad n = 1,2,3,... \tag{20}$$

where $B_n = c_n b_n$. Each function $u_n(x,t)$ is a solution to the PDE (8) and the BCs (10). But, in general, they will not individually satisfy the IC (9),

$$u_n(x,0) = B_n \sin(n\pi x) = f(x).$$

We now apply the principle of superposition: if u_1 and u_2 are two solutions to the PDE (8) and BC (10), then $c_1 u_1 + c_2 u_2$ is also a solution, for any constants c_1, c_2. This relies on the linearity of the PDE and BCs. We will, of course, soon make this more precise....

Since each $u_n(x,0)$ is a solution of the PDE, then the principle of superposition says any finite sum is also a solution. To solve the IC, we will probably need all the solutions u_n, and form the infinite sum (convergence properties to be checked),

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t). \tag{21}$$

$u(x,t)$ satisfies the BCs (10) since each $u_n(x,t)$ does. Assuming term-by-term differentiation holds (to be checked) for the infinite sum, then $u(x,t)$ also satisfies the PDE (8). To satisfy the IC, we need to find B_n's such that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x). \tag{22}$$

This is the Fourier Sine Series of $f(x)$.

To solve for the B_n's, we use the orthogonality property for the eigenfunctions $\sin(n\pi x)$,

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\, dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases} = \frac{1}{2}\delta_{mn} \tag{23}$$

where δ_{mn} is the kronecker delta,

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

The orthogonality relation (23) is derived by substituting

$$2\sin(m\pi x)\sin(n\pi x) = \cos((m-n)\pi x) - \cos((m+n)\pi x)$$

into the integral on the left hand side of (23) and noting

$$\int_0^1 \cos(m\pi x)\, dx = \delta_{m0}.$$

The orthogonality of the functions $\sin(n\pi x)$ is analogous to that of the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in 2-space; the integral from 0 to 1 in (23) above is analogous to the dot product in 2-space.

To solve for the B_n's, we multiply both sides of (22) by $\sin(m\pi x)$ and integrate from 0 to 1:

$$\int_0^1 \sin(m\pi x)\, f(x)\, dx = \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi x) \sin(m\pi x)\, dx$$

Substituting (23) into the right hand side yields

$$\int_0^1 \sin(m\pi x)\, f(x)\, dx = \sum_{n=1}^{\infty} B_n \frac{1}{2} \delta_{nm}$$

By definition of δ_{nm}, the only term that is non-zero in the infinite sum is the one where $n = m$, thus

$$\int_0^1 \sin(m\pi x)\, f(x)\, dx = \frac{1}{2} B_m$$

Rearranging yields

$$B_m = 2 \int_0^1 \sin(m\pi x)\, f(x)\, dx. \tag{24}$$

The full solution is, from (20) and (21),

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}, \tag{25}$$

where B_n are given by (24).

To derive the solution (25) of the Heat Equation (8) and corresponding BCs (10) and IC (9), we used properties of linear operators and infinite series that need justification.

3 Example : Cooling of a rod from a constant initial temperature

Suppose the initial temperature distribution $f(x)$ in the rod is constant, i.e. $f(x) = u_0$. The solution for the temperature in the rod is (25),

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t},$$

where, from (24), the Fourier coefficients are given by

$$B_n = 2\int_0^1 \sin(n\pi x)\, f(x)\, dx = 2u_0 \int_0^1 \sin(n\pi x)\, dx.$$

Calculating the integrals gives

$$B_n = 2u_0 \int_0^1 \sin(n\pi x)\, dx = -2u_0 \frac{\cos(n\pi) - 1}{n\pi} = -\frac{2u_0}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4u_0}{n\pi} & n \text{ odd} \end{cases}$$

In other words,

$$B_{2n} = 0, \qquad B_{2n-1} = \frac{4u_0}{(2n-1)\pi}$$

and the solution becomes

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{(2n-1)} \exp\left(-(2n-1)^2 \pi^2 t\right). \tag{26}$$

3.1 Approximate form of solution

The number of terms of the series (26) needed to get a good approximation for $u(x,t)$ depends on how close t is to 0. The series is

$$u(x,t) = \frac{4u_0}{\pi}\left(\sin(\pi x)\, e^{-\pi^2 t} + \frac{\sin(3\pi x)}{3} e^{-9\pi^2 t} + \cdots\right).$$

The ratio of the first and second terms is

$$\begin{aligned} \frac{|\text{second term}|}{|\text{first term}|} &= \frac{e^{-8\pi^2 t}}{3} \frac{|\sin 3\pi x|}{|\sin \pi x|} \\ &\le e^{-8\pi^2 t} \qquad \text{using} \qquad |\sin nx| \le n\,|\sin x| \\ &\le e^{-8} \qquad \text{for} \qquad t \ge \frac{1}{\pi^2} \\ &< 0.00034 \end{aligned}$$

It can also be shown that the first term dominates the sum of the rest of the terms, and hence

$$u(x,t) \approx \frac{4u_0}{\pi} \sin(\pi x)\, e^{-\pi^2 t}, \quad \text{for} \quad t \ge \frac{1}{\pi^2}. \tag{27}$$

What does $t = 1/\pi^2$ correspond to in physical time? In physical time $t' = l^2 t/\kappa$ (recall our scaling - here we use t as the dimensionless time and t' as dimensional time), $t = 1/\pi^2$ corresponds to:

$t' \approx 15$ minutes,	for a 1 m rod of copper	($\kappa \approx 1.1$ cm^2 sec^{-1})
$t' \approx 169$ minutes,	for a 1 m rod of steel	($\kappa \approx 0.1$ cm^2 sec^{-1})
$t' \approx 47$ hours,	for a 1 m rod of glass	($\kappa \approx 0.006$ cm^2 sec^{-1})

At $t = 1/\pi^2$, the temperature at the center of the rod ($x = 1/2$) is, from (27),

$$u(x,t) \approx \frac{4}{\pi e} u_0 = 0.47 u_0.$$

Thus, after a (scaled) time $t = 1/\pi^2$, the temperature has decreased by a factor of 0.47 from the initial temperature u_0.

3.2 Geometrical visualization of the solution

To analyze qualitative features of the solution, we draw various types of curves in 2D:

1. Spatial temperature profile, given by

$$u = u(x, t_0)$$

where t_0 is a fixed value of x. These profiles are curves in the ux-plane.

2. Temperature profiles in time,

$$u = u(x_0, t)$$

where x_0 is a fixed value of x. These profiles are curves in the ut-plane.

3. Curves of constant temperature in the xt-plane (level curves),

$$u(x,t) = C$$

where C is a constant.

Note that the solution $u = u(x,t)$ is a 2D surface in the 3D uxt-space. The above families of curves are the different cross-sections of this solution surface. Drawing the 2D cross-sections is much simpler than drawing the 3D solution surface.

Sketch typical curves: when sketching the curves in 1-3 above, we draw a few typical curves and any special cases. While math packages such as Matlab can be used to compute the curves from, say, 20 terms in the full power series solution (26), the emphasis in this course is to use simple considerations to get a rough idea of what the solution looks like. For example, one can use the first term approximation (27), simple physical considerations on heat transfer, and the fact that the solution $u(x,t)$ is continuous in x and t, so that if t_1 is close to t_1, $u(x,t_1)$ is close to $u(x,t_2)$.

3.2.1 Spatial temperature profiles

For fixed $t = t_0$, the first term approximate solution (27) is

$$u(x,t) \approx \frac{4u_0}{\pi} e^{-\pi^2 t_0} \sin(\pi x), \qquad t \geq 1/\pi^2. \tag{28}$$

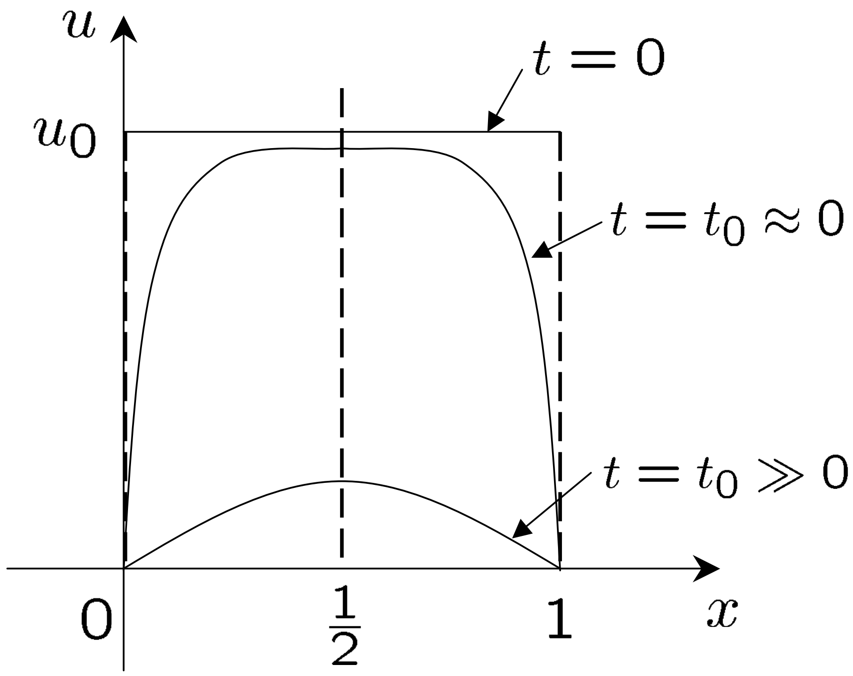

Figure 1: Spatial temperature profiles $u(x, t_0)$.

This suggests the center of the rod, $x = 1/2$, is a line of symmetry for $u(x, t)$, i.e.

$$u\left(\frac{1}{2} + s, t\right) = u\left(\frac{1}{2} - s, t\right)$$

and, for each fixed time, the location of the maximum (minimum) temperature if $u_0 > 0$ ($u_0 < 0$). We can prove the symmetry property by noting that the original PDE/BC/IC problem is invariant under the transformation $x \to 1 - x$. Note also that, from (26),

$$u_x(x, t) = 4u_0 \sum_{n=1}^{\infty} \cos\left((2n-1)\pi x\right) \exp\left(-(2n-1)^2 \pi^2 t\right)$$

Thus $u_x(1/2, t) = 0$ and $u_{xx}(1/2, t) < 0$, so the 2nd derivative test implies that $x = 1/2$ is a local max.

In Figure 1, we have plotted two typical profiles, one at early times $t = t_0 \approx 0$ and the other at late times $t = t_0 \gg 0$, and two special profiles, the initial temperature at $t = 0$ ($u = u_0$) and the temperature as $t \to \infty$ ($u = 0$). The profile for $t = t_0 \gg 0$ is found from the first term approximation (28). The line of symmetry $x = 1/2$ is plotted as a dashed line for reference in Figure 1.

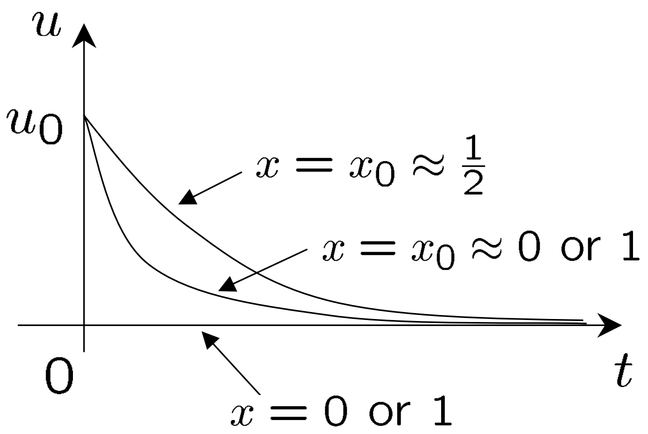

Figure 2: Temperature profiles in time $u(x_0, t)$.

3.2.2 Temperature profiles in time

Setting $x = x_0$ in the approximate solution (27),

$$u(x_0, t) \approx \left(\frac{4u_0}{\pi} \sin(\pi x_0) \right) e^{-\pi^2 t}, \qquad t \geq 1/\pi^2.$$

Two typical profiles are sketched in Figure 2, one near the center of the rod ($x_0 \approx 1/2$) and one near the edges ($x_0 \approx 0$ or 1). To draw these we noted that the center of the rod cools more slowly than points near the ends. One special profile is plotted, namely the temperature at the rod ends ($x = 0, 1$).

4 Equilibrium temperature profile (steady-state)

Intuition tells us that if the ends of the rod are held at 0^o C and there are no heat sources or sinks in the rod, the temperature in the rod will eventually reach 0. The solution above confirms this. However, we do not have to solve the full problem to determine the asymptotic or long-time behavior of the solution.

Instead, the equilibrium or steady-state solution $u = u_E(x)$ must be independent of time, and will thus satisfy the PDE and BCs with $u_t = 0$,

$$u_E'' = 0, \quad 0 < x < 1; \quad u_E(0) = u_E(1) = 0.$$

The solution is $u_E = c_1 x + c_2$ and imposing the BCs implies $u_E(x) = 0$. In other words, regardless of the initial temperature distribution $u(x,0) = f(x)$ in the rod, the temperature eventually goes to zero.

4.1 Rate of decay of $u(x,t)$

How fast does u approach $u_E = 0$? From our estimate (82) above,

$$|u_n(x,t)| \le Be^{-n^2\pi^2 t}, \qquad n = 1, 2, 3, ...$$

where B is a constant. Noting that $e^{-n^2\pi^2 t} \le e^{-n\pi^2 t} = \left(e^{-\pi^2 t}\right)^n$, we have

$$\left|\sum_{n=1}^{\infty} u_n(x,t)\right| \le \sum_{n=1}^{\infty} |u_n(x,t)| \le B\sum_{n=1}^{\infty} r^n = \frac{Br}{1-r}, \qquad r = e^{-\pi^2 t} < 1 \ (t > 0).$$

The last step is the geometric series result $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$, for $|r| < 1$. Thus

$$\left|\sum_{n=1}^{\infty} u_n(x,t)\right| \le \frac{Be^{-\pi^2 t}}{1-e^{-\pi^2 t}}, \qquad t > 0. \tag{29}$$

Therefore

- $u(x,t)$ approaches the steady-state $u_E(x) = 0$ exponentially fast (i.e. the rod cools quickly)
- the first term in the series, $\sin(\pi x)\, e^{-\pi^2 t}$ (term with smallest eigenvalue $\lambda = \pi$) determines the rate of decay of $u(x,t)$
- the B_n's may also affect the rate of approach to the steady-state, for other problems

4.2 Error of first-term approximation

Using the method in the previous section, we can compute the error between the first-term and the full solution (26),

$$\begin{aligned} |u(x,t) - u_1(x,t)| &= \left| \sum_{n=1}^{\infty} u_n(x,t) - u_1(x,t) \right| \\ &= \left| \sum_{n=2}^{\infty} u_n(x,t) \right| \leq \sum_{n=2}^{\infty} |u_n(x,t)| \leq B \sum_{n=2}^{\infty} r^n = \frac{Br^2}{1-r} \end{aligned}$$

where $r = e^{-\pi^2 t}$. Hence the solution $u(x,t)$ approaches the first term $u_1(x,t)$ exponentially fast,

$$|u(x,t) - u_1(x,t)| \leq \frac{Be^{-2\pi^2 t}}{1 - e^{-\pi^2 t}}, \qquad t > 0. \tag{30}$$

With a little more work we can get a much tighter (i.e. better) upper bound. We consider the sum

$$\sum_{n=N}^{\infty} |u_n(x,t)|$$

Substituting for $u_n(x,t)$ gives

$$\sum_{n=N}^{\infty} |u_n(x,t)| \leq B \sum_{n=N}^{\infty} e^{-n^2\pi^2 t} = Be^{-N^2\pi^2 t} \sum_{n=N}^{\infty} e^{-\left(n^2 - N^2\right)\pi^2 t} \tag{31}$$

Here's the trick: for $n \geq N$,

$$n^2 - N^2 = (n+N)(n-N) \geq 2N(n-N) \geq 0.$$

Since e^{-x} is a decreasing function, then

$$e^{-\left(n^2-N^2\right)\pi^2 t} \leq e^{-2N(n-N)\pi^2 t} \tag{32}$$

for $t > 0$. Using the inequality (32) in Eq. (31) gives

$$\begin{aligned} \sum_{n=N}^{\infty} |u_n(x,t)| &\leq Be^{-N^2\pi^2 t} \sum_{n=N}^{\infty} e^{-2N(n-N)\pi^2 t} \\ &= Be^{N^2\pi^2 t} \sum_{n=N}^{\infty} \left(e^{-2N\pi^2 t}\right)^n = Be^{N^2\pi^2 t} \frac{\left(e^{-2N\pi^2 t}\right)^N}{1 - e^{-2N\pi^2 t}}. \end{aligned}$$

Simplifying the expression on the right hand side yields

$$\sum_{n=N}^{\infty} |u_n(x,t)| \leq \frac{Be^{-N^2\pi^2 t}}{1 - e^{-2N\pi^2 t}} \tag{33}$$

For $N = 1$, (33) gives the temperature decay

$$|u(x,t)| = \left|\sum_{n=N}^{\infty} u_n(x,t)\right| \leq \sum_{n=N}^{\infty} |u_n(x,t)| \leq \frac{Be^{-\pi^2 t}}{1 - e^{-2\pi^2 t}}, \tag{34}$$

which is slightly tighter than (29). For $N = 2$, (33) gives the error between $u(x,t)$ and the first term approximation,

$$|u(x,t) - u_1(x,t)| = \left|\sum_{n=2}^{\infty} u_n(x,t)\right| \leq \sum_{n=2}^{\infty} |u_n(x,t)| \leq \frac{Be^{-4\pi^2 t}}{1 - e^{-4\pi^2 t}}$$

which is a much better upper bound than (30).

Note that for an initial condition $u(x,0) = f(x)$ that is is symmetric with respect to $x = 1/2$ (e.g. $f(x) = u_0$), then $u_{2n} = 0$ for all n, and hence error between $u(x,t)$ and the first term $u_1(x,t)$ is even smaller,

$$|u(x,t) - u_1(x,t)| = \left|\sum_{n=3}^{\infty} u_n(x,t)\right| \leq \sum_{n=3}^{\infty} |u_n(x,t)|$$

Applying result (33) with $N = 3$ gives

$$|u(x,t) - u_1(x,t)| \leq \frac{Be^{-9\pi^2 t}}{1 - e^{-6\pi^2 t}}.$$

Thus, in this case, the error between the solution $u(x,t)$ and the first term $u_1(x,t)$ decays as $e^{-9\pi^2 t}$ - very quickly!

7 Variations on the basic Heat Problem

We now consider variations to the basic Heat Problem, including different types of boundary conditions and the presence of sources and sinks.

Boundary conditions

Type I BCs (Dirichlet conditions)

Type I, or Dirichlet, BCs specify the temperature $u(x,t)$ at the end points of the rod, for $t > 0$,

$$\begin{aligned} u(0,t) &= g_1(t), \\ u(1,t) &= g_2(t). \end{aligned}$$

Type I Homogeneous BCs are

$$\begin{aligned} u(0,t) &= 0, \\ u(1,t) &= 0. \end{aligned}$$

The physical significance of these BCs for the rod is that the ends are kept at 0^o C. The solution to the Heat Equation with Type I BCs was considered in class. After separation of variables $u(x,t) = X(x)T(t)$, the associated Sturm-Liouville Boundary Value Problem for $X(x)$ is

$$X'' + \lambda X = 0; \qquad X(0) = X(1) = 0.$$

The eigenfunctions are $X_n(x) = B_n \sin(n\pi x)$.

7.1.2 Type II BCs (Newmann conditions)

Type II, or Newmann, BCs specify the rate of change of temperature $\partial u/\partial x$ (or heat flux) at the ends of the rod, for $t > 0$,

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= g_1(t), \\ \frac{\partial u}{\partial x}(1,t) &= g_2(t). \end{aligned}$$

Type II Homogeneous BCs are

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0, \\ \frac{\partial u}{\partial x}(1,t) &= 0. \end{aligned}$$

The physical significance of these BCs for the rod is that the ends are insulated. These lead to another relatively simple solution involving a cosine series (see problem 6 on PS 1). After separation of variables $u(x,t) = X(x)T(t)$, the associated Sturm-Liouville Boundary Value Problem for $X(x)$ is

$$X'' + \lambda X = 0; \qquad X'(0) = X'(1) = 0.$$

The eigenfunctions are $X_0(x) = A_0 = const$ and $X_n(x) = A_n \cos(n\pi x)$.

7.1.3 Type III BCs (Mixed)

The general Type III BCs are a mixture of Type I and II, for $t > 0$,

$$\begin{aligned} \alpha_1 \frac{\partial u}{\partial x}(0,t) + \alpha_2 u(0,t) &= g_1(t), \\ \alpha_3 \frac{\partial u}{\partial x}(1,t) + \alpha_4 u(1,t) &= g_2(t). \end{aligned}$$

After separation of variables $u(x,t) = X(x)T(t)$, the associated Sturm-Liouville Boundary Value Problem for $X(x)$ is

$$X'' + \lambda X = 0,$$

$$\begin{aligned} \alpha_1 X'(0) + \alpha_2 X(0) &= 0 \\ \alpha_3 X'(0) + \alpha_4 X(0) &= 0 \end{aligned}$$

The associated eigenfunctions depend on the values of the constants $\alpha_{1,2,3,4}$.

Example 1. $\alpha_1 = \alpha_4 = 1$, $\alpha_2 = \alpha_3 = 0$. Then

$$X'' + \lambda X = 0; \qquad X'(0) = X(1) = 0$$

and the eigenfunctions are $X_n = A_n \cos\left(\frac{2n-1}{2}\pi x\right)$. Note: the constant $X_0 = A_0$ is not an eigenfunction here.

Example 2. $\alpha_1 = \alpha_4 = 0$, $\alpha_2 = \alpha_3 = 1$. Then

$$X'' + \lambda X = 0; \qquad X(0) = X'(1) = 0$$

and the eigenfunctions are $X_n = B_n \sin\left(\frac{2n-1}{2}\pi x\right)$. Note: the constant $X_0 = A_0$ is not an eigenfunction here.

Note: Starting with the BCs in Example 1 and rotating the rod about $x = 1/2$, you'd get the BCs in Example 2. It is not surprising then that under the change of variables $\tilde{x} = 1-x$, Example 1 becomes Example 2, and vice versa. The eigenfunctions also possess this symmetry, since

$$\sin\left(\frac{2n-1}{2}\pi(1-x)\right) = (-1)^n \cos\left(\frac{2n-1}{2}\pi x\right).$$

Since we can absorb the $(-1)^n$ into the constant B_n, the eigenfunctions of Example 1 become those of Example 2 under the transformation $\tilde{x} = 1-x$, and vice versa.

7.2 Solving the Heat Problem with Inhomogeneous (time-independent) BCs

Consider the Heat Problem with inhomogeneous Type I BCs,

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < 1 \\ u(0,t) &= 0, \qquad u(1,t) = u_1 = const, \qquad t > 0, \\ u(x,0) &= 0, \qquad 0 < x < 1. \end{aligned} \tag{45}$$

Directly applying separation of variables $u(x,t) = X(x)T(t)$ is not useful, because we'd obtain $X(1)T(t) = u_1$ for $t > 0$. The strategy is to rewrite the solution $u(x,t)$ in terms of a new variable $v(x,t)$ such that the new problem for v has homogeneous BCs!

Step 1. Find the steady-state, or equilibrium solution $u_E(x)$, since this by definition must satisfy the PDE and the BCs,

$$\begin{aligned} u_E'' &= 0, \qquad 0 < x < 1 \\ u_E(0) &= 0, \qquad u_E(1) = u_1 = const. \end{aligned}$$

Solving for u_E gives $u_E(x) = u_1 x$.

Step 2. Transform variables by introducing a new variable $v(x,t)$,

$$v(x,t) = u(x,t) - u_E(x) = u(x,t) - u_1 x. \tag{46}$$

Substituting this into the Heat Problem (45) gives a new Heat Problem,

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1 \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0, \\ v(x,0) &= -u_1 x, \qquad 0 < x < 1. \end{aligned} \tag{47}$$

Notice that the BCs are now homogeneous, and the IC is now inhomogeneous. Notice also that we know how to solve this - since it's the basic Heat Problem! Based on our work, we know that the solution to (47) is

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}, \qquad B_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \tag{48}$$

where $f(x) = -u_1 x$. Substituting for $f(x)$ and integrating by parts, we find

$$\begin{aligned} B_n &= -2u_1 \int_0^1 x\sin(n\pi x)\,dx \\ &= -2u_1\left(-\left.\frac{x\cos(n\pi x)}{n\pi}\right|_{x=0}^{1} + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx \right) \\ &= \frac{2u_1(-1)^n}{n\pi} \end{aligned} \tag{49}$$

Step 3. Transform back to $u(x,t)$, from (46), (48) and (49),

$$u(x,t) = u_E(x) + v(x,t) = u_1 x + \frac{2u_1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x)\, e^{-n^2\pi^2 t}. \tag{50}$$

The term $u_E(x)$ is the steady state and the term $v(x,t)$ is called the transient, since it exists initially to satisfy the initial condition but vanishes as $t \to \infty$. You can check for yourself by direct substitution that Eq. (50) is the solution to the inhomogeneous Heat Problem (45), i.e. the PDE, BCs and IC.

7.3 Heat sources

7.3.1 Derivation

To add a heat source to the derivation of the Heat Equation, we modify the energy balance equation to read,

$$\begin{array}{c}\text{change of}\\ \text{heat energy of}\\ \text{segment in time } \Delta t\end{array} = \begin{array}{c}\text{heat in from}\\ \text{left boundary}\end{array} - \begin{array}{c}\text{heat out from}\\ \text{right boundary}\end{array} + \begin{array}{c}\text{heat generated}\\ \text{in segment}\end{array}.$$

Let $Q(x,t)$ be the heat generated per unit time per unit volume at position x in the rod. Then the last term in the energy balance equation is just $QA\Delta x\Delta t$. Applying Fourier's Law (1) gives

$$\begin{aligned} c\rho A\Delta x u(x,t+\Delta t) - c\rho A\Delta x u(x,t) &= \Delta t A\left(-K_0\frac{\partial u}{\partial x}\right)_x - \Delta t A\left(-K_0\frac{\partial u}{\partial x}\right)_{x+\Delta x} \\ &\quad + QA\Delta x\Delta t \end{aligned}$$

The last term is new; the others we had for the rod without sources. Dividing by $A\Delta x\Delta t$ and rearranging yields

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = \frac{K_0}{c\rho}\left(\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right) + \frac{Q}{c\rho}.$$

Taking the limit $\Delta t, \Delta x \to 0$ gives the Heat Equation with a heat source,

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} + \frac{Q}{c\rho} \tag{51}$$

Introducing non-dimensional variables $\tilde{x} = x/l$, $\tilde{t} = \kappa t/l^2$ gives

$$\frac{\partial u}{\partial \tilde{t}} = \frac{\partial^2 u}{\partial \tilde{x}^2} + \frac{l^2 Q}{\kappa c\rho} \tag{52}$$

Defining the dimensionless source term $q = l^2 Q/(\kappa c\rho)$ and dropping tildes gives the dimensionless Heat Problem with a source,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q \tag{53}$$

7.3.2 Solution method

The simplest case is that of a constant source $q = q(x)$ in the rod. The Heat Problem becomes,

$$\begin{aligned} u_t &= u_{xx} + q(x), \qquad 0 < x < 1, \\ u(0,t) &= b_1, \qquad u(1,t) = b_2, \qquad t > 0, \\ u(x,0) &= f(x), \qquad 0 < x < 1. \end{aligned} \tag{54}$$

If $q(x) > 0$, heat is generated at x in the rod; if $q(x) < 0$, heat is absorbed.

The solution method is the same as that for inhomogeneous BCs: find the equilibrium solution $u_E(x)$ that satisfies the PDE and the BCs,

$$\begin{aligned} 0 &= u_E'' + q(x), \qquad 0 < x < 1, \\ u_E(0) &= b_1, \qquad u_E(1) = b_2. \end{aligned}$$

Then let

$$v(x,t) = u(x,t) - u_E(x).$$

Substituting $u(x,t) = v(x,t) + u_E(x)$ into (54) gives a problem for $v(x,t)$,

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1, \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0, \\ v(x,0) &= f(x) - u_E(x), \qquad 0 < x < 1. \end{aligned}$$

Note that $v(x,t)$ satisfies the homogeneous Heat Equation (PDE) and homogeneous BCs, i.e. the basic Heat Problem. Solve the Heat Problem for $v(x,t)$ and then obtain $u(x,t) = v(x,t) + u_E(x)$.

Note that things get complicated if the source is time-dependent - we won't see that in this course.