2

ELEMENTS OF FUNCTIONAL ANALYSIS

In this chapter, we collect various definitions and theorems which are key tools for solving partial differential equations. Some proofs are omitted and may be found in [1, 1].

2.1 WEAK CONVERGENCES

Let X be a Banach space and X' denote its dualwith norm

$$\|f\|_{X'} = \sup_{x\in X, \|x\|_X \leq 1} |\langle f, x\rangle|.$$

The bidual X'' is the dual of X' with norm

$$\|\xi\|_{X'} = \sup_{f\in X', \|f\|_{X'} \leq 1} |\langle \xi, f\rangle|, \qquad \xi \in X''.$$

There is a canonical injection $J : X \to X''$ defined as follows: given $x \in X$, the map $f \mapsto \langle f, x\rangle$ is a continuous linear functional on X, thus it is an element of X'' , which we denote by Jx. We have

$$\langle Jx, f\rangle_{X'',X'} = \langle f, x\rangle_{X',X}, \qquad \forall x \in X, f \in X'.$$

Exercise 1 *Check that J is linear and that J is an isometry, that is, $\|Jx\|_{X''} = \|x\|_X$.*

2.1.1 *Reflexive Spaces, Separable Spaces*

It may happen that J is not surjective from X onto X''. However, it is convenient to identify X with a subspace of X using J. If X turns out to be surjective then one says that E is reflexive, and E is identified with E.

When X is reflexive, X'' is usually identified with X.

Remark 2.1 *Many important spaces in analysis are reflexive.*

- *Finite-dimensional spaces are reflexive (since* $\dim E = \dim E' = \dim E''$*).*
- L^p *spaces are reflexive for* $1 < p < \infty$.
- *Hilbert spaces are reflexive.*
- *However,* L^1*and* L^∞*are not reflexive.*

Definition 2.1 *We say that a metric space* X *is separable if there exists a subset* $D \subset X$ *that is countable and dense.*

Remark 2.2 *Many important spaces in analysis are separable. Finite-dimensional spaces are separable.* L^p *spaces are separable for* $1 \leq p < \infty$. L^∞ *is not separable.*

Proposition 2.1 *Let* X *be a separable metric space and let* $Y \subset X$ *be any subset. Then* Y *is also separable.*

Example 2.1 *Since* $L^2(\Omega)$ *is separable, then* $H^1(\Omega)$ *and* $H_0^1(\Omega)$ *are separables.*

2.1.2 *Weak Convergences*

Let (x_n) be a sequence of X. We denote

$$x_n \rightharpoonup x$$

the weak convergence of x_n to x for the weak topology on X denoted by $\sigma(X, X')$.

Proposition 2.2 *We have*

- $x_n \rightharpoonup x$ *for* $\sigma(X, X') \Leftrightarrow \langle f, x_n \rangle \to \langle f, x \rangle, \forall f \in X'$.
- $x_n \to x$ *strongly, then* $x_n \rightharpoonup x$ *weakly for* $\sigma(X, X')$.
- *If* $x_n \to x$ *weakly for* $\sigma(X, X')$*, then* $\|x_n\|$*is bounded and* $\|x\| < \liminf \|x_n\|$.
- *If* $x_n \to x$ *weakly for* $\sigma(X, X')$ *and if* $f_n \to f$ *strongly in* X'*, then*

$$\langle f_n, x_n \rangle \to \langle f, x \rangle.$$

But we can not pass to the limit if bothe convergences are weak.

2.1.3 *Weak* Convergence*

Let X be a space Banach, X' its dual, endowed by the dual norm

$$\|f\|_{X'} = \sup_{x \in X', \|x\| \leq 1} |\langle f, x \rangle|,$$

Let (f_n) a sequence of X', we note[1]

$$f_n \overset{*}{\rightharpoonup} f$$

the weak convergence of f_n to f for the weak* topology of X' denoted $\sigma(X', X)$.

1 The star is used because some authors denote the dual by X^*.

Remark 2.3 *there is another weake topology on X' denoted $\sigma(X', X'')$.*

Proposition 2.3 *Let (f_n) be a sequence of X'. We have*

- $f_n \overset{*}{\rightharpoonup} f \Leftrightarrow \langle f_n, x\rangle \to \langle f, x\rangle, \forall x \in X$.
- *$f_n \to f$ strongly, then $f_n \rightharpoonup f$ weakly for $\sigma(X', X'')$.*
- *If $f_n \to f$ for $\sigma(X', X'')$, then $f_n \overset{*}{\rightharpoonup} f$ for $\sigma(X', X)$. (so $\sigma(X', X)$ weaker).*
- *If $f_n \overset{*}{\rightharpoonup} f$ for $\sigma(X', X)$ then $\|f_n\|$is bounded and $\|f\| < \liminf \|f_n\|$.*
- *If $f_n \overset{*}{\rightharpoonup} f$ weakly for $\sigma(X', X)$ and if $x_n \to x$ strongly in X then*
  $$\langle f_n, x_n\rangle \to \langle f, x\rangle .$$

The following theorem is the cornerstone of the variational method.

Theorem 2.4 *Let X be a reflexive Banach space. Let $\{x_n\}$ be a bounded sequence in X Then it is possible to extract from $\{x_n\}$ a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x$ weakly in X.*

This is a result of weak compactness that may be restated *'The unit ball in a reflexive Banach space is weakly compact'*. There exists a property of weak* compactness of the unit ball of the dual of a separable normed space.

Theorem 2.5 *Let X be a separable Banach space. Let $\{f_n\}$ be a bounded sequence in X'. Then it is possible to extract from $\{f_n\}$ a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ weakly* in X'.*

2.1.4 *A Common Situation in Applications*

Suppose that it is possible to show that x (resp. f) is independent of the subsequence $\{x_m\}$ (resp. $\{f_m\}$) extracted (because we know that x (resp. f) is the unique solution of our problem). The following two propositions are then useful.

Proposition 2.6 *Let X a be reflexive space, $\{x_n\}$ is a sequence in X such that*
i) $\|x_n\| \le C < +\infty, \forall n \in \mathbb{N}$, C constant,
ii) $\{x_n\}$ admit a unique accumulation point x for the weak topology.
Then the whole sequence $\{x_n\}$ converge to x weakly in X.

Proof. If x_n does not converge towards x, for the weak topology, then it is possible to find $\varepsilon > 0, f \in X$ and a subsequence $\{x_{n_k}\}$ extracted from $\{x_n\}$ such that

$$|\langle f, x_{n_k} - x\rangle| \ge \varepsilon > 0, \qquad \forall n_k \tag{2.1}$$

From i), $\|x_{n_k}\| \le C < +\infty$ and X is a reflexive Banach space. Therefore, we may extract from $\{x_{n_k}\}$ a subsequence $\{x_{n_l}\}$ which converges towards an element of X which is none other than x from ii). We therefore have

$$\lim_{n_l \to 0} |\langle f, x_{n_l} - x\rangle| = 0, \qquad \forall n_k.$$

This contradicts (2.1). ∎

We have an analogue result for the weak* topology.

Proposition 2.7 *Let X a separable Banach space, $\{f_n\}$ is a sequence in X' such that*
i) $\|f_n\|_* \le C < +\infty, \forall n \in \mathbb{N}$, C constant,
ii) $\{f_n\}$ admit a unique accumulation point f unique for the weak topology.*
Then the whole sequence $\{f_n\}$ converge to f weakly in X'.*

2.2 SOBOLEV SPACES

To simplify matters we assume throughout this course we always suppose that Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary Γ. We assume that the reader is familiar with the notions of measurable functions and integrable functions $f :\to \mathbb{R}$.

Definition 2.2 *The Sobolev space of order* 1 *on* Ω *is defined by*

$$H^1(\Omega) := \left\{ u \in L^2(\Omega), \partial_{x_i} u \in L^2(\Omega), \forall i = 1, \cdots, n \right\}.$$

This is a separable Hilbert pace endowed by the scalar product

$$(u,v)_{H^1(\Omega)} := (u,v)_{L^2(\Omega)} + \sum_{i=1}^{n} (\partial_{x_i} u, \partial_{x_i} v)_{L^2(\Omega)} = \int_\Omega u\ v dx + \sum_{i=1}^{n} \int_\Omega \partial_{x_i} u\ \partial_{x_i} v dx,$$

and its associated norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \|\partial_{x_i} u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Remark 2.4 *Clearly we have the strict continuous inclusion*

$$H^1(\Omega) \subset L^2(\Omega).$$

We denote by $H_0^1(\Omega)$ the adherence of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$, i.e.

$$\overline{\mathscr{D}(\Omega)} = H_0^1(\Omega),$$

and by $H^{-1}(\Omega)$ its topological dual i.e. the space of continuos linear form on $H_0^1(\Omega)$.

Remark 2.5 *Since* $\mathscr{D}(\Omega) \subset H_0^1(\Omega)$ *then* $H^{-1}(\Omega) \subset \mathscr{D}'(\Omega)$, *the space of distributions*[2].

Poincaré's inequality

If Ω is bounded, there exists a constant $C_P > 0$, such that

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega),$$

and the mapping

$$u \mapsto \|\nabla u\|_{L^2(\Omega)},$$

is a norm on $H_0^1(\Omega)$ induced by the norm $\|\cdot\|_{H^1(\Omega)}$.

Remark 2.6 *Poincaré's inequality does not hold for* $H^1(\Omega)$, *take for instance* $u(x) = 1, \forall x \in \Omega$.

2 Note that $(H^1(\Omega))'$, is not a space of distributions.

Trace Theorem

We design by $\mathscr{D}(\overline{\Omega})$ the restrictions of functions of $\mathscr{D}(\mathbb{R}^n)$. Then we can define the trace $\gamma_0 : \mathscr{D}(\overline{\Omega}) \to C^0(\Gamma)$ of a function $u \in \mathscr{D}(\overline{\Omega})$ by

$$u \mapsto \gamma_0(u) = u_{|\Gamma}, \quad u \in \mathscr{D}(\overline{\Omega}).$$

Since $\mathscr{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$, this application can be prolonged by continuity to a linear application , also noted

$$\gamma_0 : H^1(\Omega) \to L^2(\Gamma).$$

For a proof, see for instance [1, 3]. We have in particular

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) \mid \gamma_0(v) = 0 \right\}.$$

Green Formulas

If $u, v \in H^1(\Omega)$ then we have

$$\int_\Omega \frac{\partial u(x)}{\partial x_i} v(x) dx = \int_{\partial\Omega} u(x) v(x) \nu_i d\Gamma - \int_\Omega u(x) \frac{\partial v(x)}{\partial x_i} dx, \; 1 \le i \le n,$$

where ν_i design the i^{eme} cosinus director of the normal vector ν on Γ directed to the exterior of Ω. If $n = 1$, this formula is reduced to the integration by part formula.

Theorem 2.8 *If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then*

$$-\int_\Omega \triangle u(x) v(x) dx = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx - \int_{\partial\Omega} \frac{\partial u(x)}{\partial \nu} v(x) d\Gamma,$$

where $\nabla u = (\frac{\partial u}{\partial x_i})_{1 \le i \le n}$ is the gradient vector of u and $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$.

See Appendix A.3 for more integration formulas.

2.3 FUNCTIONS VALUED IN BANACH SPACES

The functional setting for evolution problems requires spaces which involve time. Given a function $u = u(x,t)$, it is convenient to separate the roles of space and time.

Let X be a space of Banach with a norm $\|\cdot\|_X$ and T a positive real. Assume that $t \in [0, T]$ and for a.e. t, the function $u(\cdot, t)$ belongs to a Banach space X (e.g. $L^2(\Omega)$ or $H_0^1(\Omega)$). Then, we may consider u as a function of the real variable t with values into X:

$$u : [0, T] \to X.$$

When we adopt this convention, we write $u(t)$ and $\dot{u}(t)$ instead of $u(x,t)$ and $u_t(x,t)$.

2.3.1 *Spaces of continuous functions*

We start by introducing the set $C([0,T];X)$ of the continuous functions $u:[0,T]\to X$, endowed with the norm

$$\|u(t)\|_{C([0,T];X)}=\sup_{0\le t\le T}\|u(t)\|_X$$

Clearly $C([0,T];X)$ is a Banach space. We say that $v:[0,T]\to X$ is the (strong) derivative v of u if

$$\left\|\frac{u(t+h)-u(t)}{h}-v(t)\right\|_X\to 0,\quad \text{as } h\to 0 \text{ for each } t\in[0,T]$$

The symbol $C^1([0,T];X)$ denotes the Banach space of functions whose derivative exists and belongs to $C([0,T];X)$, endowed with the norm

$$\|u(t)\|_{C^1([0,T];X)}=\|u(t)\|_{C([0,T];X)}+\|\dot{u}(t)\|_{C([0,T];X)}$$

Similarly we can define the spaces $C^k([0,T];X)$ and $C^\infty([0,T];X)$, while $\mathscr{D}(]0,T[;X)$ denotes the subspace of $C^\infty([0,T];X)$ of the functions compactly supported in $]0,T[$.

2.3.2 $L^p(0,T;X)$ *Spaces*, $1\le p\le\infty$.

We note $L^p(0,T;X)$, the set of measurable functions defined on $(0,T)$ with values in X such that $t\to\|u(t)\|_X^p$ is summable on $(0,T)$. This is a Banach space endowed by the norm

$$\|u\|_{L^p(0,T;X)}=\left(\int_0^T\|u\|_X^p\right)^{\frac{1}{p}}<+\infty, 1\le p<\infty.$$

If $p=\infty$, $L^\infty(0,T;X)$ is a Banach space endowed by the norm

$$\|u\|_{L^\infty(0,T;X)}=\sup_{0\le t\le T} ess\,\|u(t)\|_X<+\infty$$

If $p=2$ and $X=H$ is a Hilbert space with a scalare product $(\cdot,\cdot)_H$, then $L^2(0,T;H)$ is a Hilbert space for the scalare product

$$(u,v)_{L^2(0,T;H)}=\int_0^T(u(t),v(t))_H dt.$$

Proposition 2.9 *We have the following results*[3] *:*

3 We recall that (See [1])

	Reflexif	Separable	Dual space
L^p with $1<p<\infty$	Yes	Yes	$L^{p'}$
L^1	No	Yes	L^∞
L^∞	No	No	Strictly bigger then L^1

- *If X is a separable space, then $L^p(0,T;X)$ is also separable,* $1 < p < \infty$.
- *The dual space of $L^p(0,T;X)$ can be identified to $L^{p'}(0,T;X')$ where p' satisfies* $\frac{1}{p} + \frac{1}{p'} = 1$, *with the convetion* $p' = \infty$ *if* $p = 1$.
- *If X is a reflexive space, then $L^p(0,T;X)$ is also reflexive space,* $1 < p < \infty$.
- $\mathscr{D}(]0,T[\,;X)$ *is dense in* $L^p(0,T;X), 1 \leq p < \infty$.
- *If X, Y design two Banach spaces, $X \subset Y$, with continuous injection, then there exists a continuous injection of $L^p(0,T;X)$ in $L^p(0,T;Y)$.*

Moreover, the following important theorem holds, due to Bochner:

Theorem 2.10 (Bochner) *A measurable function $u : [0,T] \to X$ is summable in $[0,T]$ if and only if the real function $t \to \|u(t)\|_X$ is summable on $[0,T]$. Moreover*

$$\left\| \int_0^T u(t)dt \right\|_X \leq \int_0^T \|u(t)\|_X \, dt \tag{2.2}$$

and

$$\left\langle f, \int_0^T u(t)dt \right\rangle_X = \int_0^T \langle f, u(t) \rangle_X \, dt, \ \forall f \in X'. \tag{2.3}$$

The inequality (2.2) is well known in the case of real or complex functions. By Riesz's Representation Theorem, (2.3) shows that *the action of any element of X' commutes with the integral.*

Exercise 2 *If $u \in L^1(0,T;X)$, the function $t \to \int_0^t u(s)ds$ is continuous and*

$$\frac{d}{dt} \int_0^t u(s)ds = u(t) \quad a.e.in \ (0,T).$$

Hint: use DCT on the sequence $\chi_{[t_n,t]} \|u_n(t)\|_X$.

Another useful result: the weak convergence in $L^2(0,T;X)$ "preserves boundedness in $L^\infty(0,T;X)$".

Proposition 2.11 *Let $\{u_k\} \subset L^2(0,T;X)$, such that $u_k \rightharpoonup u$ in $L^2(0,T;X)$. Assume that*

$$\sup_{t \in [0,T]} \|u_k(t)\|_X \leq C$$

with C independent of k. Then, also,

$$\sup_{t \in [0,T]} \|u(t)\|_X \leq C.$$

Exercise 3 *Soit $\theta \in L^2(0,T)$ et $v \in L^2(0,1)$. Démontrer que*

$$\eta(t,x) = \theta(t)\, v(x) \in L^2\left(0,T;L^2(0,1)\right).$$

2.3.3 *Sobolev spaces involving time*

Let V be a Hilbert space. To define Sobolev spaces, we need the notion of derivative in the sense of distributions for functions $u \in L^1_{loc}(0,T;V)$.

Definition 2.3 *We denote by* $\mathscr{D}'(0,T;V)$ *the space of distributions on* $]0,T[$ *with values* V *defined as*

$$\mathscr{D}'(0,T;V) = \mathscr{L}(\mathscr{D}(]0,T[),V).$$

We note for $f \in \mathscr{D}'(]0,T[;V)$ and $\varphi \in \mathscr{D}(]0,T[)$,

$$f(\varphi) = \langle f, \varphi \rangle, \quad \text{(element of } V\text{)}$$

The application $\varphi \to \langle f, \varphi \rangle$ is continue of $\mathscr{D}(]0,T[)$ in V, endowed by the topology of the uniform convergence on bounded sets $\mathscr{D}\left(]0,T[\right)$.

Lemma 2.1 *Let* $u \in L^1_{loc}(]0,T[;V)$, *then the application*

$$u \to \int_0^T u\ \varphi dt, \qquad \forall \varphi \in D(]0,T[)$$

is a distribution on $]0,T[$ *with value in* V.

Exercise 4 *Check it !*

To define Sobolev spaces, we need the notion of derivative in the sense of distributions for functions $u \in L^1_{loc}(]0,T[;V)$. In general

Definition 2.4 *The weak derivative* $\dot{u}$ *is defined by the linear operator*

$$u \mapsto \langle \dot{u}, \varphi \rangle = -\int_0^T u(t)\, \dot{\varphi}(t)\, dt \tag{2.4}$$

from $\mathscr{D}(]0,T[)$ *in* V

Proof. Indeed

$$\|(\dot{u},\varphi)\|_V = \left\| \int_0^T u(t)\ \ \dot{\varphi}(t)\, dt \right\|_V \leq \int_0^T |\dot{\varphi}(t)|\, \|u(t)\|_V\, dt \leq C \max |\dot{\varphi}(t)|$$

for every $\varphi \in \mathscr{D}(]0,T[)$, where C is the integral of $\|u(t)\|_V$ over the support of $\dot{\varphi}(t)$. Therefore $\dot{u}$ is continuous with respect to the convergence in $\mathscr{D}(]0,T[)$ and defines an element of $\mathscr{D}'(]0,T[;V)$. ■

Another definition equivalent to (2.4) is

$$\int_0^T \varphi(t)\,(\dot{u},v)_V\, dt = -\int_0^T \dot{\varphi}(t)\,(u,v)_V\, dt, \quad \forall v \in V, \varphi \in \mathscr{D}(]0,T[). \tag{2.5}$$

Definition 2.5 *We denote by* $H^1(0,T;V)$ *the Sobolev space of the functions* $u \in L^2(0,T;V)$ *such that* $\dot{u} \in L^2(0,T;V)$.

This is a Hilbert space with inner product

$$(u,v)_{H^1(0,T;V)} = \int_0^T (u(t),v(t))_V + (\dot{u}(t),\dot{v}(t))_V dt.$$

Since functions in $H^1(a,b)$ are continuous in $[a,b]$, it makes sense to consider the value of u at any point of $[a,b]$. For $H^1(0,T;V)$ we have the following theorem

Theorem 2.12 *Let $u \in H^1(0,T;V)$. Then, $u \in C([0,T];V)$ and*

$$\max_{0\leq t\leq T} \|u(t)\| \leq C \|u\|_{H^1(0,T;V)},$$

where C is a constant depending on T. Moreover, the fundamental theorem of calculus holds:

$$u(t) = u(s) + \int_s^t u'(r)dr \quad \forall t,s \in [0,T].$$

2.3.4 *$H^1(0,T;V,W)$ spaces*

In study of evolution equation, it is quite often that

$$u \in L^p(0,T;V) \text{ and } \dot{u} \in L^q(0,T;W), \text{ where } V \subset W.$$

In this case, it makes sense to define the Sobolev space

$$H^1(0,T;V,W) = \left\{u \in L^2(0,T;V) \text{ such that } \dot{u} \in L^2(0,T;W)\right\},$$

where $\dot{u}$ is intended in the sense of distribution in $\mathscr{D}'(0,T;W)$, endowed with the norm

$$\|u\|_{H^1(0,T;V,W)} = \left(\|u\|^2_{L^2(0,T;V)} + \|\dot{u}\|^2_{L^2(0,T;W)}\right)^{1/2}$$

Indeed, the linear functional

$$\langle \dot{u},\varphi\rangle = \int_0^T \varphi(t)\,\dot{u}dt = -\int_0^T \dot{\varphi}(t)\,udt$$

defines a distribution $\dot{u} \in \mathscr{D}'(0,T;W)$. It is enough to observe that, since $V \subset W$, we have, for some constant C

$$\begin{aligned}
\|\langle \dot{u},\varphi\rangle\|_W &= \left\|-\int_0^T \dot{\varphi}(t)\,udt\right\|_W \\
&\leq \max|\dot{\varphi}(t)| \int_0^T \|u(t)\|_W dt \leq \max|\dot{\varphi}(t)| \int_0^T \|u(t)\|_V dt \\
&\leq C\max|\dot{\varphi}(t)| \sqrt{T}\left(\int_0^T \|u(t)\|_V^2 dt\right)^{1/2} \leq C' \max|\dot{\varphi}(t)|
\end{aligned}$$

and therefore $\dot{u}$ is continuous with respect to the convergence in $\mathscr{D}(0,T)$.

2.3.5 *Hilbert triplets*

Consider two Hilbert spaces: V, the space where we seek the solution, and V', which the derivative of the solution or the source term belong to. Let us introduce a third space H, "intermediate" between V and V'. For better clarity, it is convenient to use the symbol $\langle\cdot,\cdot\rangle_*$ to denote the duality between V' and V. In the applications to boundary value problems, usually $H = L^2(\Omega)$, while V is a Sobolev space.

In practice, we often meet a pair of Hilbert spaces V, H with the following properties:

1. $V \subset H$, i.e. there exists C such that

$$\|u\|_H \leq C\,\|u\|_V, \quad \forall v \in V \tag{2.6}$$

2. V is dense in H.

Using Riesz's Representation Theorem we may identify H with H'. Also, we may *continuously embed* H into V', so that any element in H can be thought as an element of V'. To see it, observe that, for any fixed $u \in H$, the functional $T_u : V \to \mathbb{R}$, defined by

$$\langle T_u, v\rangle_* = (u,v)_H, \quad v \in V, \tag{2.7}$$

is continuous in V. In fact, the Cauchy-Schwarz inequality and (2.6) give

$$|(u,v)_H| \leq \|u\|_H\,\|v\|_H \leq C\,\|u\|_H\,\|v\|_V\,.$$

Thus, the map $u \to T_u$ is continuous from H into V', with $\|T_u\|_{V'} \leq C\,\|u\|_H$. Moreover, if $T_u = 0$, then

$$0 = \langle T_u, v\rangle_* = (u,v)_H, \quad \forall v \in V$$

which forces $u = 0$, by the density of V in H. Thus, the map $u \to T_u$ is *one to one* and defines the continuous embedding of H into V'. This allows the identification of $u \in H$ with $T_u \in V'$. In particular, instead of (2.7), we can write

$$\langle T_u, v\rangle_* = \langle u, v\rangle_* = (u,v)_H, \quad \forall v \in V, \tag{2.8}$$

regarding u on the left in the last equality as an element of V' and on the right as an element of H.

Summarizing, we have

$$V \subset H \subset V'$$

with dense embeddings. We call $\{V, H, V'\}$ a Hilbert triplet and H is called the pivot space.

Remark 2.7 *The space V can be taken a reflexive Banach space.*

2.3.6 $H^1(0,T;V,V')$ *spaces*

We shall see in the next Chapter that the natural functional setting for Evolution problems is precisely the space $H^1(0,T;V,V')$.

The following result is fundamental[4]

Theorem 2.13 *Let $\{V,H,V'\}$ be a Hilbert triplet, with V and H separable. Then*
a) $C^\infty([0,T];V)$ is dense in $H^1(0,T;V,V')$.
b) $H^1(0,T;V,V') \subset C([0,T];H)$ that is

$$\|u\|_{C([0,T];H)} \leq C(T)\,\|u\|_{H^1(0,T;V,V')}$$

c) If $u,v \in H^1(0,T;V,V')$, the following integration by parts formula holds:

$$\int_s^t \langle \dot{u}(r), v(r)\rangle_* + \langle \dot{v}(r), u(r)\rangle_* \, dr = (u(t), v(t))_H - (u(s), v(s))_H \tag{2.9}$$

for all $s,t \in [0,T]$.

Idea of proof for b).

$$\begin{aligned} |u(t)|_H^2 &= \int_0^t \frac{d}{ds}\left(|u(s)|_H^2\right) ds = \int_0^t \frac{d}{ds}(u(s),u(s))_H\, ds \\ &= 2\int_0^t (\dot{u}(s),u(s))_H\, ds = 2\int_0^t \langle \dot{u}(s),u(s)\rangle_*\, ds, \text{ using (2.8)} \\ &\leq 2\int_0^t \|\dot{u}(s)\|_{V'}\,\|u\|_V\, ds \end{aligned}$$

by applying the inequality $2ab \leq a^2+b^2$, it becomes:

$$|u(t)|_H^2 \leq \int_0^t \|\dot{u}(s)\|_{V'}^2 + \|u\|_V^2\, ds \leq \int_0^T \|\dot{u}(s)\|_{V'}^2 + \|u\|_V^2\, ds$$

from which

$$|u(t)|_H^2 \leq \|u\|_{H^1(0,T;V,V')}$$

and we take the sup on $t \in [0,T]$. See [1] for the complete proof. ■

Remark 2.8 *From the integration by parts formula we infer,*

$$\frac{d}{dt}(u(t), v(t))_H = \langle \dot{u}(t), v(t)\rangle_* + \langle \dot{v}(t), u(t)\rangle_* \tag{2.10}$$

a.e. $t \in [0,T]$ and (letting $u = v$ in (2.9))

$$\int_s^t \frac{d}{dt}\|u(r)\|_H^2\, dr = \|u(t)\|_H^2 - \|u(s)\|_H^2.$$

4 See [1]

2.4 THE GALERKIN METHOD

2.4.1 *Galerkin Approximation of a Hilbert Space*

Let V be a separable Hilbert space and $\{V_m\}_{m\in\mathbb{N}^*}$ a family of finite dimensional vector spaces satisfying the axioms:

i) $V_m \subset V \quad (\dim V_m < +\infty)$,
ii) $V_m \to V$ when $m \to +\infty$ in the following sense :

There exists $\mathcal{V}$ dense subspace of V, such that, for all $v \in \mathcal{V}$, we can find a sequence[5] $\{v_m\}_{m\in\mathbb{N}^*}$ satisfying:

$$\forall m, v_m \in V_m \text{ and } v_m \to v \text{ in } V \quad \text{as } m \to +\infty.$$

The space V_m is called the Galerkin approximation of order m ($m = \dim V_m$) of V.

Example 2.2 *The orthonormal basis on $L^2(0,\pi)$ defined by*

$$v_m(x) = \sqrt{2/\pi}\sin(mx), m \geq 1, \quad or \quad v_m(x) = \sqrt{2/\pi}\cos(mx), m \geq 0.$$

For the first case $V_m = span\left\{\sqrt{2/\pi}\sin(kx), k = 1, m\right\}$ and $\mathcal{V} = span\left\{\sqrt{2/\pi}\sin(mx)\right\}_{m\geq 1}$. It well known that

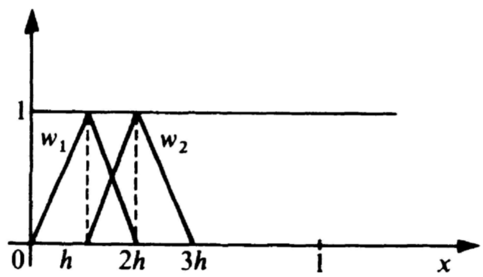
$$\forall v \in L^2(0,\pi), \exists v_m \text{ such that } v_m \to v \text{ in } L^2(0,\pi) \quad \text{as } m \to +\infty,$$

it suffices to take v_m =partial sum of Fourier series of v. The basis in this case is fixed, i.e. we add new elements to the old ones as n increases and Thus V_m is increasing.

Example 2.3 *Consider $Q =]0,1[$ and let $V = H_0^1(\Omega)$. Let $m \in \mathbb{N}^*$ and $h = 1/m$. We define (see Figure) $w_1, w_2, ..., w_m$ satisfying*

$$\begin{cases} w_j \text{ piecewise linear,} \\ w_j(jh) = 1, w_j((j\pm1)h) = 0 \\ w_j(x) = 0 \text{ if } x \notin [(j-1)h, (j+1)h], j = 1, ..., m-1. \end{cases}$$

We take $V_m = span\{w_1, w_2, ..., w_{m-1}\}$ which is a space of dimension $m-1$.



Remark 2.9 *The general framework of the last example corresponds to approximation by finite elements. The family of functions $\{w_j\}$ changes completely with the parameter $h = 1/m$. It is therefore a different situation from that of the first example.*

Remark 2.10 *For time-dependent problems this method is sometimes referred to as the Faedo-Galerkin.*

5 i.e., we know effectively how to construct a sequence.

2.4.2 *The Outline of a Galerkin Method*

Let P be the exact problem for which we intend to look for the existence of a solution in a space of functions constructed over a separable Hilbert (or a reflexive Banach) space V. The plan of study will be the following:
I. Variational formulation of problem P.
II. Uniqueness of the solution.
III. Existence of the solution. After a Galerkin approximation V_m of V, then we define an approximate problem P_m in the finite dimensional space V_m having a unique solution U_m.
The procedure of study is the following.

1. We define the solution u_m of problem P_m.
2. We establish some a priori estimates on u_m, which show that $\{u_m\}$ is bounded, independently of m, in certain normed spaces.
3. By using the results of weak compactness of the unit ball in a Hilbert space (or a reflexive Banach space), it is possible to extract from $\{u_m\}$ a subsequence $\{u_{m_k}\}$ which has a limit in the weak (or weak *) topology of spaces, Then let u be the limit obtained.
4. We show that u is a solution of problem P, therefore the solution sought from uniqueness.
5. Strong convergence results (optional).

IV. Properties of stability with respect to the given data or continuity of the solution with respect to the data.

Remark 2.11 *The Galerkin method of approximation is not limited only to Hilbert spaces and is interesting particularly for and problems with coefficients depending on time and nonlinear problems.*

REFERENCES

[1] H. BREZIS, *Analyse Functionnelle. Theorie and Applications*, Masson, (1987).

[2] R. DAUTRAY, J. L. LIONS, *Mathematical Analysis and Numerical Science and Technology*, Berlin Heidelberg (1992).

[3] P. A. RAVIART, J. M. THOMAS, *Introduction à l'Analyse Numerique des EDP*, Masson (1992).

[4] S. SALSA, *Partial Differential Equation in Action*, Springer-Verlag, (2008).