Practical Work 8: Theorems and Proofs in Functional Analysis

Author: $\mathfrak{B}$. $\mathfrak{B}$asti (Master PDEs and Applications)

Date: `\today`.

1. The Fourier Transform

Definition 1 (Fourier Transform). *For $f \in L^1(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ is defined as*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx.$$

Remark 1. *The Fourier transform converts differentiation into multiplication. More precisely, if f is sufficiently smooth, then*

$$\widehat{\partial_{x_j} f}(\xi) = 2\pi i \xi_j \hat{f}(\xi).$$

This property makes the Fourier transform extremely useful for solving linear partial differential equations.

Theorem 2 (Plancherel's Theorem). *The Fourier transform extends uniquely to a unitary operator on $L^2(\mathbb{R}^n)$ and satisfies*

$$\left\| \hat{f} \right\|_{L^2} = \| f \|_{L^2}.$$

Proof. One first proves the identity for Schwartz functions. The result then extends to all of $L^2(\mathbb{R}^n)$ by density, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (see [3]). □

2. Sobolev Spaces

Definition 2 (Sobolev Space [2]). *For $k \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(\Omega)$ consists of all functions $u \in L^p(\Omega)$ whose weak derivatives up to order k also belong to $L^p(\Omega)$.*

Remark 2. *Sobolev spaces generalize the notion of classical differentiability. Even functions that are not differentiable in the classical sense may possess weak derivatives and therefore belong to a Sobolev space.*

Theorem 3 (Sobolev Embedding [1]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. If $kp > n$, then we have the continuous embedding*

$$W^{k,p}(\Omega) \hookrightarrow C^0\left(\overline{\Omega}\right).$$

Lemma 1 (Rellich–Kondrachov [1,2]). *If Ω is bounded with Lipschitz boundary, the embedding*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for $1 \le q < p^*$, where

$$p^* = \frac{np}{n-p}.$$

3. Hilbert Space Theory

Theorem 4 (Projection Theorem). *Let H be a Hilbert space and $K \subset H$ a closed convex subset. For any $x \in H$, there exists a unique $y \in K$ such that*

$$\|x - y\| = \inf_{z \in K} \|x - z\|.$$

Corollary 1. *Every closed subspace M of a Hilbert space H admits an orthogonal complement $M^\perp$ such that*

$$H = M \oplus M^\perp.$$

Remark 3. *This decomposition allows every element $x \in H$ to be written uniquely as $x = m + m^\perp$ with $m \in M$ and $m^\perp \in M^\perp$. Such orthogonal decompositions are fundamental in approximation theory and spectral analysis (see [1]).*

Bibliography

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011.

[2] L. C. Evans, *Partial Differential Equations*, AMS, 2010.

[3] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.

Happy LaTeX. Enjoy!

Displaying LaTeX Code

Here is the LaTeX code used in this document.

```
\documentclass[a4paper,11pt]{article}
\usepackage{amsfonts} \usepackage[utf8]{inputenc}
\usepackage{amsmath, amssymb} \usepackage{graphicx}
\usepackage{geometry} \usepackage{hyperref}
\usepackage{amsthm}

\geometry{a4paper, top=1cm, bottom=1.9cm, left=2.8cm, right=2.6cm}
\def\baselinestretch{1.3}
\newtheorem{theorem}{Theorem}[section]
\newtheorem{lemma}[theorem]{Lemma}
\newtheorem{corollary}[theorem]{Corollary}
\newtheorem{definition}[theorem]{Definition}

\begin{document}

\title{\textbf{PW 7: Theorems and Proofs in Functional Analysis}}
\author{\textbf{$\mathfrak{B}.$ $\mathfrak{B}$asti} (Master PDEs and Applications)}
\date{\today}
\maketitle

\section{The Fourier Transform}

\begin{definition}[Fourier Transform]
For $f\in L^{1}\left( \mathbb{R}^{n}\right) $, the Fourier transform $\hat{f}$
is defined as
\begin{equation*}
 \hat{f}\left( \xi \right)
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 =
 \int_{\mathbb{R}^{n}} f(x)e^{-2\pi i x\cdot \xi}\,dx .
\end{equation*}
\end{definition}

\begin{remark}
The Fourier transform converts differentiation into multiplication.
More precisely, if $f$ is sufficiently smooth, then
\[
\widehat{\partial_{x_j}f}(\xi)=2\pi i \xi_j\,\hat f(\xi).
\]
This property makes the Fourier transform extremely useful for solving
linear partial differential equations.
\end{remark}

\begin{theorem}[Plancherel’s Theorem]
The Fourier transform extends uniquely to a unitary operator on
$L^{2}\left( \mathbb{R}^{n}\right)$ and satisfies
\begin{equation*}
 \left\Vert \hat{f}\right\Vert _{L^{2}}
 =
 \left\Vert f\right\Vert _{L^{2}}.
\end{equation*}
\end{theorem}

\begin{proof}
One first proves the identity for Schwartz functions. The result then
extends to all of $L^{2}(\mathbb{R}^{n})$ by density, since
$\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$
(see \cite{SteinShakarchi}).
\end{proof}

\section{Sobolev Spaces}

\begin{definition}[Sobolev Space \cite{Evans}]
For $k\in \mathbb{N}$ and $1\leq p\leq \infty $, the Sobolev space
$W^{k,p}\left( \Omega \right)$ consists of all functions
$u\in L^{p}\left(\Omega\right)$ whose weak derivatives up to order $k$
also belong to $L^{p}\left(\Omega\right)$.
\end{definition}

\begin{remark}
Sobolev spaces generalize the notion of classical differentiability.
Even functions that are not differentiable in the classical sense may
possess weak derivatives and therefore belong to a Sobolev space.
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\end{remark}

\begin{theorem}[Sobolev Embedding \cite{Brezis}]
Let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with smooth
boundary. If $kp>n$, then we have the continuous embedding
\begin{equation*}
 W^{k,p}\left(\Omega\right)
 \hookrightarrow
 C^{0}\left(\overline{\Omega}\right).
\end{equation*}
\end{theorem}

\begin{lemma}[Rellich--Kondrachov \cite{Brezis,Evans}]
If $\Omega$ is bounded with Lipschitz boundary, the embedding
\[
W^{1,p}\left(\Omega\right)
\hookrightarrow
L^{q}\left(\Omega\right)
\]
is compact for $1\le q<p^\ast$, where
\[
p^\ast=\frac{np}{n-p}.
\]
\end{lemma}

\section{Hilbert Space Theory}

\begin{theorem}[Projection Theorem]
Let $H$ be a Hilbert space and $K\subset H$ a closed convex subset.
For any $x\in H$, there exists a unique $y\in K$ such that
\begin{equation*}
 \left\Vert x-y\right\Vert
 =
 \inf_{z\in K}\left\Vert x-z\right\Vert .
\end{equation*}
\end{theorem}

\begin{corollary}
Every closed subspace $M$ of a Hilbert space $H$ admits an orthogonal
complement $M^\perp$ such that
\[
H = M \oplus M^\perp .
\]
\end{corollary}
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\begin{remark}
This decomposition allows every element $x\in H$ to be written uniquely
as $x = m + m^\perp$ with $m\in M$ and $m^\perp\in M^\perp$.
Such orthogonal decompositions are fundamental in approximation theory
and spectral analysis (see \cite{Brezis}).
\end{remark}

\begin{thebibliography}{9}

\bibitem{Brezis}
H. Brezis,
\textit{Functional Analysis, Sobolev Spaces and Partial Differential Equations},
Springer, 2011.

\bibitem{Evans}
L. C. Evans,
\textit{Partial Differential Equations},
AMS, 2010.

\bibitem{SteinShakarchi}
E. M. Stein and R. Shakarchi,
\textit{Fourier Analysis: An Introduction},
Princeton University Press, 2003.

\end{thebibliography}

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