Final Exam in LaTeX*

Your Full Name†

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Abstract

This exam covers linear evolution equations, focusing on semigroup theory, the Hille-Yosida theorem, and the energy method. Prepared in LaTeX (`article` class, 11pt, `a4paper`), the document utilizes `amsmath`, `amssymb`, and `geometry` with specific margins (top = 2 cm, bottom = 2 cm, left = 3 cm, and right = 2.8 cm). Students must provide rigorous justifications within the defined functional spaces.

1 Linear Semigroups

Definition 1.1 (C_0-Semigroup [2]) *A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is called a strongly continuous semigroup (or C_0-semigroup) if $T(0) = I$, $T(t + s) = T(t)T(s)$, and $\lim\limits_{t\to 0^+} T(t)x = x$ for every $x \in X$.*

2 The Heat Equation

Definition 2.1 (Evolution Solution) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider the initial-boundary value problem:*

$$\begin{cases} \partial_t u - \Delta u = 0, & \textit{in } \Omega \times (0, \infty), \\ u = 0, & \textit{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = g(x), & \textit{in } \Omega. \end{cases}$$

A function $u \in C([0, \infty); L^2(\Omega)) \cap C^1((0, \infty); L^2(\Omega))$ is a solution if it satisfies the abstract Cauchy problem $u'(t) = Au(t)$ where $A = \Delta$ with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$.

Theorem 2.2 (Hille-Yosida Theorem) *A linear operator A is the infinitesimal generator of a C_0-semigroup of contractions on a Hilbert space X if and only if A is maximal dissipative and $D(A)$ is dense in X.*

Remark 2.3 *The Hille-Yosida theorem provides the necessary and sufficient conditions for the existence of a unique solution to the parabolic problem, as discussed in [1].*

*For Master 1 PDEs and Applications

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3 Energy Estimates

For the heat equation, we define the energy functional $E(t)$ as follows:

$$E(t) = \frac{1}{2} \int_\Omega |u(x,t)|^2 \, dx.$$

By testing the equation with u, we obtain the fundamental dissipation inequality [1, 2]:

$$\frac{d}{dt} E(t) = - \int_\Omega |\nabla u|^2 \, dx \leq 0.$$

References

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011.

[2] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

Good Luck with your Exam!