Final Exam: LaTeX for Master 1 PDEs and Applications

Duration: 2 Hours **Date:** May 13^{th}, 2025

Instructions

Prepare the following document using LaTeX and compile it with `TeXStudio`. The document must:

- Use the class `article` with the options `a4paper` and `11pt`.
- Include the packages `amsmath`, `amssymb`, `amsfonts`, and `geometry`.
- Use the page margins: top $= 2\,\text{cm}$, bottom $= 2\,\text{cm}$, left $= 3\,\text{cm}$, right $= 2.8\,\text{cm}$.
- Contain a title, an abstract, a table of contents, sections, theorem environments, and references.

Your final document should be approximately **two pages** long.

Exercise 1: Document Structure

Create a title page containing:

- The title: **Exam in LaTeX, Master 1 PDEs and Applications**
- Your full name as the author
- A footnote indicating your department
- The date of the exam

Add a table of contents and an abstract describing the purpose of the document.

Exercise 2: Document Content

1. Create a section entitled **Sobolev Spaces**. Inside this section, define the following concept.

Definition 1 (Sobolev Space [3]). *For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ consists of all functions $u \in L^p(\Omega)$ whose weak derivatives up to order k also belong to $L^p(\Omega)$.*

Definition 2 (Weak Solution). *Let $\Omega \subset \mathbb{R}^n$ be an open domain. A function $u \in W_0^{1,2}(\Omega)$ is called a weak solution of*

$$-\Delta u = f \quad \text{in } \Omega$$

if

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

2. Create a subsection entitled **Lax–Milgram Theorem**, and insert its principle.

Theorem 1 (Lax–Milgram Theorem). *Let V be a Hilbert space, $a(\cdot,\cdot)$ a continuous and coercive bilinear form on $V \times V$, and L a continuous linear functional on V. Then there exists a unique $u \in V$ such that*

$$a(u,v) = L(v) \quad \text{for all } v \in V.$$

Proof. Define the operator $A : V \to V^*$ by

$$(Au)(v) = a(u,v), \qquad v \in V.$$

The continuity of a implies that A is bounded, while coercivity yields $\langle Au, u\rangle \geq \alpha \|u\|_V^2$ for some $\alpha > 0$, hence A is invertible. Therefore, for any $L \in V^*$ there exists a unique $u \in V$ such that $Au = L$, which is equivalent to $a(u,v) = L(v)$ for all $v \in V$ (see [1, 2]). □

Exercise 3: Document Completing

Create a new section entitled **Galerkin approximation**. Then write:
Let $\{\phi_1, \phi_2, \dots\}$ be a basis of $W_0^{1,2}(\Omega)$. Then the Galerkin approximation:

$$u_n = \sum_{i=1}^{n} c_i \phi_i$$

satisfying

$$\int_\Omega \nabla u_n \cdot \nabla \phi_j \, dx = \int_\Omega f \phi_j \, dx, \quad j = 1, \dots, n.$$

Exercise 4: Bibliography

Create a bibliography using the following references:

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011.

[2] L. C. Evans, *Partial Differential Equations*, AMS, 2010.

[3] R. A. Adams and J. J. Fournier, *Sobolev Spaces*, 2nd ed., Academic Press, 2003.

Good Luck in the LaTeX Exam!