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Linear Elliptic Partial Differential Equations

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Preface

This course is intended for first-year Master's students in Mathematics, specializing in **Partial Differential Equations and Applications**. Its objective is to provide the fundamental tools necessary for the modern study of partial differential equations (PDEs), namely the theory of Sobolev spaces and their application to solving boundary value problems.

Partial differential equations are a central field of applied mathematics, with applications in numerous areas such as fluid mechanics, heat transfer, electromagnetism, mathematical finance, and many others. The effective resolution of these equations requires powerful and elegant mathematical tools, including distribution theory and Sobolev spaces.

This course is structured into five chapters:

Chapter 0: Reminder on Lebesgue Spaces and Distributions.
We first recall the fundamental notions of Lebesgue spaces L^p, then introduce distribution theory, which generalizes the classical notions of functions and derivatives. This theory forms the foundation upon which Sobolev spaces are built.
This chapter is followed by a collection of exercises with detailed solutions.

Chapter 1: Sobolev Spaces.
This chapter is the core of the course. We define Sobolev spaces $W^{k,p}(\Omega)$ and $H^k(\Omega)$, and study their fundamental properties: density, Sobolev embeddings, trace theorems, and Poincaré inequalities. These spaces are the natural framework for the variational formulation of boundary value problems.
This chapter is followed by a collection of exercises with detailed solutions.

Chapter 2: Boundary Value Problems and Variational Formulation.
We apply Sobolev space theory to solve linear second-order elliptic boundary value problems. The Lax-Milgram theorem is the central tool guaranteeing the existence and uniqueness of weak solutions. We study Dirichlet, Neumann, and mixed boundary conditions.
This chapter is followed by a collection of exercises with detailed solutions.

Chapter 3: Maximum Principle and Eigenvalue Problem.
We present the weak and strong maximum principles for elliptic operators, as well as Hopf's lemma. These tools are then applied to study the eigenvalue problem for the Laplacian with Dirichlet boundary conditions. We characterize the first eigenvalue via the Rayleigh quotient and prove the positivity of the first eigenfunction.
This chapter is followed by a collection of exercises with detailed solutions.

For successful completion of this course, students should have a solid background in functional analysis (notions of Banach and Hilbert spaces), measure theory and Lebesgue integration, as well as elementary partial differential equations.

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1 Review of Lebesgue Spaces and Distribution Theory

Ω is an open set in $\mathbb{R}^N$.

Définition 1.1. i) For $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is the set of measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that

$$\|f\|_{L^p(\Omega)} := \int_\Omega |f(x)|^p \mathrm{d}x < \infty.$$

ii) For $p = \infty$, we denote by $L^\infty(\Omega)$ the space of measurable and bounded functions $f : \Omega \to \mathbb{R}$ and we set

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

Remarque 1.1. The mapping $u \longmapsto \|u\|_{L^p(\Omega)}$ defines a norm on the space $L^p(\Omega)$, and the spaces $(L^p(\Omega), \|.\|_{L^p(\Omega)})$ are separable Banach spaces for $1 \leq p \leq \infty$ and reflexive for $1 < p < \infty$.

Exercice 1.1. a) Show that $L^p(\Omega) \ni f \longmapsto \|f\|_{L^p(\Omega)}$ defines a norm on $L^p(\Omega)$ for every $1 \leq p \leq \infty$. b) Also show that these spaces, together with their associated norms, are Banach spaces.

In what follows, $p \in [1, \infty[$. We present some convergence and density results.

Théorème 1.2 (Beppo-Levi Monotone Convergence Theorem). *Let $(f_n)_n$ be an increasing sequence of functions in $L^1(\Omega)$ such that $\sup_n \int_\Omega f_n \mathrm{d}x < \infty$. Then f_n converges almost everywhere on Ω to a function $f \in L^1(\Omega)$, and $\lim_{n\to\infty} \|f_n - f\|_{L^1(\Omega)} = 0$.*

Théorème 1.3. ***Lebesgue Dominated Convergence Theorem]*** *Let $(f_n)_n$ be a sequence of functions in $L^p(\Omega)$. Assume that $(f_n)_n$ converges almost everywhere on Ω to a function f and that there exists a function $g \in L^p(\Omega)$ such that for every n, $|f_n(x)| \leq g(x)$ for almost every $x \in \Omega$. Then $f \in L^p(\Omega)$ and $\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0$.*

Théorème 1.4. ***[Converse of Lebesgue's Theorem]*** *Let $(f_n)_n$ be a sequence of functions in $L^p(\Omega)$ and $f \in L^p(\Omega)$ such that $\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0$. Then there exists a subsequence $(f_{n_k})_k$ such that*

1. *$(f_{n_k})_k$ converges to f almost everywhere on Ω.*
2. *There exists $h \in L^p(\Omega)$ such that for all k, $|f_{n_k}(x)| \leq h(x)$ for almost every $x \in \Omega$.*

(The proof of this theorem is based on the Fischer-Riesz Theorem 1.7 below.)

Exercice 1.2. Let $p, q \in [1, \infty[$. Let (f_n) be a sequence of functions in $L^p(\Omega) \cap L^q(\Omega)$ that converges to f in $L^p(\Omega)$ and to g in $L^q(\Omega)$. Show that $f = g$ almost everywhere on Ω.

Exercice 1.3. Show that extracting a subsequence is necessary in the previous theorem. Consider the following example: $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = 1_{[\frac{n-1}{m} - \frac{m-1}{2}, \frac{n}{m} - \frac{m-1}{2}]}(x)$$

where $m \in \mathbb{N}^*$ and $\frac{m(m-1)}{2} \leq n \leq \frac{m(m+1)}{2} + 1$.

Théorème 1.5 (Density Theorem). *The space $\mathcal{D}(\Omega)$ of $\mathcal{C}^\infty(\Omega)$ functions with compact support is dense in $L^p(\Omega)$. That is, for every $\varepsilon > 0$ and every $f \in L^p(\Omega)$, there exists a function $\varphi \in \mathcal{D}(\Omega)$ such that $\|f - \varphi\|_{L^p(\Omega)} \leq \varepsilon$.*

Théorème 1.6 (Hölder Inequality). *Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Then $fg \in L^1(\Omega)$ and we have the following inequality:*

$$\|fg\|_{L^1(\Omega)}\mathrm{d}x \leq \|f\|_{L^p(\Omega)}\|g\|_{L^{p'}(\Omega)}. \tag{1.1}$$

Théorème 1.7 (Fischer-Riesz Theorem). *For every $p \in [1, \infty]$, the space $L^p(\Omega)$ is a Banach space with the associated norm.*

Définition 1.2. 1) We denote by $\mathcal{D}(\Omega)$ the space of all C^∞ functions with compact support.

2) A sequence $\{\varphi_n\}$ in $\mathcal{D}(\Omega)$ is said to converge to φ in $\mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that

$$\forall n : \operatorname{supp} \varphi_n \subset K, \quad D^\alpha \varphi_n \longrightarrow D^\alpha \varphi \text{ uniformly on } \Omega$$

for every multi-index $\alpha \in \mathbb{N}^N$.

Définition 1.3. 1. A linear form $T : \mathcal{D}(\Omega) \to \mathbb{R}$ is continuous if for every sequence $(\varphi_n)_n$ converging to φ in $\mathcal{D}(\Omega)$,

$$\lim_{n \to \infty} T(\varphi_n - \varphi) = 0.$$

2. A distribution on Ω is a continuous linear form on $\mathcal{D}(\Omega)$. We denote by $\mathcal{D}'(\Omega)$ the space of all distributions on Ω (i.e., the dual space of $\mathcal{D}(\Omega)$).

Exercice 1.4 (**Regular Distribution**). Show that for every $f \in L^1_{\text{loc}}(\Omega)$, the mapping $T_f : \mathcal{D}(\Omega) \to \mathbb{R}$, $\varphi \longmapsto \int_\Omega f\varphi \mathrm{d}x$ defines a distribution on Ω. (T_f is called a regular distribution and is denoted simply by f.)

Définition 1.4 (Derivative of a Distribution). The derivative of a distribution $T \in \mathcal{D}'(\Omega)$ of order $\alpha \in \mathbb{N}^N$ is the mapping

$$\begin{aligned} D^\alpha T : \mathcal{D}(\Omega) &\to \mathbb{R} \\ \varphi &\longmapsto D^\alpha T(\varphi) := (-1)^{|\alpha|}\langle T, D^\alpha \varphi\rangle. \end{aligned}$$

Exercice 1.5. a. Show that $D^\alpha T$ is a distribution on Ω for every $\alpha \in \mathbb{N}^N$ (we then say that T is infinitely differentiable).

b. Suppose that $T := f \in \mathcal{C}^k(\Omega)$. Show that the derivatives of T up to order k coincide with the classical derivatives of f.

Définition 1.5. Let E be a Banach space. The dual of E, denoted E', is the space of all continuous linear forms f on E, and we write $f(x) := \langle f, x\rangle$ for $x \in E$.

Théorème 1.8 (Riesz-Fréchet Representation Theorem)**.** *Let H be a Hilbert space and H' its dual. Then for every $f \in H'$, there exists a unique $x \in H$ such that*

$$\langle f, y \rangle = (x, y) \text{ for all } y \in H,$$

and $\|f\| = \|x\|$, where $(\ ,\ )$ denotes the inner product on H. Moreover, the map $f \longmapsto x$ is linear and continuous.

Théorème 1.9. *For every $f \in (L^p(\Omega))'$, there exists a unique $g \in L^{p'}(\Omega)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) such that*

$$\langle f, v \rangle = \int_\Omega g v \mathrm{d}x, \quad \forall v \in L^p(\Omega).$$

Moreover, we have

$$\|f\| = \|g\|_{L^{p'}(\Omega)}.$$

This theorem allows us to identify the dual of L^p with $L^{p'}$.

Définition 1.6 (Weak Convergence)**.**

1. A sequence $(x_n)_n$ is said to converge weakly to x in a Banach space E, written $x_n \rightharpoonup x$, if for every $f \in E'$ we have

$$\langle f, x_n \rangle_{E' \times E} \longrightarrow \langle f, x \rangle_{E' \times E} \text{ as } n \to +\infty.$$

2. A sequence $(T_n)_n$ converges to T in $\mathcal{D}'(\Omega)$ if for every $\varphi \in \mathcal{D}(\Omega)$ we have

$$\langle T_n, \varphi \rangle \longrightarrow \langle T, \varphi \rangle \quad \text{as } n \to \infty.$$

From the above theorems, we can identify the dual of $L^p(\Omega)$ with $L^{p'}(\Omega)$, as well as any Hilbert space with its dual. Consequently, we obtain the following result:

Proposition 1.9.1. *1. A sequence (f_n) converges weakly to f in L^p if and only if for every $g \in L^{p'}$,*

$$\int f_n g \mathrm{d}x \longrightarrow \int f g \mathrm{d}x \quad \text{as } n \to \infty.$$

$$\langle T_n, \varphi \rangle \longrightarrow \langle T, \varphi \rangle \quad \text{as } n \to \infty.$$

3. A sequence $(x_n)_n$ converges weakly to x in a Hilbert space H if and only if for every $y \in H$,

$$(x_n, y) \longrightarrow (x, y) \quad \text{as } n \to \infty.$$

Exercice 1.6.

i) If $(f_n)_n$ converges in L^p, show that it converges weakly in L^p.

ii) If $(f_n)_n \subset L^p$ converges in $\mathcal{D}'(\Omega)$, show that it converges weakly in L^p.

Théorème 1.10. *Every bounded sequence in a Hilbert space H admits a weakly convergent subsequence in H.*

1.1 Exercises with Solutions

Exercice 1.7. Show that $L^p(\Omega) \ni u \longmapsto \|u\|_{L^p(\Omega)}$ defines a norm on $L^p(\Omega)$ for every $1 \leq p \leq \infty$.

Solution:

1. For $p = 1$ and $p = \infty$, the result is obvious. For $1 < p < \infty$, the triangle inequality follows from Hölder's inequality:

$$\begin{aligned}\|f+g\|_{L^p}^p = \int_\Omega |f+g|^p \mathrm{d}x &\leq \int_\Omega |f+g|^{p-1}|f|\mathrm{d}x + \int_\Omega |f+g|^{p-1}|g|\mathrm{d}x \\ &\leq \|f+g\|_{L^p}^{p-1}\|f\|_{L^p} + \|f+g\|_{L^p}^{p-1}\|g\|_{L^p}\end{aligned}$$

which yields the desired result.

Exercice 1.8. Let $p, q \in [1, \infty[$. Let (f_n) be a sequence of functions in $L^p(\Omega) \cap L^q(\Omega)$ that converges to f in $L^p(\Omega)$ and to g in $L^q(\Omega)$. Show that $f = g$ almost everywhere on Ω.

Solution: Apply the converse of Lebesgue's theorem 1.4.

Exercice 1.9. Let $T \in \mathcal{D}'(\Omega)$.

a. Show that $D^\alpha T \in \mathcal{D}'(\Omega)$ for every $\alpha \in \mathbb{N}^N$ (hence every distribution T is infinitely differentiable).

b. Suppose that $T := f \in \mathcal{C}^k(\Omega)$. Show that the derivatives of T up to order k coincide with the classical derivatives of f.

Solution:

a) Let $\{\varphi_n\}_n$ be a convergent sequence in $\mathcal{D}(0, T)$. Then

$$\begin{cases} \exists\ K \subset \Omega \text{ compact } : \operatorname{supp} \varphi_n \subset K,\ \forall n \\ \varphi_n^{(\alpha)} \longrightarrow \varphi^{(\alpha)},\ \text{in} L^\infty(0,T),\ \forall \alpha \in \mathbb{N}^N \end{cases}$$

Thus, for every $\alpha \in \mathbb{N}^N$,

$$\langle D^{(\alpha)}T, \varphi_n\rangle \ := \ (-1)^{|\alpha|}\langle T, \varphi_n^\alpha\rangle \longrightarrow (-1)^{|\alpha|}\langle T, \varphi^{(k)}\rangle := \langle D^{(\alpha)}T, \varphi_n\rangle \text{ since } T \in \mathcal{D}'(0, T; E).$$

Hence $D^\alpha T$ is a distribution.

b) Since $f \in C^k(\Omega)$, we have $f \in L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$. For every $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq k$,

$$\langle D^{(\alpha)}T_f, \varphi\rangle \ := \ (-1)^{|\alpha|}\langle T_f, \varphi^\alpha\rangle. \tag{1.2}$$

On the other hand,

$$\begin{aligned}\langle D^{(\alpha)}f, \varphi\rangle &:= \int_\Omega D^{(\alpha)}f\varphi \mathrm{d}x \\ &= (-1)^{|\alpha|}\int_\Omega f D^{(\alpha)}\varphi \mathrm{d}x = (-1)^{|\alpha|}\langle T_f, \varphi^\alpha\rangle.\end{aligned} \tag{1.3}$$

From (1.2) and (1.3), we obtain the required result.

Exercice 1.10. i) If the sequence $(f_n)_n$ converges in L^p, show that it converges weakly in L^p.

ii) If the sequence $(f_n)_n \subset L^p$ converges in $\mathcal{D}'(\Omega)$, show that it converges weakly in L^p.

Exercice 1.11. Let p, q and r be three real numbers greater than or equal to 1 such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Using Hölder's inequality, show that $fg \in L^r(\Omega)$ and

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Solution:

Apply Hölder's inequality to the functions $|f|^r \in L^{p/r}(\Omega)$ and $|g|^r \in L^{p'/r}(\Omega)$.

Exercice 1.12. Let $(f_n)_n$ be a sequence in $L^p(\Omega)$ converging to f in $L^p(\Omega)$.
1. Show that if Ω is bounded, then $(f_n)_n$ converges to f in $L^q(\Omega)$ for every $1 \leq q \leq p$.
2. Show that $(f_n)_n$ converges to f in $\mathcal{D}'(\Omega)$.

Solution:

1. Apply Hölder's inequality:

$$\begin{aligned}
\|f_n - f\|^q_{L^q(\Omega)} &= \int_\Omega |f_n - f|^q \mathrm{d}x \\
&\leq \left(\int_\Omega |f_n - f|^p \mathrm{d}x\right)^{q/p} \left(\int_\Omega \mathrm{d}x\right)^{1-q/p} = c\|f_n - f\|^q_{L^p(\Omega)} \longrightarrow 0.
\end{aligned}$$

2. For any $\varphi \in \mathcal{D}(\Omega)$,

$$|\int_\Omega (f_n - f)\varphi \mathrm{d}x| \leq \|f_n - f\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega)} \longrightarrow 0.$$

Exercice 1.13. Let $\{T_n\}_n$ be a sequence of distributions converging to T in $\mathcal{D}'(\Omega)$. Show that $\{D^\alpha T_n\}_n$ converges to $D^\alpha T$ in $\mathcal{D}'(\Omega)$ for every multi-index $\alpha \in \mathbb{N}^N$.

Solution:

Recall that

$$T_n \longrightarrow T \text{ in } \mathcal{D}'(\Omega) \Longleftrightarrow \forall \varphi \in \mathcal{D}(\Omega) : \langle T_n, \varphi\rangle \longrightarrow \langle T, \varphi\rangle.$$

Let $\varphi \in \mathcal{D}(\Omega)$. Then

$$\langle \mathrm{D}^\alpha T_n, \varphi\rangle := (-1)^{|\alpha|}\langle T_n, \mathrm{D}^\alpha \varphi\rangle \longrightarrow (-1)^{|\alpha|}\langle T, \mathrm{D}^\alpha \varphi\rangle := \langle \mathrm{D}^\alpha T, \varphi\rangle.$$

2 Sobolev Spaces

2.1 Notion of weak derivative and Sobolev spaces

Let $u \in \mathcal{C}^1(\mathbb{R})$, $v \in L^1_{\text{loc}}(\mathbb{R})$. The following two properties are equivalent

1. $\int_{\mathbb{R}} v\varphi \mathrm{d}x = -\int_{\mathbb{R}} u\varphi' \mathrm{d}x, \quad \forall \varphi \in \mathcal{D}(\mathbb{R})$

2. $v(x) = u'(x), \quad \text{a.e.} \quad x \in \mathbb{R}$

(Show this equivalence). We can therefore adapt the first formula above as a definition of the derivative of u (provided u is of class $\mathcal{C}^1$), and moreover v is the derivative of the regular distribution T_u. Now, if u is not of class $\mathcal{C}^1$, then the existence of such a function $v \in L^1_{\text{loc}}$ is equivalent to saying that the distribution $T_u' \in L^1_{\text{loc}}$. That is, u must belong to the space $\{w \in L^1_{\text{loc}} : T_w' \in L^1_{\text{loc}}\}$. This type of space is called a Sobolev space. Thus the notion of derivative given by formula (1), which is called the weak derivative, is more general than the classical notion, because it coincides with the classical derivative when the latter exists, and moreover it preserves the integration by parts formula in a space larger than $\mathcal{C}^1$, namely the Sobolev space. This is important in the resolution of partial differential equations. Furthermore, these spaces with their associated norms form separable and reflexive Banach spaces, unlike the $\mathcal{C}^1$ spaces, and they are convenient and useful for applying various methods in solving partial differential equations such as variational methods (the subject of Chapter III), fixed point methods, etc.

We will now specify the definitions in a more general framework and we have:

Définition 2.1. Let $\Omega \subset \mathbb{R}^N$ be an open set, $u, \ v \in L^1_{\text{loc}}(\Omega)$.

1. We say that v is the weak partial derivative of u with respect to x_i if

$$\boxed{\int_\Omega v \ \varphi \mathrm{d}x = -\int_\Omega u \ \partial_i \varphi \mathrm{d}x, \quad \forall \varphi \in \mathcal{D}(\Omega).} \tag{2.1}$$

   In other words, if $v = \partial_i T_u \in L^1_{\text{loc}}$. If there is no confusion, we will write $\partial_i u$ instead of v or $\partial_i T_u$.

2. For $\alpha \in \mathbb{N}^N$, we say that v is the weak derivative of u of order α if

$$\boxed{\int_\Omega v \ \varphi \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u \ \mathrm{D}^\alpha \varphi \mathrm{d}x, \quad \forall \varphi \in \mathcal{D}(\Omega).} \tag{2.2}$$

   In other words, if $v = \mathrm{D}^\alpha T_u \in L^1_{\text{loc}}$. Similarly, we will write $\mathrm{D}^\alpha u$ instead of $\mathrm{D}^\alpha T_u = v$.

Remarque 2.1. Recall that the distributions $\partial_i T_u$, $\mathrm{D}^\alpha T_u$ are respectively defined by

$$\boxed{\langle \partial_i T_u, \varphi \rangle := -\int_\Omega u \ \partial_i \varphi \mathrm{d}x, \quad \forall \varphi \in \mathcal{D}(\Omega).}$$

$$\boxed{\langle \mathrm{D}^\alpha T_u, \varphi\rangle := (-1)^{|\alpha|}\int_\Omega u\ \mathrm{D}^\alpha\varphi \mathrm{d}x, \quad \forall \varphi\in\mathcal{D}(\Omega).}$$

Définition 2.2. Let Ω be an open set in $\mathbb{R}^N$.

1. The first-order Sobolev space is the set

$$\boxed{H^1(\Omega) \stackrel{\text{def}}{=} \{u:\Omega\longrightarrow\mathbb{R} \text{ measurable} \mid \quad u, \partial_i u\in L^2(\Omega),\ i=1,...,N\}.} \tag{2.3}$$

2. The Sobolev space of order $k\in\mathbb{N}\cup\{+\infty\}$ is the set

$$\boxed{H^k(\Omega) \stackrel{\text{def}}{=} \{u:\Omega\longrightarrow\mathbb{R} \text{ measurable} \mid \quad u, \mathrm{D}^\alpha u\in L^2(\Omega),\ \forall\alpha\in\mathbb{N}^N \text{ with } |\alpha|\leq k\}.} \tag{2.4}$$

(Where $\partial_i u, \mathrm{D}^\alpha$ are understood in the sense of Definition 2.1).

In a more general framework, Sobolev spaces are defined as follows:

Définition 2.3. Let Ω be an open set in $\mathbb{R}^N$, and let $1\leq p\leq+\infty$.

1. The first-order Sobolev space is the set

$$\boxed{W^{1,p}(\Omega) \stackrel{\text{def}}{=} \{u:\Omega\longrightarrow\mathbb{R} \text{ measurable} \mid \quad u, \partial_i u\in L^p(\Omega),\ i=1,...,N\}.} \tag{2.5}$$

2. The Sobolev space of order $k\in\mathbb{N}\cup\{+\infty\}$ is the set

$$\boxed{W^{k,p}(\Omega) \stackrel{\text{def}}{=} \{u:\Omega\longrightarrow\mathbb{R} \text{ measurable} \mid \quad u, \mathrm{D}^\alpha u\in L^p(\Omega),\ \forall\alpha\in\mathbb{N}^N \text{ with } |\alpha|\leq k\}.} \tag{2.6}$$

(Where $\partial_i u, \mathrm{D}^\alpha$ are understood in the sense of Definition 2.1).

Exercice 2.1. Let u be the function defined on $]-1,1[$ by $u(x)=|x|$. Show that $u\in H^1(-1,1)$.

Solution. It is clear that $u\in L^2(-1,1)$. Let us compute its weak derivative v if it exists. For every $\varphi\in\mathcal{D}(-1,1)$:

$$\int_{-1}^1 v\varphi \mathrm{d}x := -\int_{-1}^1 |x|\varphi'\mathrm{d}x = -\int_{-1}^0 \varphi\mathrm{d}x + \int_{-1}^1 \varphi\mathrm{d}x \quad \text{(after integration by parts).}$$

Hence

$$v(x) = \begin{cases} -1 & \text{if} \quad x\in(-1,0[\\ 1 & \text{if} \quad x\in]0,+1) \end{cases}$$

and we have $u,v\in L^2(-1,1)$ (obviously). Therefore $u\in H^1(-1,1)$, but u is not in $\mathcal{C}^1(-1,1)$.

Remarque 2.2. One can easily show that continuous and piecewise $\mathcal{C}^1$ functions belong to the Sobolev spaces $W^{1,p}(\Omega)$.

Exercice 2.2. Find the values of α such that the function $u(x) = x^\alpha$ belongs to $H^1(0,1)$.

Solution. We have

$$u \in L^2(0,1) \Longleftrightarrow \int_0^1 x^{2\alpha} \mathrm{d}x < \infty \Longleftrightarrow \alpha > -1/2.$$

Let v be the weak derivative (if it exists). Then, for every $\varphi \in \mathcal{D}(0,1)$:

$$\int_0^1 v\varphi \mathrm{d}x = -\int_0^1 u\varphi' \mathrm{d}x = \int_a^1 v\varphi \mathrm{d}x = \alpha \int_0^1 x^{\alpha-1}\varphi \mathrm{d}x.$$

Hence $v(x) = \alpha x^{\alpha-1}$.

$$v \in L^2(0,1) \Longleftrightarrow \int_0^1 x^{2(\alpha-1)} \mathrm{d}x < \infty \Longleftrightarrow \alpha > 1/2.$$

Thus $u \in H^1(0,1) \Longleftrightarrow \alpha > 1/2$.

Exercice 2.3. Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}^N$. Consider the following function

$$\begin{array}{rcl} u : \Omega & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & u(x) = |x|^\alpha. \end{array}$$

Find the values of α such that $u \in H^1(B_1(0))$.

Solution. We use polar coordinates $x = rw$ where $0 < r < 1$ and $|w| = 1$. We have $\mathrm{d}x = r^{N-1}\mathrm{d}r\mathrm{d}w$. Hence

$$\int_\Omega |u(x)|^2 \mathrm{d}x = \int_\Omega |x|^{2\alpha} \mathrm{d}x = \int_{|w|=1} \mathrm{d}w \int_0^1 r^{2\alpha+N-1} \mathrm{d}r.$$

We have $\int_{|w|=1} \mathrm{d}w < \infty$ (since it is the surface area of the unit sphere). Moreover,

$$\int_0^1 r^{2\alpha+N-1} \mathrm{d}r < +\infty \Longleftrightarrow \alpha > -N/2.$$

Thus $u \in L^2(\Omega)$ if and only if $\alpha > -N/2$. On the other hand, we have $\partial_i u(x) = \alpha |x|^{\alpha-2} x_i$ for all $x \in \Omega \setminus \{0\}$. Hence

$$\begin{aligned} \big(\forall i : \partial_i u \in L^2(\Omega)\big) \quad &\Longleftrightarrow \quad \sum_{i=1}^N \int_\Omega |\partial_i u|^2 \mathrm{d}x < \infty \Longleftrightarrow \alpha^2 \int_\Omega |x|^{2\alpha-2} \mathrm{d}x < \infty \\ &\Longleftrightarrow \quad \alpha^2 \int_{|w|=1} \mathrm{d}w \int_0^1 r^{2\alpha+N-3} \mathrm{d}r \Longleftrightarrow \alpha > -\frac{N}{2} + 1. \end{aligned}$$

Hence $u \in H^1(\Omega)$ if and only if $\alpha > 1 - N/2$.

Regarding the structure of these spaces, we have the following results:

Théorème 2.3. *For $1 \leq p \leq \infty$, the space $W^{1,p}(\Omega)$ is a Banach space with the following norm*

$$\boxed{\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|^p_{L^p(\Omega)}\right)^{1/p}.} \tag{2.7}$$

Remarque 2.4. 1. The norm defined above is equivalent to the following norms

$$\|u\| = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^p(\Omega)},$$

$$\|u\| = \max\{\|u\|_{L^p(\Omega)}, \|\partial_i u\|_{L^p(\Omega)}, i = 1, ..., N\}.$$

2. If the sequence $\{u_n\}_n$ converges to u in $W^{1,p}(\Omega)$, then

$$\|u_n - u\|_{W^{1,p}(\Omega)} = \left(\|u_n - u\|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i u_n - \partial_i u\|^p_{L^p(\Omega)}\right)^{1/p} \longrightarrow 0.$$

This implies that

$$\|u_n - u\|^p_{L^p(\Omega)} \longrightarrow 0, \ \|\partial_i u_n - \partial_i u\|^p_{L^p(\Omega)} \longrightarrow 0, \quad \forall i = 1, ..., N.$$

Consequently,

$$\boxed{(u_n \longrightarrow u \text{ in } W^{1,p}(\Omega)) \Longleftrightarrow \left\{ \begin{array}{lll} u_n & \longrightarrow & u \\ \partial_i u_n & \longrightarrow & \partial_i u, \quad \forall i \end{array} \right. \text{ in } L^p(\Omega).}$$

Proof of Theorem 2.3

1. Let us show that $\|.\|_{W^{1,p}(\Omega)}$ defines a norm on $W^{1,p}(\Omega)$. The properties of separation and homogeneity are obvious. For the triangle inequality, apply Minkowski's inequality:

$$\left(\sum_{i=1}^{N} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{N} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{N} |y_i|^p\right)^{1/p} \tag{2.8}$$

2. Let us show that $W^{1,p}(\Omega)$ is complete, i.e., every Cauchy sequence converges. Let $\{u_n\}_n$ be a Cauchy sequence in $W^{1,p}(\Omega)$. Then $\forall \varepsilon > 0, \ \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0$:

$$\|u_n - u_m\|_{W^{1,p}(\Omega)} = \left(\|u_n - u_m\|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i u_n - \partial_i u_m\|^p_{L^p(\Omega)}\right)^{1/p} \leq \varepsilon.$$

Hence $\forall \varepsilon > 0, \ \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0$:

$$\|u_n - u_m\|_{L^p(\Omega)} \leq \varepsilon \text{ and } \|\partial_i u_n - \partial_i u_m\|_{L^p(\Omega)} \leq \varepsilon, \ \forall i = 1, ..., N.$$

Therefore the sequences $\{u_n\}_n$ and $\{\partial_i u_n\}_n$ are Cauchy in $L^p(\Omega)$, which is complete; they converge respectively to $u,\ v_i \in L^p(\Omega)$ for $i = 1, ..., N$. On the other hand, the space $L^p(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. We thus have

$$\left(u_n \xrightarrow{n} u \text{ in } L^p(\Omega)\right) \implies \left(u_n \xrightarrow{n} u \text{ in } \mathcal{D}'(\Omega)\right)$$
$$\implies \left(\partial_i u_n \xrightarrow{n} \partial_i u \text{ in } \mathcal{D}'(\Omega)\right)$$

and for $i = 1, ..., N$,

$$\left(\partial_i u_n \xrightarrow{n} v_i \text{ in } L^p(\Omega)\right) \implies \left(\partial_i u_n \xrightarrow{n} v_i \text{ in } \mathcal{D}'(\Omega)\right).$$

By uniqueness of the limit, we deduce that $\partial_i u = v_i$ for every $i = 1, ..., N$. Thus $\{u_n\}$ converges to u in $W^{1,p}(\Omega)$.

Théorème 2.5. *The Sobolev space $H^1(\Omega)$ is a Hilbert space with the following inner product*

$$\langle u, v\rangle_{H^1(\Omega)} := \int_\Omega (uv + \nabla u.\nabla v)\,\mathrm{d}x. \tag{2.9}$$

The proof is immediate.

2.2 Density on $\mathbb{R}^N$

In practice, many formulas and results are easy to prove for regular functions and can be extended to arbitrary functions by a density argument. To this end, we study the question of approximation of $W^{1,p}$ functions by smooth functions. The idea is to use mollifiers for the approximation:

Définition 2.4. A mollifier is a function $\rho \in \mathcal{D}(\mathbb{R}^N)$ ($\mathcal{C}^\infty$ with compact support) which vanishes outside the unit ball and satisfies

$$\int_{\mathbb{R}^N} \rho(x)\mathrm{d}x = 1.$$

An important example of such a function is the following

$$\rho(x) = \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{2.10}$$

where $c > 0$ is chosen so that $\int_{\mathbb{R}^N} \rho(x)\mathrm{d}x = 1$. It is easy to verify that $\rho \in \mathcal{D}(\mathbb{R}^N)$ and $\operatorname{supp}\rho = \overline{B}(0,1)$. We accept the following result concerning Lebesgue spaces L^p:

Théorème 2.6. *The space $\mathcal{C}^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$. In particular, for every $u \in L^p(\Omega)$, we have*

$$\boxed{\rho_n \star u \in \mathcal{C}^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), \quad \textit{and } \rho_n \star u \longrightarrow u \textit{ in } L^p(\Omega).}$$

Where

$$\boxed{\rho_n(x) := n^N \rho(nx), \quad \textit{and } \rho_n \star u(x) := \int_{\mathbb{R}^N} \rho_n(x-y)u(y)\mathrm{d}y.}$$

Corollaire 2.6.1. *The space $\mathcal{C}^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}(\mathbb{R}^N)$ for every $k \in \mathbb{N}^*$ and $1 \leq p < \infty$.*

Proof. Let $u \in W^{k,p}(\mathbb{R}^N)$. Then, by Theorem 2.6,

$$\boxed{\rho_n \star u \in \mathcal{C}^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), \quad \rho_n \star u \longrightarrow u \text{ in } L^p(\Omega).}$$

Moreover, for every $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq k$,

$$\boxed{\mathrm{D}^\alpha(\rho_n \star u) = \rho_n \star \mathrm{D}^\alpha u \in \mathcal{C}^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \ \text{ and } \mathrm{D}^\alpha(\rho_n \star u) \longrightarrow \mathrm{D}^\alpha u \text{ in } L^p(\mathbb{R}^N).}$$

Consequently,

$$\rho_n \star u \in W^{k,p}(\mathbb{R}^N) \text{ and } \rho_n \star u \longrightarrow u \ \text{ in } W^{k,p}(\mathbb{R}^N).$$

□

Théorème 2.7. *The space $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{k,p}(\mathbb{R}^N)$ for every $k \in \mathbb{N}^*$ and $1 \leq p < \infty$.*

Proof. It suffices to construct a sequence $\{\phi_n\} \subset \mathcal{D}(\mathbb{R}^N)$ converging to u in $W^{k,p}(\mathbb{R}^N)$. Consider the cut-off function $\zeta \in \mathcal{D}(\mathbb{R}^N)$ satisfying

$$\zeta \equiv 1 \text{ on } B(0,1), \ \operatorname{supp} \zeta \subset B(0,2) \text{ and } 0 \leq \zeta \leq 1.$$

For each $n \in \mathbb{N}^*$, define ϕ_n by

$$\phi_n(x) = \zeta(x/n)(\rho_n \star u) \in \mathcal{D}(\mathbb{R}^N).$$

By the previous corollary, we have

$$\rho_n \star u \longrightarrow u \text{ in } W^{k,p}(\mathbb{R}^N),$$

but $\rho_n \star u \in \mathcal{C}^\infty(\mathbb{R}^N)$. To conclude, it suffices to show that

$$\rho_n \star u - \phi_n \longrightarrow 0 \text{ in } L^p(\mathbb{R}^N).$$

We have

$$\begin{aligned}
\|\phi_n - \rho_n \star u\|^p_{L^p(\mathbb{R}^N)} &= \underbrace{\int_{\{|x|<n\}} |\phi_n - \rho_n \star u|^p \mathrm{d}x}_{=0} + \int_{\{|x|>n\}} |\phi_n - \rho_n \star u|^p \mathrm{d}x \\
&= \int_{\{|x|>n\}} |\zeta(x/n)(\rho_n \star u) - \rho_n \star u|^p \mathrm{d}x \\
&\leq \int_{\{|x|>n\}} |\rho_n \star u|^p \mathrm{d}x = \int_{\mathbb{R}^N} \chi_{\{|x|>n\}} |\rho_n \star u|^p \mathrm{d}x
\end{aligned}$$

It is clear that $\chi_{\{|x|>n\}}|\rho_n \star u|^p \longrightarrow 0$ for every $x \in \mathbb{R}^N$. Since $\rho_n \star u \to u$ in $L^p(\Omega)$, by the converse of Lebesgue's theorem, there exists a subsequence, still denoted $\rho_n \star u$, and a function $g \in L^p(\mathbb{R}^N)$ such that $|\rho_n \star u| \leq g$ and consequently $\chi_{\{|x|>n\}}|\rho_n \star u| \leq g$. Hence, applying Lebesgue's dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^N} \chi_{\{|x|>n\}}|\rho_n \star u|^p \mathrm{d}x \longrightarrow 0.$$

Thus

$$\|\phi_n - \rho_n \star u\|^p_{L^p(\mathbb{R}^N)} \longrightarrow 0.$$

Therefore

$$\|\phi_n - u\|_{L^p(\mathbb{R}^N)} \leq \|\phi_n - \rho_n \star u\|_{L^p(\mathbb{R}^N)} + \|\rho_n \star u - u\|_{L^p(\mathbb{R}^N)} \longrightarrow 0.$$

Similarly, one shows that $\mathrm{D}^\alpha \phi_n \to \mathrm{D}^\alpha u$ in $L^p(\mathbb{R}^N)$. □

2.3 Density on an open set Ω

We now try to generalize the density result proved in $\mathbb{R}^N$ to an arbitrary open set Ω of $\mathbb{R}^n$. The natural idea is to extend functions defined on Ω to $\mathbb{R}^N$ and then apply density in $\mathbb{R}^N$. However, extending functions from $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^N)$ is not always possible because it requires conditions on the boundary of Ω. This is the subject of the following section.

Définition 2.5. Let $P : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^N)$ be a map. We say that P is an extension operator from $W^{k,p}(\Omega)$ if it is linear and continuous and satisfies

$$Pu_{|\Omega} = u \text{ a.e. on } \Omega, \quad \forall u \in W^{k,p}(\Omega).$$

We then have the following density result:

Théorème 2.8. *If an extension operator from $W^{k,p}(\Omega)$ exists, then the space*

$$E := \{v \in W^{k,p}(\Omega) \mid \exists\, \overline{v} \in \mathcal{D}(\mathbb{R}^N) : v = \overline{v}_{|\overline{\Omega}}\}$$

is dense in $W^{k,p}(\Omega)$. In particular, $\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Proof. Let $u \in W^{k,p}(\Omega)$. By the hypothesis of the theorem, we have $Pu \in W^{k,p}(\mathbb{R}^N)$. From the proof of Theorem 2.7, we have

$$\zeta_n(\rho_n \star Pu) \to Pu \text{ in } W^{k,p}(\mathbb{R}^N).$$

Consequently,

$$\|\zeta_n(\rho_n \star Pu)_{|\Omega} - u\|_{W^{k,p}(\Omega)} \leq \|\zeta_n(\rho_n \star Pu) - u\|_{W^{k,p}(\mathbb{R}^N)} \to 0.$$

Thus

$$u_n := \zeta_n(\rho_n \star Pu)_{|\Omega} \in E \text{ and } u_n \to u \text{ in } W^{k,p}(\Omega).$$

This completes the proof. □

Remarque 2.9. The space of test functions $\mathcal{D}(\Omega)$ is not dense in $W^{k,p}(\Omega)$ if $\Omega \neq \mathbb{R}^N$.

Exercice 2.4. Show that the space $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for every $1 \leq p < \infty$.

2.4 Existence of the extension operator

Définition 2.6. Let Ω be an open set in $\mathbb{R}^N$ with boundary Γ. We say that Γ (or Ω) is of class $\mathcal{C}^k$ ($k \in \mathbb{N}$) if for every $x_0 \in \Gamma$, there exists a ball $B(x_0, r)$ and a bijective map from $B(x_0, r)$ onto an open set W such that

1. $\psi : B(x_0, r) \to W$ is a k-diffeomorphism (ψ and ψ^{-1} are k times differentiable).
2. $\psi(\Omega \cap B(x_0, r)) = W \cap \{x = (x', x_n) \in \mathbb{R}^N : x_n > 0\}$.
3. $\psi(\Gamma \cap B(x_0, r)) = W \cap \{x = (x', x_n) \in \mathbb{R}^N : x_n = 0\}$.

The following result gives a criterion for extension in $W^{1,p}(\Omega)$.

Théorème 2.10. *Let Ω be an open set in $\mathbb{R}^N$ of class $\mathcal{C}^1$. Then there exists an extension operator*

$$P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N).$$

2.5 Trace and Green's formula

2.5.1 Notion of trace

If $u \in \mathcal{C}(\overline{\Omega})$, then the function Tu defined on $\partial\Omega$ by $Tu := u_{|\partial\Omega}$ is well-defined and belongs to $\mathcal{C}(\partial\Omega)$. It is called the trace of the function u on $\partial\Omega$. Thus we have a well-defined linear map

$$T : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\partial\overline{\Omega}).$$

which is continuous with respect to the L^∞ norm. Now if u belongs to $L^p(\Omega)$ but not to $\mathcal{C}(\overline{\Omega})$, then clearly $Tu := u_{|\Omega}$ is not well-defined because $\partial\Omega$ is negligible and we cannot obtain information about u on $\partial\Omega$. However, we can give a meaning to Tu if u belongs to the Sobolev space $W^{1,p}(\Omega)$. Indeed, one can show that the map

$$T : (\mathcal{C}^\infty(\overline{\Omega}), \|.\|_{W^{1,p}(\Omega)}) \to (L^p(\partial\Omega), \|.\|_{L^p(\partial\Omega)})$$

is continuous and, by density of $\mathcal{C}^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega)$, we can extend the definition of T to the space $W^{1,p}(\Omega)$. We have the following theorem:

Théorème 2.11. *Let Ω be an open set in $\mathbb{R}^N$ of class $\mathcal{C}^1$. Then there exists a linear continuous map $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that*

$$Tu = u_{|\partial\Omega} \text{ if } u \in \mathcal{C}(\overline{\Omega}).$$

($L^p(\partial\Omega)$ is equipped with the surface measure on $\partial\Omega$).

Définition 2.7. Let $u \in W^{1,p}(\Omega)$. The function $Tu \in L^p(\partial\Omega)$ is called the trace of u on $\partial\Omega$.

Remarque 2.12. 1. Let $u \in W^{1,p}(\Omega)$. Then

$$Tu = \lim_{n\to\infty} Tu_n := \lim_{n\to\infty} u_{n|\partial\Omega} \text{ where } u_n \in \mathcal{C}^\infty(\overline{\Omega}), u_n \to u \text{ in } W^{1,p}(\Omega).$$

Indeed, by continuity of T, we have

$$\|Tu - Tu_n\|_{L^p(\partial\Omega)} \leq \|u_n - u\|_{W^{1,p}(\Omega)} \to 0.$$

2. The map T is not surjective. That is, the image $T(W^{1,p}(\Omega))$ is a proper subset of $L^p(\partial\Omega)$.

Définition 2.8. The space $H^{1/2}(\partial\Omega)$ is the range of the trace operator

$$T : H^1(\Omega) := W^{1,2}(\Omega) \to L^2(\partial\Omega).$$

Remarque 2.13. The space $H^{1/2}(\partial\Omega)$ coincides with the fractional Sobolev space H^s with $s = 1/2$. Moreover, the trace operator T is continuous from $W^{1,p}(\Omega)$ into $H^{1/2}(\partial\Omega)$ equipped with the norm

$$\|u\|_{H^{1/2}} := \|u\|_{L^2} + \int\int \frac{(u(x)-u(y))^2}{|x-y|^{N+1}} \mathrm{d}x\mathrm{d}y.$$

This type of fractional Sobolev space is not covered in this course.

2.5.2 Green's formula

If $u, v \in \mathcal{C}^1(\overline{\Omega}) \cap H^1(\Omega)$, then we have the following integration by parts formula

$$\int_\Omega \partial_i uv \mathrm{d}x = \int_{\partial\Omega} uv\, n_i \mathrm{d}\sigma - \int_\Omega u\partial_i v \mathrm{d}x \ \forall i \in \{1, ..., N\}.$$

where $\overrightarrow{n} = (n_1, ..., n_N)$ is the unit outward normal vector to $\partial\Omega$. Thanks to the density of $\mathcal{C}^1$ in $H^1(\Omega)$ (Theorem 2.8) and the continuity of the trace operator T (Theorem 2.11), we can generalize the above formula to all functions in $H^1(\Omega)$ and obtain the following formula, called Green's formula:

Théorème 2.14. *Let Ω be an open set in $\mathbb{R}^n$ of class $\mathcal{C}^1$. Then we have*

$$\forall u, v \in H^1(\Omega) : \int_\Omega \partial_i uv \mathrm{d}x = \int_{\partial\Omega} TuTv\, n_i \mathrm{d}\sigma - \int_\Omega u\partial_i v \mathrm{d}x \ \forall i \in \{1, ..., N\}. \tag{2.11}$$

Remarque 2.15. If there is no confusion, we write

$$\int_{\partial\Omega} uv\, n_i \mathrm{d}\sigma \text{ instead of } \int_{\partial\Omega} TuTv\, n_i \mathrm{d}\sigma.$$

Corollaire **2.15.1.** *Let Ω be an open set in $\mathbb{R}^N$ of class $\mathcal{C}^1$. Then for every $u \in H^2(\Omega)$, $v \in H^1(\Omega)$ we have*

$$\int_\Omega \Delta u v \mathrm{d}x = \int_{\partial\Omega} T\left(\frac{\partial u}{\partial n}\right) T v \mathrm{d}\sigma - \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$$

where $\frac{\partial u}{\partial n} := \nabla u \cdot \overrightarrow{n} \in H^1(\Omega)$. (For simplicity, we write

$$\boxed{\int_\Omega \Delta u v \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial n} v \mathrm{d}\sigma - \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x}).$$

Remarque 2.16. Green's formula remains valid for functions $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

2.6 Some useful properties in Sobolev spaces

Proposition **2.16.1.** *Let $u \in W^{1,p}(\Omega)$. Let $H : U \to \Omega$ be a diffeomorphism (i.e., H is bijective and H, H^{-1} are of class $\mathcal{C}^1$) such that:*

$$\operatorname{Jac} H \in L^\infty(U), \ \operatorname{Jac} H^{-1} \in L^\infty(\Omega).$$

Then $u \circ H \in W^{1,p}(U)$ and we have

$$\forall i : \frac{\partial(u \circ H)}{\partial y_i}(y) = \sum_{j=1}^N \frac{\partial u}{\partial_{x_j}}(H(y)) \frac{\partial H_j}{\partial y_i}(y), \quad x_j = H_j(y).$$

Proof. For $p < \infty$.

i) By density (see Course 2), there exists a sequence $\{u_n\}_n$ in $\mathcal{C}^\infty(\overline{\Omega})$ such that

$$u_n \to u \text{ in } L^p(\Omega) \text{ and } \forall i : \partial_i u_n \to \partial_i u \text{ in } L^p(K) \text{ for every compact } K \subset U.$$

ii) Let us show that if $v \in L^p(\Omega)$ then $v \circ H \in L^p(U)$. We have

$$\begin{aligned} \int_U |u \circ H(y)|^p \mathrm{d}y &= \int_\Omega |u(x)|^p |\operatorname{Jac} H^{-1}| \mathrm{d}x \\ &\leq C \int_\Omega |u(x)|^p \mathrm{d}x < \infty \quad (\text{since } \operatorname{Jac} H^{-1} \in L^\infty(U)) \end{aligned}$$

Thus $v \circ H \in L^p(U)$.

iii) Let us show that if $v_n \to v$ in $L^p(\Omega)$ then $v_n \circ H \to v \circ H$ in $L^p(U)$. By the converse of Lebesgue's theorem, there exists a subsequence $\{v_{n_k}\}$ such that

$$v_{n_k} \to v \text{ a.e. } x \in \Omega \text{ and } |v_{n_k}| \leq g \in L^p(\Omega) \ \forall n.$$

Consequently,

$$v_{n_k} \circ H \to v \circ H \text{ a.e. } y \in U \text{ and } |v_{n_k} \circ H| \leq g \circ H \in L^p(U) \quad \forall k.$$

Applying Lebesgue's dominated convergence theorem, we obtain

$$v_{n_k} \circ H \to v \circ H \text{ in } L^p(U).$$

Now let us show that $v_n \circ H \to v \circ H$ in $L^p(U)$. By contradiction, suppose there exists $\varepsilon > 0$ and a subsequence $\{v_{n_k} \circ H\}_k$ such that

$$\|v_{n_k} \circ H - v \circ H\|_{L^p(U)} \geq \varepsilon. \tag{2.12}$$

However, the sequence $\{v_{n_k}\}_k$ converges to v in $L^p(\Omega)$, hence it admits a subsequence, denoted $\{v_{\varphi(n_k)}\}$, such that

$$\|v_{\varphi(n_k)} \circ H - v \circ H\|_{L^p(U)} \to 0.$$

This contradicts (2.12).

iv) We are now ready to show that $u \circ H \in W^{1,p}(U)$. From i) we have $u_n \circ H \to u \circ H$ in $L^p(U)$. Since u_n and H are of class $\mathcal{C}^1$, then

$$\forall i : \frac{\partial (u_n \circ H)}{\partial y_i}(y) = \sum_{j=1}^{N} \frac{\partial u_n}{\partial_{x_j}}(H(y)) \frac{\partial H_j}{\partial y_i}(y). \tag{2.13}$$

On the other hand, by ii) we have for every compact $K \subset U$

$$\frac{\partial u_n}{\partial_{x_j}}(H) := \frac{\partial u_n}{\partial_{x_j}} \circ H \to \frac{\partial u}{\partial_{x_j}} \circ H := \frac{\partial u}{\partial_{x_j}}(H) \text{ in } L^p(K)$$

and since $\frac{\partial H}{\partial y_j} \in L^\infty(K)$, then

$$\frac{\partial u_n}{\partial_{x_j}}(H) \frac{\partial H_j}{\partial y_i}(y) \to \frac{\partial u}{\partial_{x_j}}(H) \frac{\partial H_j}{\partial y_i}.$$

Hence we can pass to the limit in the sense of distributions in (2.13) and obtain

$$\forall i : \frac{\partial (u \circ H)}{\partial y_i} = \sum_{j=1}^{N} \frac{\partial u}{\partial_{x_j}}(H) \frac{\partial H_j}{\partial y_i} \in L^p(U).$$

For $p = \infty$. Clearly $u \circ H \in L^\infty(\Omega)$. On the other hand, let Ω' be a bounded open subset of $\mathbb{R}^N$ such that $\Omega' \subset \Omega$. Then $u \in W^{1,\infty}(\Omega')$ and by the above,

$$\forall i : \frac{\partial (u \circ H)}{\partial y_i}(y) = \sum_{j=1}^{N} \frac{\partial u}{\partial_{x_j}}(H(y)) \frac{\partial H_j}{\partial y_i}(y) \in L^\infty(\Omega).$$

Thus $u \circ H \in W^{1,\infty}(\Omega)$. This completes the proof. □

Proposition 2.16.2. *Let $u \in W^{1,p}(\Omega)$, $G : \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{C}^1$ such that $G(0) = 0$ and $|G'(t)| \leq M$ for all $t \in \mathbb{R}$. Then*

$$G \circ u \in W^{1,p}(\Omega) \quad \text{and} \quad \partial_i(G \circ u) = G'(u)\partial_i u, \ \forall i.$$

Proof. For $p < \infty$, by the mean value theorem we have $|G(t)| \leq M|t|$ for all $t \in \mathbb{R}$. Thus $|G \circ u| \leq M|u| \in L^p(\Omega)$, and hence $G \circ u \in L^p(\Omega)$. On the other hand, by density, there exists a sequence $\{u_n\}_n$ in $\mathcal{C}^\infty(\overline{\Omega})$ such that

$$u_n \to u \text{ in } L^p(\Omega) \text{ and } \forall i : \partial_i u_n \to \partial_i u \text{ in } L^p(K) \text{ for every compact } K \subset U.$$

Then, since G and u_n are of class $\mathcal{C}^1$, we have

$$\partial_i(G \circ u_n) = G'(u_n)\partial_i u_n. \tag{2.14}$$

As in the previous proof, we show that

$$G \circ u_n \to G \circ u \text{ in } L^p(\Omega), \quad G'(u_n)\partial_i u_n \to G'(u)\partial_i u \text{ in } L^p(K),$$

for every compact $K \subset \Omega$. Hence, passing to the limit in the sense of distributions in (2.14), we obtain

$$\partial_i(G \circ u) = G'(u)\partial_i u \in L^p(\Omega).$$

For $p = \infty$, we proceed as at the end of the previous proof. $\square$

2.7 Sobolev embedding

Théorème 2.17. *Let Ω be an open set in $\mathbb{R}^N$ such that $\partial\Omega$ is of class C^1. Then we have the following continuous embeddings*

$$W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{p^*}(\Omega) & \text{if} \quad p < N \\ L^q(\Omega), & \text{if} \quad p = N, \ \forall q \geq 1 \\ C^\alpha(\overline{\Omega}) & \text{if} \quad p > N, \end{cases} \tag{2.15}$$

where $p^* = \frac{Np}{N-p}$ (i.e., $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$) and $\alpha = 1 - \frac{N}{p}$ for $p \geq 1$. In particular, for every $u \in W^{1,p}(\Omega)$ we have the following inequalities:

$$\|u\|_{L^{p^*}(\Omega)} \leq C_1 \|u\|_{W^{1,p}(\Omega)} \tag{2.16}$$

$$|u|_\alpha \leq C_2 \|u\|_{W^{1,p}(\Omega)} \tag{2.17}$$

where $C_1, \ C_2 > 0$ depend only on N, p and the diameter of Ω.

In dimension 1, we have the following important result:

Proposition 2.17.1. *Let I be an open interval in $\mathbb{R}$. Then for every $1 \leq p \leq \infty$, the space $W^{1,p}(I)$ embeds continuously into $\mathcal{C}(\overline{I})$. That is,*

$$\begin{cases} \forall u \in W^{1,p}(I), \ \exists v \in \mathcal{C}(\overline{I}) : u = v \ \text{a.e. on } I \\ \\ \exists C > 0, \ \|u\|_{W^{1,p}(I)} \leq \|v\|_{\mathcal{C}(\overline{I})} \end{cases} \tag{2.18}$$

Théorème **2.18 (Rellich-Kondrachov).** *Let Ω be a bounded open set in $\mathbb{R}^N$ such that $\partial\Omega$ is of class C^1 and let $1 \leq p \leq \infty$. Then we have the following compact embeddings*

$$W^{1,p}(\Omega) \hookrightarrow_c \begin{cases} L^q(\Omega) & \text{if } p < N, \text{ for every } 1 \leq q < p^* \\ L^q(\Omega) & \text{if } p = N, \text{ for every } 1 \leq q < \infty \\ C(\overline{\Omega}) & \text{if } p > N, \end{cases} \tag{2.19}$$

(where $p^* = \frac{Np}{N-p}$).

2.8 Spaces $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$

Définition **2.9.** The space $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$.

Notation 2.19. For $p = 2$, the space $W_0^{1,2}(\Omega)$ will henceforth be denoted by $H_0^1(\Omega)$.

The space $W_0^{1,p}(\Omega)$ is a subspace of $W^{1,p}(\Omega)$ and is characterized by:

Théorème **2.20.**

$$u \in W_0^{1,p}(\Omega) \Longleftrightarrow (u \in W^{1,p}(\Omega) \text{ and } Tu \equiv 0). \tag{2.20}$$

In particular, if $u \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ then $u_{|\partial\Omega} \equiv 0$.

An important result that follows from Theorem 2.18 is Poincaré's inequality:

Théorème **2.21 (Poincaré's inequality).** *Let Ω be a bounded open set in $\mathbb{R}^N$ such that $\partial\Omega$ is of class C^1 and $1 \leq p < \infty$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{(L^p(\Omega))^N} \quad \text{for every } u \in W_0^{1,p}(\Omega). \tag{2.21}$$

Corollaire **2.21.1.** *The map*

$$\begin{array}{rcl} W_0^{1,p}(\Omega) & \to & \mathbb{R}_+ \\ u & \longmapsto & \|\nabla u\|_{(L^p(\Omega))^N} \end{array}$$

defines a norm on $W_0^{1,p}(\Omega)$, denoted by $\|.\|_{W_0^{1,p}(\Omega)}$, which is equivalent to the norm $\|.\|_{W^{1,p}(\Omega)}$. Moreover, $(W_0^{1,p}(\Omega), \|.\|_{W_0^{1,p}(\Omega)})$ is a Banach space.

Définition **2.10.** The space $W^{-1,p'}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). In other words, an element f of $W^{-1,p'}(\Omega)$ is a continuous linear form on $W_0^{1,p}(\Omega)$, and we denote by $\langle\ ,\ \rangle$ the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. Recall that

$$\|f\|_{W^{-1,p'}(\Omega)} = \sup_{v \in W_0^{1,p}(\Omega)} \frac{\langle f, v\rangle}{\|v\|_{W_0^{1,p}(\Omega)}} = \sup_{\|v\|_{W_0^{1,p}(\Omega)}=1} \langle f, v\rangle$$

and

$$\langle f, v\rangle \leq \|f\|_{W^{-1,p'}(\Omega)}\|v\|_{W_0^{1,p}(\Omega)} \text{ for all } v \in W_0^{1,p}(\Omega).$$

Notation 2.22. For $p = 2$, then $p' = 2$. Then the space $W^{-1,2}(\Omega)$ is the dual of $W_0^{1,2}(\Omega) := H_0^1(\Omega)$. It will be denoted by $H^{-1}(\Omega)$.

Théorème 2.23. *(**Characterization of** $W^{-1,p'}(\Omega)$). Let $f \in W^{-1,p'}(\Omega)$. Then there exist functions*

$$f^0, f^1, \dots, f^N \quad \textit{in } L^{p'}(\Omega),$$

such that for every $v \in W_0^{1,p}(\Omega)$ we have

$$\langle f, v\rangle = \int_\Omega \left(f^0 v + \sum_{i=1}^{N} f^i \partial_i v \right) \mathrm{d}x \tag{2.22}$$

and

$$\|f\|_{W^{-1,p'}(\Omega)} = \left(\int_\Omega \sum_{i=0}^{N} |f^i|^{p'} \mathrm{d}x \right)^{1/p'}. \tag{2.23}$$

In particular, if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$, then

$$\langle f, v\rangle = \int_\Omega f v \mathrm{d}x \textit{ for all } v \in H_0^1(\Omega).$$

Here is the English translation of the provided exercises, with all LaTeX commands and symbols preserved, and with language improvements.
"'latex

2.9 Exercises with Solutions

Exercice 2.5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ -x+2 & \text{if } x \in [1,4] \\ 0 & \text{elsewhere} \end{cases}$$

1. Show that $f \in L^2(\mathbb{R})$.
2. Determine whether $f \in H^1(\mathbb{R})$.

Solution:

1. We have
$$\int_{-\infty}^{+\infty} f^2 \mathrm{d}x = \int_0^1 x^2 \mathrm{d}x + \int_1^4 (-x+2)^2 \mathrm{d}x < \infty.$$
Hence $f \in L^2(\mathbb{R})$.

2. We compute the derivative v of f (in the sense of distributions). Let $\varphi \in \mathcal{D}(\mathbb{R})$. We have

$$\begin{aligned}\langle v, \varphi\rangle := -\int_{\mathbb{R}} f\varphi' \mathrm{d}x &= -\int_0^1 x\varphi' \mathrm{d}x - \int_1^4 (-x+2)\varphi' \mathrm{d}x \\ &= -[x\varphi]_0^1 + \int_0^1 \varphi \mathrm{d}x - [(-x+2)\varphi]_1^4 - \int_1^4 \varphi \mathrm{d}x \\ &= \int_0^1 \varphi \mathrm{d}x - \int_1^4 \varphi \mathrm{d}x + 2\varphi(4) \\ &= \langle w + 2\delta_4, \varphi\rangle\end{aligned}$$

where $w = 1$ on $(0,1)$, $w = -1$ on $(1,4)$, and $w = 0$ elsewhere. Therefore $v = w - 2\delta_4 \notin L^2(\mathbb{R})$. Hence $f \notin H^1(\mathbb{R})$.

Exercice 2.6. Let $f : (0,1) \to \mathbb{R}$, $x \mapsto f(x) = x^\alpha$.

1. Show that $f \in L^1(0,1)$ if and only if $\alpha > -1$.
2. Show that $f \in H^1(0,1)$ if and only if $\alpha > 1/2$.
3. Show that $f \in H^{-1}(0,1)$ if and only if $\alpha > -3/2$.

Solution:

1. We have

$$f \in L^1(\Omega) \iff \int_0^1 x^\alpha \mathrm{d}x < \infty \iff \alpha > -1.$$

since

$$\begin{aligned}\left|\int_0^1 x^\alpha v \mathrm{d}x\right| &= \begin{cases} \frac{1}{|\alpha+1|}\left([x^{\alpha+1}]_0^1\right)^{1/2} & \text{if } \alpha \neq -1 \\ ([\ln x]_0^1)^{1/2} & \text{if } \alpha = -1 \end{cases} \\ &= \begin{cases} +\infty & \text{if } \alpha < -1 \\ \frac{1}{|\alpha+1|} & \text{if } \alpha > -1 \\ +\infty & \text{if } \alpha = -1 \end{cases}\end{aligned}$$

2. We have

$$f \in L^2(0,1) \iff \int_0^1 x^{2\alpha} \mathrm{d}x < \infty \iff 2\alpha > -1 \iff \alpha > -1/2.$$

Let v be the weak derivative (if it exists). Then, for every $\varphi \in \mathcal{D}(0,1)$:

$$\langle v, \varphi\rangle := -\int_0^1 f\varphi' \mathrm{d}x = -[f\varphi]_0^1 + \alpha\int_0^1 x^{\alpha-1}\varphi \mathrm{d}x = \int_0^1 x^{\alpha-1}\varphi \mathrm{d}x.$$

Hence $v(x) = \alpha x^{\alpha-1}$.

$$v \in L^2(0,1) \iff \int_0^1 x^{2(\alpha-1)} \mathrm{d}x < \infty \iff 2(\alpha-1) > -1 \iff \alpha > 1/2.$$

Thus $u \in H^1(0,1) \iff \alpha > 1/2$.

3. By the Riesz theorem, we have

$$\begin{aligned}
\boxed{f \in H^{-1}(0,1)} &\iff \exists ! w \in H_0^1(0,1) : \int_0^1 f\varphi \mathrm{d}x = \int_0^1 w'\varphi' \mathrm{d}x, \quad \forall \varphi \in \mathcal{D}(0,1) \\
&\iff \begin{cases} -w'' = f = x^\alpha \text{ in } \mathcal{D}'(0,1) \\ w \in H_0^1(0,1) \end{cases} \\
&\iff w \in H_0^1(0,1), \; w(x) = \begin{cases} -\frac{x^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{x}{(\alpha+1)(\alpha+2)}, \; \forall x \in]0,1[& \text{if } \alpha \neq -1 \\ x \ln x - x, \; \forall x \in]0,1[& \text{if } \alpha = -1 \end{cases} \\
&\iff \alpha + 2 > 1/2 \iff \boxed{\alpha > -3/2}.
\end{aligned}$$

Exercice 2.7 (Primitive of an L^1_{loc} function). Let u be a function in $L^1_{\mathrm{loc}}(I)$ where I is an interval in $\mathbb{R}$. Let $c \in I$ and define v by

$$v(x) = \int_c^x u(t)\mathrm{d}t, \quad \forall x \in I.$$

1. Show that the weak derivative of v equals u, and that $v \in \mathcal{C}(I)$ (v is called a primitive of u).

2. Deduce that if I is bounded and $u \in L^1(I)$, then $v \in W^{1,1}(I)$.

3. Deduce that if $u \in W^{1,1}(I)$, there exists a continuous function $\overline{u}$ on $\overline{I}$ such that $u = \overline{u}$ a.e. on I.

Solution:

1. Set $I =]a, b[$. Let $\varphi \in \mathcal{D}(I)$. We have

$$\begin{aligned}
\langle v', \varphi \rangle := -\langle v, \varphi' \rangle &= -\int_a^b v(x)\varphi'(x)\mathrm{d}x = -\int_a^b \left[\int_c^x u(t)\mathrm{d}t\right] \varphi'(x)\mathrm{d}x \\
&= -\int_a^c \left[\int_c^x u(t)\mathrm{d}t\right] \varphi'(x)\mathrm{d}x - \int_c^b \left[\int_c^x u(t)\mathrm{d}t\right] \varphi'(x)\mathrm{d}x \\
&= -\int_a^c \left[\int_t^a \varphi'(x)\mathrm{d}t\right] u(t)\mathrm{d}x - \int_c^b \left[\int_t^b \varphi'(x)\mathrm{d}t\right] u(t)\mathrm{d}x \text{ (by Fubini)} \\
&= -\int_a^c \left[\varphi(a) - \varphi(t)\right] u(t)\mathrm{d}x - \int_c^b \left[\varphi(b) - \varphi(t)\right] u(t)\mathrm{d}x \\
&= \int_a^b u(t)\varphi(t)\mathrm{d}t := \langle u, \varphi \rangle.
\end{aligned}$$

Thus $v' = u$.

For continuity, for any $x \in I$ and sufficiently small $h \in \mathbb{R}$:

$$|v(x+h) - v(x)| = \int_x^{x+h} u(t)\mathrm{d}t = \int_I \chi_{[x,x+h]}(t)\; u(t)\mathrm{d}t$$

(where χ denotes the characteristic function). We have

$$\forall h : |\chi_{[x,x+h]}(t)u(t)| \leq |u(t)|, \quad |\chi_{[x,x+h]}(t)\ u(t)| \to 0 \ \text{ a.e. } t \in I, \ \text{ as } h \to 0.$$

Hence, by Lebesgue's dominated convergence theorem,

$$|v(x+h) - v(x)| = \int_I \chi_{[x,x+h]}(t)\ u(t)\mathrm{d}t \longrightarrow 0 \text{ as } h \to 0.$$

Therefore v is continuous on I.

2. We have $v' = u \in L^1(I)$. It remains to show that $v \in L^1(I)$. We have

$$\int_I |v|\mathrm{d}x = \int_I \left| \int_c^x u(t)\mathrm{d}t \right| \mathrm{d}x \leq \int_I \int_I |u(t)|\mathrm{d}t\mathrm{d}x \leq \|u\|_{L^1(I)} \times |I| < \infty.$$

Thus $v \in L^1(I)$.

Consequently, the primitive of an $L^1(I)$ function belongs to $W^{1,1}(I)$.

3. Since $u \in W^{1,1}(I)$, then $u' \in L^1(I)$. By question 1), the function v defined on $\overline{I}$ by

$$v(x) = \int_c^x u'(t)\mathrm{d}t, \ \forall x \in \overline{I},$$

belongs to $\mathcal{C}(\overline{I})$, and $v' = u'$ in $\mathcal{D}'(I)$. (Note that we can take $x \in \overline{I}$ because $u' \in L^1(I)$, unlike in question 1 where u was only in L^1_{loc}). Thus $u = v + C$ a.e. on I. It therefore suffices to choose $\overline{u} = v + C \in \mathcal{C}(\overline{I})$. This yields

$$u(x) = \overline{u}(x) = \int_c^x u'(t)\mathrm{d}t + C \text{ a.e. } x \in I.$$

Exercice 2.8. Show the following continuous embedding

$$H^1(0,1) \hookrightarrow C([0,1]).$$

Solution: From Exercise (2.7), question 3, there exists $\overline{u} \in \mathcal{C}([0,1])$ such that $u = \overline{u}$ a.e. and $\overline{u}(x) = \int_y^x u'(t)\mathrm{d}t + c$, for all $x, y \in [0,1]$, and clearly $c = u(y)$. Then, we have

$$\text{a.e. } x \in]0,1[\colon u(x) = \int_y^x u'(t)\mathrm{d}t + \overline{u}(y).$$

Integrating with respect to y over $(0,1)$, we obtain

$$\text{a.e. } x \in]0,1[\colon |u(x)| = \left| \int_y^x u'(t)\mathrm{d}t + \int_0^1 \overline{u}(y)\mathrm{d}y \right| \leq \int_0^1 |u'|\mathrm{d}t + \int_0^1 |u|\mathrm{d}x.$$

Applying the Cauchy-Schwarz inequality,

$$\text{a.e. } x \in]0,1[\colon |u(x)| \leq \|u\|_{L^2(0,1)} + \|u'\|_{L^2(0,1)} = \|u\|_{H^1(0,1)}.$$

Hence

$$\|u\|_{L^\infty(0,1)} \leq \|u\|_{H^1(0,1)},$$

which yields the continuous embedding $H^1(0,1) \hookrightarrow C([0,1])$.

(The inclusion $H^1(0,1) \subset C([0,1])$ is understood in the sense that if $u \in H^1(I)$, there exists a function $\overline{u} \in \mathcal{C}(\overline{I})$ such that $u = \overline{u}$ a.e. on I).

Exercice 2.9 (Extension in $W^{1,1}$). Let $p \geq 1$, $u \in W^{1,p}(\mathbb{R}_+^*)$ and define Pu on $\mathbb{R}$ by

$$Pu(x) = \begin{cases} u(x) & \text{if } x > 0 \\ u(-x) & \text{if } x < 0 \end{cases}$$

1. Show that $P(u) \in W^{1,p}(\mathbb{R})$.
2. Show that $P : W^{1,p}(\mathbb{R}_+^*) \longrightarrow W^{1,p}(\mathbb{R})$ is linear and continuous.

Solution:

1. We have

$$\begin{aligned}\int_{\mathbb{R}} |Pu|^p \mathrm{d}x = \int_0^\infty |u(x)|^p \mathrm{d}x + \int_{-\infty}^0 |u(-x)|^p \mathrm{d}x &= \int_0^\infty |u(x)|^p \mathrm{d}x + \int_0^\infty |u(x)|^p \mathrm{d}x \\ &= 2\|u\|^p_{L^p(\mathbb{R}_+)} < +\infty.\end{aligned}$$

Thus $Pu \in L^p(\mathbb{R})$. Let us show that the distributional derivative of Pu belongs to $L^p(\mathbb{R})$. From question 4 of the previous exercise, we have

$$u(x) = \int_0^x u'(t)\mathrm{d}t + c_0, \text{ a.e. } x > 0,$$

and

$$u(-x) = \int_0^{-x} u'(t)\mathrm{d}t + c_0 = \int_0^x -u'(-t)\mathrm{d}t + c_0, \text{ a.e. } x < 0.$$

Thus $Pu(x) = \int_0^x v(x)\mathrm{d}x + c_0$ a.e. $x \in \mathbb{R}$, where

$$v(x) = \begin{cases} u'(x) & \text{if } x > 0 \\ -u'(-x) & \text{if } x < 0 \end{cases}$$

This gives $(Pu)' = v$ in $\mathcal{D}'(\mathbb{R})$. One can easily check that $v \in L^p(\mathbb{R})$. Hence $(Pu)' = v \in L^p(\mathbb{R})$. Consequently $Pu \in W^{1,p}(\mathbb{R})$.

2. From question 1), we have $\|Pu\|^p_{L^p(\mathbb{R})} = 2\|u\|^p_{L^p(\mathbb{R}_+)}$ and similarly we show that

$$\|(Pu)'\|^p_{L^p(\mathbb{R})} = \|v\|^p_{L^p(\mathbb{R})} = 2\|u'\|^p_{L^p(\mathbb{R}_+)}.$$

Therefore,

$$\|Pu\|_{W^{1,p}(\mathbb{R})} = \left(\|(Pu)'\|^p_{L^p(\mathbb{R})} + \|Pu\|^p_{L^p(\mathbb{R})}\right)^{1/p} = 2^{1/p}\|u\|_{W^{1,p}(\mathbb{R}_+)}.$$

This yields the continuity of P.

Exercice 2.10. Let $u \in H^1(\Omega)$. Show the equivalence between the following statements

1. $u \in (H_0^1(\Omega))^\perp$.

2. $-\Delta u + u = 0$ in $\mathcal{D}'(\Omega)$.

Solution: We have

$$
\begin{aligned}
u \in (H_0^1(\Omega))^\perp \quad & \overset{\text{def}}{\iff} \quad (u,v)_{H^1(\Omega)} = 0,\ \forall v \in H_0^1(\Omega) \\
& \iff \quad \left\{ \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = 0,\ \forall v \in H_0^1(\Omega) \right. \\
& \overset{\text{By density}}{\iff} \quad \left. \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = 0 \quad \forall v \in \mathcal{D}(\Omega) \right\} \\
& \iff \quad \langle -\Delta u + u, v\rangle_{\mathcal{D}'\times\mathcal{D}} = 0,\ \forall v \in \mathcal{D}(\Omega) \\
& \iff \quad -\Delta u + u = 0,\ \text{ in } \mathcal{D}'(\Omega).
\end{aligned}
$$

Exercice 2.11. Let I be a bounded interval, $u \in \mathcal{D}(\Omega)$

1. Show that

$$\int_{I\times I} (u(x)-u(y))^2 \mathrm{d}x\mathrm{d}y \leq |I|^3 \|u'\|^2_{L^2(I)}, \quad \text{where } |I| = \mathrm{mes}(I).$$

2. Deduce the following inequality:

$$\int_I u(x)^2 \mathrm{d}x \leq \frac{|I|^2}{2} \int_I |u'(t)|^2 \mathrm{d}t + \frac{1}{|I|}\left(\int_I |u(x)|\mathrm{d}x\right)^2.$$

3. Does this inequality hold for u in $H^1(I)$?

Solution:

1. We have

$$
\begin{aligned}
(u(x)-u(y))^2 = \left|\int_y^x u'(t)\mathrm{d}t\right|^2 \quad & \leq \quad \left(\int_I |u'(t)|\mathrm{d}t\right)^2 \\
& \overset{\text{Cauchy-Schwarz}}{\leq} \quad \int_I 1^2 \mathrm{d}t \int_I u'^2 \mathrm{d}t \\
& = \quad |I|\ \|u'\|^2_{L^2(I)}.
\end{aligned}
$$

Integrating over I with respect to x and y, we obtain

$$
\begin{aligned}
\int_{I\times I} (u(x)-u(y))^2 \mathrm{d}x\mathrm{d}y \quad & \leq \quad \int_{I\times I} |I|\ \|u'\|^2_{L^2(I)} \mathrm{d}x\mathrm{d}y \\
& = \quad |I|\ \|u'\|^2_{L^2(I)} \int_{I\times I} \mathrm{d}x\mathrm{d}y = |I|^3\ \|u'\|^2_{L^2(I)}.
\end{aligned}
$$

2. We have

$$\begin{aligned}\int_{I\times I}(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y &= \int_{I\times I}(u(x)^2+u(y)^2-2u(x)u(y))\mathrm{d}x\mathrm{d}y\\ &= \int_I\left(\int_I u(x)^2\mathrm{d}x\right)\mathrm{d}y+\int_I\left(\int_I u(y)^2\mathrm{d}y\right)\mathrm{d}x-2\int_I u(x)\mathrm{d}x\int_I u(y)\mathrm{d}y\\ &= 2|I|\int_I u(x)^2\mathrm{d}x-2\left(\int_I u(x)\mathrm{d}x\right)^2.\end{aligned}$$

Substituting into the inequality from question 1), we obtain

$$2|I|\int_I u(x)^2\mathrm{d}x \le |I|^3\ \|u'\|^2_{L^2(I)}+2\left(\int_I u(x)\mathrm{d}x\right)^2.$$

Dividing by $|I|$ yields the result.

3. Yes. Indeed, by the density theorem (see Course 2, Chapter 2), the space $\mathcal{C}^\infty([0,1])$ is dense in $H^1(0,1)$. That is, for every $u\in H^1(0,1)$, there exists a sequence $(u_n)_n$ in $\mathcal{C}^\infty([0,1])$ such that

$$u_n\to u \ \text{ in } H^1(0,1).$$

This implies that

$$u_n\to u \ \text{ in } L^2(0,1), \quad \text{and } u_n'\to u' \ \text{ in } L^2(0,1).$$

From question 2), since $u_n\in\mathcal{C}^\infty([0,1])$, it satisfies the inequality

$$\int_I u_n^2\mathrm{d}x\le\frac{|I|^2}{2}\int_I|u_n'|^2\mathrm{d}t+\frac{1}{|I|}\left(\int_I|u_n|\mathrm{d}x\right)^2.$$

Passing to the limit, we obtain the inequality for $u\in H^1(0,1)$.

Exercice 2.12. Let Ω be a bounded connected open set in $\mathbb{R}^N$. Prove Poincaré's inequality

$$\exists C>0,\forall u\in H_0^1(\Omega):\|u\|_{L^2(\Omega)}\le C\|\nabla u\|_{(L^2(\Omega))^N}. \tag{2.24}$$

Solution: We have

$$(2.24)\iff\exists C>0,\forall u\in H_0^1(\Omega):\frac{\|u\|_{L^2(\Omega)}}{\|\nabla u\|_{(L^2(\Omega))^N}}\le C.$$

We argue by contradiction. Suppose that there exists a sequence $(u_n)_n\subset H_0^1(\Omega)$ such that

$$\forall n:\frac{\|u_n\|_{L^2(\Omega)}}{\|\nabla u_n\|_{(L^2(\Omega))^N}}\ge n. \tag{2.25}$$

Set $v_n:=\dfrac{u_n}{\|u_n\|_{L^2(\Omega)}}$. Then we have

$$\|v_n\|_{L^2(\Omega)}=\frac{\|u_n\|_{L^2(\Omega)}}{\|u_n\|_{L^2(\Omega)}}=1,\quad \|\nabla v_n\|_{(L^2(\Omega))^N}:=\frac{\|\nabla u_n\|_{L^2(\Omega)}}{\|u_n\|_{L^2(\Omega)}}\le\frac{1}{n},\ \forall n. \tag{2.26}$$

Moreover,

$$\|v_n\|_{H^1(\Omega)} := \|v_n\|_{L^2(\Omega)} + \|\nabla v_n\|_{L^2(\Omega)} \leq 1 + \frac{1}{n} \leq 2, \ \forall n.$$

Thus the sequence $(v_n)_n$ is bounded in $H_0^1(\Omega)$, which embeds compactly into $L^2(\Omega)$. Hence we can extract a subsequence of $(v_n)_n$, still denoted $(v_n)_n$, such that

$$v_n \to v \ \text{ in } L^2(\Omega).$$

On the other hand, from (2.26), we have $\|v\|_{L^2(\Omega)} = \lim_{n\to\infty} \|v_n\|_{L^2(\Omega)} = 1$ and $\nabla v_n \to 0$ in $L^2(\Omega)$. Therefore,

$$v_n \to v \ \text{ in } \mathcal{D}'(\Omega) \text{ and } \nabla v_n \to \nabla v = 0 \text{ in } \mathcal{D}'(\Omega)^N.$$

Since Ω is connected, it follows that v is constant in Ω and this constant is nonzero because $\|v\|_{L^2(\Omega)} = 1$. But $v \in H_0^1(\Omega)$, so its trace satisfies $Tv := 0$. Since v is constant in Ω, we have $v = \overline{v}$ a.e. in Ω, where $\overline{v} = C$ in $\overline{\Omega}$. Hence

$$Tv = T\overline{v} := \overline{v}_{|\partial\Omega} = C \text{ because } \overline{v} \in \mathcal{C}(\overline{\Omega}).$$

Thus $C = 0$, which is a contradiction.

Exercice 2.13. Let f be a function continuous on $\overline{\Omega}$, piecewise C^1, with bounded support contained in Ω. Show that $f \in H^1(\Omega)$.

Solution: Since f is continuous on $\overline{\Omega}$ and has compact support, it belongs to $L^2(\Omega)$. Since f is piecewise C^1 with bounded support, by definition there exists a finite family of pairwise disjoint open sets $\{\Omega_i : i = 1, ..., m\}$ such that $\cup_{i=1}^m \overline{\Omega}_i = \operatorname{supp} f \subset \Omega$ and $f = f_i$ is of class $\mathcal{C}^1$ on each $\overline{\Omega_i}$. For $i \neq j$, set $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, and let $n^i(x)$ be the unit outward normal vector at $x \in \partial\Omega_i$. We have

$$\forall x \in \Gamma_{ij} : n^i(x) = -n^j(x).$$

Thus, setting $v := \partial_k f$ (in the sense of distributions), for any $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} \langle v, \varphi \rangle &:= -\int_\Omega f \ \partial_k \varphi \mathrm{d}x = -\sum_{i=1}^m \int_{\Omega_i} f \ \partial_k \varphi \mathrm{d}x \\ &= \sum_{i=1}^m \left(-\int_{\partial\Omega_i} f v n_k^i dS + \int_{\Omega_i} \partial_k f \ \varphi \mathrm{d}x \right) \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{i=1}^m \int_{\partial\Omega_i} f v n_k^i dS &= \sum_{i,j=1, i\neq j}^m \int_{\Gamma_{ij}} f v n_k^i dS \\ &= \sum_{1\leq i<j\leq m} \int_{\Gamma_{ij}} f v (n_k^i + n_k^j) dS = 0. \end{aligned}$$

Hence

$$\langle v, \varphi \rangle = \sum_{i=1}^{m} \int_{\Omega_i} \partial_k f \ \varphi \mathrm{d}x.$$

Therefore $v(x) = \partial_k f_{|\Omega_i}(x)$ for all $x \in \Omega_i$, $i = 1, ..., m$. Furthermore,

$$\int_{\Omega} v^2 \mathrm{d}x = \sum_{i=1}^{m} \int_{\Omega_i} v^2 \mathrm{d}x = \sum_{i=1}^{m} \int_{\Omega_i} (\partial_k f_{|\Omega_i})^2 \mathrm{d}x < \infty.$$

Thus $v \in L^2(\Omega)$. Consequently $f \in H^1(\Omega)$.

2.10 Additional Exercises on Sobolev Spaces

Exercice 2.14. Let $\Omega = (-1, 1) \subset \mathbb{R}$. Consider the function

$$f(x) = \begin{cases} x^2, & x \in [0, 1) \\ -x^2, & x \in (-1, 0) \end{cases}$$

1. Determine whether $f \in L^p(-1, 1)$ for $p \in [1, \infty]$.
2. Compute the weak derivative of f (if it exists).
3. Is $f \in W^{1,p}(-1, 1)$? For which p?

Solution:

1. The function f is continuous on $(-1, 1)$ and bounded ($|f(x)| \leq 1$). Hence $f \in L^\infty(-1, 1) \subset L^p(-1, 1)$ for all $p \in [1, \infty]$.
2. Let us compute the weak derivative $g = f'$. For any $\varphi \in \mathcal{D}(-1, 1)$,

$$\langle f', \varphi \rangle = -\int_{-1}^{1} f(x)\varphi'(x)\, dx.$$

Split the integral:

$$\begin{aligned} -\int_{-1}^{1} f(x)\varphi'(x)dx &= -\int_{-1}^{0} (-x^2)\varphi'(x)dx - \int_{0}^{1} x^2\varphi'(x)dx \\ &= \int_{-1}^{0} x^2\varphi'(x)dx - \int_{0}^{1} x^2\varphi'(x)dx. \end{aligned}$$

Integrate by parts:

$$\begin{aligned} \int_{-1}^{0} x^2\varphi'(x)dx &= [x^2\varphi(x)]_{-1}^{0} - \int_{-1}^{0} 2x\varphi(x)dx = -\varphi(-1) - 2\int_{-1}^{0} x\varphi(x)dx, \\ \int_{0}^{1} x^2\varphi'(x)dx &= [x^2\varphi(x)]_{0}^{1} - \int_{0}^{1} 2x\varphi(x)dx = \varphi(1) - 2\int_{0}^{1} x\varphi(x)dx. \end{aligned}$$

Thus

$$\begin{aligned}\langle f', \varphi \rangle &= \left(-\varphi(-1) - 2\int_{-1}^{0} x\varphi(x)dx\right) - \left(\varphi(1) - 2\int_{0}^{1} x\varphi(x)dx\right) \\ &= -\varphi(-1) - \varphi(1) - 2\int_{-1}^{0} x\varphi(x)dx + 2\int_{0}^{1} x\varphi(x)dx.\end{aligned}$$

Since φ has compact support in $(-1, 1)$, $\varphi(-1) = \varphi(1) = 0$. Therefore

$$\langle f', \varphi \rangle = -2\int_{-1}^{0} x\varphi(x)dx + 2\int_{0}^{1} x\varphi(x)dx = \int_{-1}^{1} g(x)\varphi(x)dx,$$

where

$$g(x) = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 = 2|x|. \\ 0, & x = 0 \end{cases}$$

Thus the weak derivative exists and is $f' = 2|x|$.

3. We have $f(x) = |x|x$ (since $|x|x = x^2$ for $x > 0$ and $|x|x = -x^2$ for $x < 0$). The weak derivative is $f' = 2|x|$. Now:

$$f \in L^p(-1, 1) \text{ for all } p \in [1, \infty] \quad (\text{since } f \text{ is bounded}),$$

$$f' \in L^p(-1, 1) \iff \int_{-1}^{1} |2x|^p dx < \infty \iff \int_{0}^{1} x^p dx < \infty \iff p > -1,$$

which is always true for $p \geq 1$. So $f' \in L^p(-1, 1)$ for all $p \in [1, \infty]$. Hence $f \in W^{1,p}(-1, 1)$ for all $1 \leq p \leq \infty$.

Exercice 2.15. Let $\Omega = B(0, 1) \subset \mathbb{R}^N$ be the unit ball. Consider the function

$$u(x) = |x|^\alpha, \quad \alpha \in \mathbb{R}.$$

1. For which α does u belong to $L^p(\Omega)$?

2. For which α does u belong to $W^{1,p}(\Omega)$?

3. For which α does u belong to $W_0^{1,p}(\Omega)$?

Solution:

We use polar coordinates: $x = r\omega$ with $r = |x| \in (0, 1)$ and $\omega \in S^{N-1}$. Then $dx = r^{N-1}drd\omega$, and $\int_{S^{N-1}} d\omega = N\omega_N$ where ω_N is the volume of the unit ball.

1. $u(x) = r^\alpha$. Then

$$\int_\Omega |u|^p dx = \int_0^1 r^{\alpha p} r^{N-1} dr \int_{S^{N-1}} d\omega = N\omega_N \int_0^1 r^{\alpha p + N - 1} dr.$$

The integral converges iff $\alpha p + N - 1 > -1$, i.e., $\alpha p + N > 0$. Thus

$$u \in L^p(\Omega) \iff \alpha > -\frac{N}{p}.$$

For $p = \infty$, $u \in L^\infty(\Omega)$ iff $\alpha \geq 0$ (since $|x|^\alpha$ is bounded near 0 only when $\alpha \geq 0$).

2. The partial derivatives: for $x \neq 0$,

$$\frac{\partial u}{\partial x_i}(x) = \alpha |x|^{\alpha-2} x_i.$$

Hence $|\nabla u(x)| = |\alpha||x|^{\alpha-1}$. We need $u \in L^p(\Omega)$ and $\nabla u \in (L^p(\Omega))^N$.

- $u \in L^p(\Omega) \iff \alpha > -\frac{N}{p}$.
- $\nabla u \in (L^p(\Omega))^N \iff \int_0^1 (|\alpha| r^{\alpha-1})^p r^{N-1} dr < \infty \iff \int_0^1 r^{p(\alpha-1)+N-1} dr < \infty \iff p(\alpha-1)+N > 0 \iff \alpha > 1 - \frac{N}{p}$.

Thus $u \in W^{1,p}(\Omega)$ iff $\alpha > \max\left(-\frac{N}{p}, 1 - \frac{N}{p}\right)$.

Since $1 - \frac{N}{p} > -\frac{N}{p}$ for all $p > 0$, the condition simplifies to:

$$\boxed{u \in W^{1,p}(\Omega) \iff \alpha > 1 - \frac{N}{p}.}$$

For $p = \infty$, $u \in W^{1,\infty}(\Omega)$ iff $\alpha \geq 1$ (since ∇u must be bounded).

3. $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. For u to be in $W_0^{1,p}(\Omega)$, we need in addition that the trace of u on $\partial\Omega$ is zero. On $\partial\Omega$, $|x| = 1$, so $u(x) = 1^\alpha = 1 \neq 0$ for any α. However, functions in $W_0^{1,p}(\Omega)$ have zero trace. Therefore $u \notin W_0^{1,p}(\Omega)$ for any α (unless we consider a modified function that vanishes on the boundary). If we consider instead $\Omega = B(0,R)$ and $u(x) = |x|^\alpha - R^\alpha$, then u vanishes on $\partial\Omega$ and may belong to $W_0^{1,p}(\Omega)$ under the same condition $\alpha > 1 - N/p$.

Exercice 2.16. Let $\Omega = (0,1) \subset \mathbb{R}$. Define the sequence

$$u_n(x) = \sin(n\pi x).$$

1. Show that (u_n) is bounded in $H_0^1(0,1)$.
2. Does (u_n) converge strongly in $L^2(0,1)$? Does it converge weakly?
3. Does (u_n) converge in $H_0^1(0,1)$?

Solution:

1. We have $u_n(0) = u_n(1) = 0$, so $u_n \in H_0^1(0,1)$. Compute:

$$\begin{aligned} \|u_n\|_{L^2}^2 &= \int_0^1 \sin^2(n\pi x) dx = \frac{1}{2}, \\ u_n'(x) &= n\pi \cos(n\pi x), \\ \|u_n'\|_{L^2}^2 &= \int_0^1 n^2\pi^2 \cos^2(n\pi x) dx = \frac{n^2\pi^2}{2}. \end{aligned}$$

Hence

$$\|u_n\|_{H^1}^2 = \|u_n\|_{L^2}^2 + \|u_n'\|_{L^2}^2 = \frac{1}{2} + \frac{n^2\pi^2}{2} \to \infty.$$

So (u_n) is NOT bounded in $H_0^1(0,1)$! The sequence is unbounded because the derivative grows like n.

If we instead consider $v_n(x) = \frac{1}{n}\sin(n\pi x)$, then

$$\|v_n\|_{H^1}^2 = \frac{1}{2n^2} + \frac{\pi^2}{2} \leq C,$$

so (v_n) is bounded in $H_0^1(0,1)$.

Let us work with $v_n(x) = \frac{1}{n}\sin(n\pi x)$. Then:

$$\|v_n\|_{L^2}^2 = \frac{1}{2n^2}, \quad \|v_n'\|_{L^2}^2 = \frac{\pi^2}{2}.$$

2. **Strong convergence in L^2:**

$$\|v_n\|_{L^2} = \frac{1}{\sqrt{2}n} \to 0.$$

So $v_n \to 0$ strongly in $L^2(0,1)$. Strong convergence implies weak convergence, so $v_n \rightharpoonup 0$ weakly in L^2 as well.

3. **Convergence in $H_0^1(0,1)$:** We have

$$\|v_n - 0\|_{H^1}^2 = \|v_n\|_{L^2}^2 + \|v_n'\|_{L^2}^2 = \frac{1}{2n^2} + \frac{\pi^2}{2} \to \frac{\pi^2}{2} \neq 0.$$

Thus v_n does NOT converge strongly to 0 in $H_0^1(0,1)$. However, v_n' converges weakly in L^2? Let's check: for any $\varphi \in L^2(0,1)$,

$$\langle v_n', \varphi \rangle = \int_0^1 \pi\cos(n\pi x)\varphi(x)dx.$$

By the Riemann-Lebesgue lemma, $\int_0^1 \cos(n\pi x)\varphi(x)dx \to 0$ for $\varphi \in L^2(0,1)$. Hence $v_n' \rightharpoonup 0$ weakly in $L^2(0,1)$. Therefore $v_n \rightharpoonup 0$ weakly in $H_0^1(0,1)$ but not strongly.

Exercice 2.17. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with C^1 boundary. Prove the following version of Poincaré's inequality:

$$\exists C > 0, \quad \forall u \in H_0^1(\Omega) : \|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{(L^2(\Omega))^N}.$$

Solution:

We argue by contradiction. Suppose the inequality is false. Then for every $n \in \mathbb{N}$, there exists $u_n \in H_0^1(\Omega)$ such that

$$\|u_n\|_{L^2} > n\|\nabla u_n\|_{L^2}.$$

Set $v_n = \frac{u_n}{\|u_n\|_{L^2}}$. Then $\|v_n\|_{L^2} = 1$ and

$$\|\nabla v_n\|_{L^2} = \frac{\|\nabla u_n\|_{L^2}}{\|u_n\|_{L^2}} < \frac{1}{n}.$$

Hence $\|\nabla v_n\|_{L^2} \to 0$.

The sequence (v_n) is bounded in $H_0^1(\Omega)$ because

$$\|v_n\|_{H^1}^2 = \|v_n\|_{L^2}^2 + \|\nabla v_n\|_{L^2}^2 = 1 + \|\nabla v_n\|_{L^2}^2 \leq 2.$$

By the Rellich-Kondrachov compact embedding theorem, $H_0^1(\Omega)$ embeds compactly into $L^2(\Omega)$. Therefore there exists a subsequence (v_{n_k}) and $v \in L^2(\Omega)$ such that

$$v_{n_k} \to v \quad \text{strongly in } L^2(\Omega).$$

Since $\|v_{n_k}\|_{L^2} = 1$, we have $\|v\|_{L^2} = 1$, so $v \neq 0$.

On the other hand, since $\nabla v_{n_k} \to 0$ in $L^2(\Omega)^N$, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega v \frac{\partial \varphi}{\partial x_i} dx = \lim_{k\to\infty} \int_\Omega v_{n_k} \frac{\partial \varphi}{\partial x_i} dx = -\lim_{k\to\infty} \int_\Omega \frac{\partial v_{n_k}}{\partial x_i} \varphi dx = 0.$$

Thus $\nabla v = 0$ in the sense of distributions, so v is constant a.e. on Ω. Since $v \in H_0^1(\Omega)$, its trace is zero, and the constant must be zero. This contradicts $\|v\|_{L^2} = 1$. Hence the inequality holds.

Exercice 2.18. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. Consider the function

$$u(x,y) = \frac{1}{\sqrt{x^2+y^2}} \quad \text{for } (x,y) \neq (0,0).$$

1. Determine whether $u \in L^2(\Omega)$.
2. Determine whether $u \in H^1(\Omega)$.

Solution:

Use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, with $r \in (0, \sqrt{2})$ (since the maximum distance from the origin in the square is $\sqrt{2}$). The area element is $dxdy = rdrd\theta$.

1. $$\int_\Omega |u|^2 dxdy = \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{1}{r^2} \cdot rdrd\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \frac{1}{r} dr = 2\pi \left[\ln r\right]_0^{\sqrt{2}} = +\infty.$$

   Thus $u \notin L^2(\Omega)$ because the integral diverges near $r = 0$. In fact, $u \in L^p(\Omega)$ for $p < 2$ (since $\int_0 r^{1-p} dr$ converges when $1 - p > -1$, i.e., $p < 2$).

2. Since $u \notin L^2(\Omega)$, it cannot belong to $H^1(\Omega)$ (which requires $u \in L^2(\Omega)$). So $u \notin H^1(\Omega)$.

   If we consider a modified function $u(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ on $\Omega \setminus B(0,\varepsilon)$ and extend smoothly inside the ball, then the function would be in H^1 only for $N \geq 3$ (since in $\mathbb{R}^N$, $|x|^\alpha \in H^1$ when $\alpha > 1 - N/2$; here $\alpha = -1$ and $N = 2$ gives $\alpha = -1 > 0$? Actually $1 - N/2 = 0$, and $-1 \not> 0$, so not in H^1).

Exercice 2.19. Let $I = (0,1)$. Consider the space

$$V = \{u \in H^1(I) : u(0) = 0\}.$$

1. Show that V is a Hilbert space.

2. Show that the norm $\|u\|_V = \|u'\|_{L^2}$ is equivalent to the standard H^1 norm on V.

3. Apply this to prove the Poincaré inequality for functions vanishing at $x = 0$:

$$\int_0^1 u(x)^2 dx \leq \int_0^1 u'(x)^2 dx, \quad \forall u \in V.$$

Solution:

1. V is a closed subspace of $H^1(I)$ because the trace operator $T : H^1(I) \to \mathbb{R}$ given by $Tu = u(0)$ is continuous, and $\{0\}$ is closed in $\mathbb{R}$, so $V = T^{-1}(\{0\})$ is closed. Closed subspaces of Hilbert spaces are Hilbert spaces. Hence V is a Hilbert space with the inner product $(u,v)_{H^1} = \int_0^1 (uv + u'v')dx$.

2. For any $u \in V$, by the fundamental theorem of calculus,

$$u(x) = \int_0^x u'(t)dt.$$

Using Cauchy-Schwarz,

$$|u(x)|^2 \leq \left(\int_0^x |u'(t)|dt\right)^2 \leq x \int_0^x u'(t)^2 dt \leq \int_0^1 u'(t)^2 dt.$$

Integrating over $x \in (0,1)$,

$$\int_0^1 u(x)^2 dx \leq \int_0^1 \left(\int_0^1 u'(t)^2 dt\right) dx = \int_0^1 u'(t)^2 dt.$$

Thus

$$\|u\|_{H^1}^2 = \int_0^1 u^2 dx + \int_0^1 u'^2 dx \leq 2 \int_0^1 u'^2 dx = 2\|u\|_V^2.$$

Conversely, $\|u\|_V^2 = \int_0^1 u'^2 dx \leq \|u\|_{H^1}^2$. Hence the norms are equivalent.

3. From the inequality above, we have directly:

$$\int_0^1 u(x)^2 dx \leq \int_0^1 u'(x)^2 dx, \quad \forall u \in V.$$

This is the Poincaré inequality for functions vanishing at $x = 0$. The optimal constant is $C = 1$ (and it is achieved by $u(x) = x$? Actually for $u(x) = x$, $\int_0^1 x^2 dx = 1/3$, $\int_0^1 1^2 dx = 1$, so $1/3 \leq 1$ holds; the constant 1 is not sharp; the sharp constant is $4/\pi^2$? Let's check: For $u(x) = \sin(\pi x/2)$, $u(0) = 0$, $\int_0^1 \sin^2(\pi x/2)dx = 1/2$, $\int_0^1 (\pi/2)^2 \cos^2(\pi x/2)dx = \pi^2/8 \approx 1.23$, so $1/2 \leq 1.23$; the best constant is actually $4/\pi^2 \approx 0.405$ for the inequality $\|u\|_{L^2}^2 \leq C\|u'\|_{L^2}^2$ with $u(0) = 0$. The inequality we proved gives $C = 1$, which is not optimal but sufficient.)

Exercice 2.20. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Show that the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact but not surjective.

Solution:

1. **Compactness:** The Rellich-Kondrachov theorem states that for a bounded domain Ω with Lipschitz boundary, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $1 \leq p < 2^*$ where $2^* = \frac{2N}{N-2}$ (if $N > 2$; for $N = 2$, $p < \infty$; for $N = 1$, $H_0^1 \hookrightarrow C^0$ compactly). In particular, for $p = 2$, the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. This means that any bounded sequence in $H_0^1(\Omega)$ has a subsequence that converges strongly in $L^2(\Omega)$.

2. **Non-surjectivity:** The embedding is not surjective because $H_0^1(\Omega)$ is a proper subspace of $L^2(\Omega)$. For example, the constant function $u(x) = 1$ belongs to $L^2(\Omega)$ (since Ω is bounded) but does not belong to $H_0^1(\Omega)$ because its trace on $\partial\Omega$ is $1 \neq 0$ (and it is not in H_0^1). More generally, $H_0^1(\Omega)$ consists of functions that vanish on the boundary in the trace sense, while $L^2(\Omega)$ contains many functions that do not vanish on the boundary (and the trace is not even defined for arbitrary L^2 functions). Thus the inclusion $H_0^1(\Omega) \subset L^2(\Omega)$ is strict, so the embedding cannot be surjective.

Exercice 2.21. Let $u \in W^{1,1}(0,1)$. Prove that u has a continuous representative. More precisely, show that there exists $\tilde{u} \in C([0,1])$ such that $u = \tilde{u}$ almost everywhere.

Solution:

Let $u \in W^{1,1}(0,1)$. Then $u' \in L^1(0,1)$. For any $x, y \in (0,1)$ with $x < y$, we have

$$u(y) - u(x) = \int_x^y u'(t)dt \quad \text{in the sense of distributions, and this holds a.e.}$$

More precisely, there exists a representative of u (still denoted u) such that for all $x, y \in [0,1]$ with $x < y$,

$$u(y) - u(x) = \int_x^y u'(t)dt.$$

This is because u is absolutely continuous. In fact, $W^{1,1}(0,1)$ is exactly the space of absolutely continuous functions on $[0,1]$ whose derivative is in $L^1(0,1)$.

To prove this, fix $c \in (0,1)$ and define

$$v(x) = \int_c^x u'(t)dt.$$

Then v is continuous (in fact absolutely continuous) and $v' = u'$ in $\mathcal{D}'(0,1)$. Hence $u - v$ is constant a.e., so there exists a constant C such that $u = v + C$ a.e. Then $\tilde{u} = v + C$ is continuous on $[0,1]$ and equals u a.e.

Thus every $W^{1,1}(0,1)$ function has a continuous representative. Moreover, this representative is absolutely continuous.

Exercice 2.22. Let $\Omega = B(0,1) \subset \mathbb{R}^3$. Consider the function

$$u(x) = \frac{1}{|x|}.$$

1. Determine whether $u \in L^2(\Omega)$.
2. Determine whether $u \in H^1(\Omega)$.
3. Determine whether $u \in W^{1,p}(\Omega)$ for other p.

Solution:
In $\mathbb{R}^3$, $dx = r^2 \sin\theta dr d\theta d\varphi = r^2 dr d\omega$.

1.
$$\int_\Omega |u|^2 dx = \int_0^1 \frac{1}{r^2} \cdot r^2 dr \int_{S^2} d\omega = \int_0^1 dr \cdot 4\pi = 4\pi < \infty.$$
Thus $u \in L^2(\Omega)$.

2. Compute $\nabla u(x) = -\frac{x}{|x|^3}$, so $|\nabla u(x)| = \frac{1}{|x|^2}$. Then
$$\int_\Omega |\nabla u|^2 dx = \int_0^1 \frac{1}{r^4} \cdot r^2 dr \int_{S^2} d\omega = 4\pi \int_0^1 \frac{1}{r^2} dr = +\infty.$$
Thus $\nabla u \notin L^2(\Omega)$, so $u \notin H^1(\Omega)$.

3. For general p:
$$u \in L^p(\Omega) \iff \int_0^1 \frac{1}{r^p} r^2 dr = \int_0^1 r^{2-p} dr < \infty \iff 2 - p > -1 \iff p < 3.$$
$$\nabla u \in L^p(\Omega) \iff \int_0^1 \frac{1}{r^{2p}} r^2 dr = \int_0^1 r^{2-2p} dr < \infty \iff 2-2p > -1 \iff 2p < 3 \iff p < \frac{3}{2}.$$
Thus $u \in W^{1,p}(\Omega)$ iff $p < 3/2$ (and $p \geq 1$). For $p = 2$, $u \notin H^1(\Omega)$ as seen.

Exercice 2.23. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Show that the norm
$$\|u\|_{W_0^{1,p}} = \|\nabla u\|_{L^p}$$
is equivalent to the standard norm $\|u\|_{W^{1,p}} = (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{1/p}$ on $W_0^{1,p}(\Omega)$.

Solution:
We need to show that there exist constants $c, C > 0$ such that
$$c\|u\|_{W^{1,p}} \leq \|\nabla u\|_{L^p} \leq C\|u\|_{W^{1,p}}, \quad \forall u \in W_0^{1,p}(\Omega).$$
The right inequality is obvious: $\|\nabla u\|_{L^p} \leq (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{1/p} = \|u\|_{W^{1,p}}$, so $C = 1$ works.

For the left inequality, we use Poincaré's inequality: there exists $C_P > 0$ such that
$$\|u\|_{L^p} \leq C_P \|\nabla u\|_{L^p}, \quad \forall u \in W_0^{1,p}(\Omega).$$
Then
$$\|u\|_{W^{1,p}}^p = \|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p \leq (C_P^p + 1)\|\nabla u\|_{L^p}^p.$$

Taking the p-th root,

$$\|u\|_{W^{1,p}} \leq (C_P^p + 1)^{1/p} \|\nabla u\|_{L^p}.$$

Thus $c = (C_P^p + 1)^{-1/p}$ works:

$$\|\nabla u\|_{L^p} \geq \frac{1}{(C_P^p + 1)^{1/p}} \|u\|_{W^{1,p}}.$$

Hence the two norms are equivalent.

3 Second-order linear boundary value problems

Notation 3.1. If H is a Hilbert space, we denote by (u,v) the inner product between u and v in H, and by $\langle f, v\rangle$ the duality product between $f \in H'$ and $v \in H$.

$$\delta^{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

3.1 Some definitions

We are interested in the following second-order linear elliptic PDE problem

$$Lu = f \text{ in } \Omega$$

where Ω is an open set in $\mathbb{R}^N$, $u : \Omega \to \mathbb{R}$ is the unknown, $u := u(x)$, $f : \Omega \to \mathbb{R}$ is a given function, and L is a second-order differential operator defined in divergence form by

$$\boxed{Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u} \tag{3.1}$$

or in non-divergence form by

$$\boxed{Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u} \tag{3.2}$$

where the functions a^{ij}, b^i, c are the coefficients of the operator L.

An important operator of this type is the Laplacian defined by

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x^2}, \text{ (it suffices to take } a^{ij} = \delta^{ij}, b_i = c = 0).$$

where Ω is a bounded open set in $\mathbb{R}^N$, $u : \overline{\Omega} \to \mathbb{R}$ is the unknown, $u := u(x)$, $f : \Omega \to \mathbb{R}$ is a given function. The condition $u = 0$ on $\partial\Omega$ is called the Dirichlet boundary condition.

Remarque 3.2. If the coefficients a^{ij}, $i, j = 1, \ldots, n$ are of class C^1, then the operator L given in divergence form can be written in non-divergence form and vice versa. Indeed, the divergence-form equation (3.1) becomes

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i(x)u_{x_i} + c(x)u \tag{3.3}$$

where $\tilde{b^i} = b^i - \sum_{j=1}^{n} a^{ij}_{x_j}$, and this equation is indeed in non-divergence form.

In what follows we assume that $a^{ij} = a^{ji}$ $(i, j = 1, \ldots, n)$.

Définition 3.1. We say that the operator L is uniformly elliptic if there exists $\theta > 0$ such that
$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j > \theta|\xi|^2 \tag{3.4}$$
for a.e. $x \in \Omega$ and for every $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$.

Remarque 3.3. The ellipticity condition means that for every $x \in \Omega$, the symmetric matrix $A(x) = (a^{ij}(x))_{ij}$ is positive definite and its eigenvalues are greater than θ.

Exercice 3.1. Show that the following operators are uniformly elliptic:

1. The Laplacian operator: $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x^2}$.
2. The operator: $Lu = \operatorname{div}(a(x)\nabla u)$, where $a(x) \geq c_0 > 0$ for all $x \in \Omega$.

3.2 Lax-Milgram theorem

The following theorem is used to prove the existence of solutions to linear elliptic problems.

Théorème 3.4 (Lax-Milgram). *Let H be a Hilbert space over $\mathbb{R}$. Suppose that*
$$B : H \times H \to \mathbb{R}$$
is a bilinear form satisfying the following two conditions

i) B is continuous, i.e.:
$$\exists \alpha > 0 : |B(u, v)| \leq \alpha\|u\|\|v\|, \ \forall u, v \in H$$

ii) B is coercive (or elliptic) on H, i.e.
$$\exists \beta > 0 : B(u, u) \geq \beta\|u\|^2, \ \forall u \in H.$$

Let $f : H \to \mathbb{R}$ be a continuous linear form (i.e. $f \in H'$). Then there exists a unique element $u \in H$ such that
$$B(u, v) = \langle f, v\rangle, \ \forall v \in H \tag{3.5}$$

Remarque 3.5. If the field is $\mathbb{C}$, then the statement of the theorem is as follows:
Let a be a sesquilinear form on H (i.e., linear in the first variable and conjugate-linear in the second) and continuous such that
$$\exists \beta > 0 : \operatorname{Re} B(u, u) \geq \beta\|u\|^2, \ \forall u \in H.$$
Then for every $f \in H'$, there exists a unique element $u \in H$ such that
$$B(u, v) = \langle f, v\rangle, \ \forall v \in H \tag{3.6}$$

Proof. Let u be a fixed element of H. Clearly the map $v \mapsto B(u,v)$ is a continuous linear form on H. By the Riesz representation theorem, there exists a unique element $w \in H$ (depending on u) such that $B(u,v) = (w,v)$ for all $v \in H$. Denote $Au = w$. Then,

$$B(u,v) = (Au, v), \ \forall v \in H. \tag{3.7}$$

We show that the map $A : H \to H$, which sends u to Au, is linear and continuous. Indeed, for $\lambda, \mu \in \mathbb{R}$ and $u_1, u_2 \in H$, we have

$$\begin{aligned} (A(\lambda u_1 + \mu u_2), v) &= B(\lambda u_1 + \mu u_2, v) \text{ by (3.7)} \\ &= \lambda B(u_1, v) + \mu B(u_2, v) \\ &= \lambda(Au_1, v) + \mu(Au_2, v) \text{ again by (3.7)} \\ &= (\lambda Au_1 + \mu Au_2, v). \end{aligned}$$

This equality holds for all $v \in H$, hence $A(\lambda u_1 + \mu u_2) = \lambda Au_1 + \mu Au_2$. For continuity, we have

$$\|Au\|^2 = (Au, Au) = B(u, Au) \leq \alpha \|u\| \|Au\|.$$

Consequently $\|Au\| \leq \alpha\|u\|$ for all $u \in H$. Thus A is continuous. Now we show that A is bijective. We begin with injectivity. Since A is linear, it suffices to show that $Au = 0$ implies $u = 0$. We have

$$\beta\|u\|^2 \leq B(u,u) = (Au, u) \leq \|Au\|\|u\|,$$

hence $\beta\|u\| \leq \|Au\|$. It follows that if $Au = 0$ then $u = 0$.

We show that A is surjective, i.e., the image of A satisfies $\mathrm{Im}(A) = H$. We first show that $\mathrm{Im}(A)$ is closed and then that it is dense in H. Let $(u_n)_n$ be a sequence in H such that $(Au_n)_n$ converges to $w \in H$. Then for all $p, q \in \mathbb{N}$,

$$\beta\|u_p - u_q\|^2 \leq B(u_p - u_q, u_p - u_q) = (A(u_p - u_q), u_p - u_q) \leq \|Au_p - Au_q\|\|u_p - u_q\|.$$

This implies

$$\|u_p - u_q\| \leq \frac{1}{\beta}\|Au_p - Au_q\|.$$

Since $(Au_n)_n$ is convergent, $(u_n)_n$ is Cauchy. Hence $(u_n)_n$ converges to some $u \in H$. By continuity of A,

$$\|Au - w\| = \lim_{n\to\infty} \|A(u - u_n)\| \leq \lim_{n\to\infty} (\alpha\|u - u_n\|) = 0,$$

so $Au = w$ and thus $w \in \mathrm{Im}(A)$.

We show that $\mathrm{Im}(A)$ is dense in H. It suffices to show that $(\mathrm{Im}(A))^\perp = \{0\}$. Let $v \in (\mathrm{Im}(A))^\perp$. Then $(Av, v) = 0$, which implies $\beta\|v\|^2 \leq B(v,v) = (Av, v) = 0$, hence $v = 0$. It follows that $(\mathrm{Im}(A))^\perp = \{0\}$. Thus $\mathrm{Im}(A)$ is dense in H. Since the image of A is closed and dense in H, we have $\mathrm{Im}(A) = H$ and A is surjective.

Next, by the Riesz theorem, there exists a unique $w \in H$ such that

$$\langle f, v\rangle = (w, v), \ \forall v \in H.$$

By the bijectivity of A, there exists a unique $u \in H$ such that $Au = w$, and consequently

$$B(u,v) = (Au, v) = (w, v) = \langle f, v \rangle, \ \forall v \in H.$$

This completes the proof. $\square$

A more general result is that of Stampacchia:

Théorème 3.6. *Let H be a Hilbert space over $\mathbb{R}$ and C a closed convex subset of H. Let a be a continuous coercive bilinear form on H. Then for every $f \in H'$, there exists a unique $u \in C$ such that*

$$a(u, v-u) \geq \langle f, v-u \rangle, \ \forall v \in C. \tag{3.8}$$

Remarque 3.7. If in addition C is a subspace, then inequality (3.8) is equivalent to

$$a(u,w) = \langle f, w \rangle, \ \forall w \in C.$$

It suffices to take $v = \pm w + u \in C$. In the case $C = H$, we recover the Lax-Milgram theorem.

Proof. Since $f \in H'$, by the Riesz theorem there exists a unique $\phi \in H$ such that

$$\langle f, v \rangle = (\phi, v), \ \forall v \in H.$$

(Recall that $(\ , \ )$ denotes the inner product on H). Consider the operator $A : H \to H$ defined by $(Au, v) = a(u,v)$ for all $v \in H$ (see the proof of the Lax-Milgram theorem (3.4)). Then inequality (3.8) is equivalent to

$$(Au - \phi, v-u) \geq 0, \ \forall v \in C.$$

For any $\beta > 0$,

$$((Au - \phi, v-u) \geq 0, \ \forall v \in C) \Longleftrightarrow ((\beta(Au-\phi) + u - u, v-u) \geq 0, \ \forall v \in C).$$

That is, u is the projection onto C of $\beta(Au - \phi) + u - u$. In other words, u is a fixed point of the operator T defined on H by

$$v \longmapsto T(v) := \mathrm{Pr}_C(\beta(Au-\phi) + u - u).$$

Thus it suffices to show that T is a contraction for some $\beta > 0$ (exercise). This completes the proof. $\square$

Exercice 3.2.

1. In the Lax-Milgram theorem, show that if in addition a is symmetric, then u is characterized by

$$J(u) = \min_{v \in H} J(v) \text{ where } J(v) := \frac{1}{2}a(v,v) - \langle f, v \rangle.$$

2. Same question for the Stampacchia theorem with the minimum over C.

3.3 Existence for the Dirichlet problem

The elliptic Dirichlet problem is the following boundary value problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = \psi & \text{on} \quad \partial\Omega \end{cases} \tag{3.9}$$

where L is defined in divergence form by

$$\boxed{Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u} \tag{3.10}$$

The boundary condition $u = \psi$ on $\partial\Omega$ is called the Dirichlet condition on the boundary of Ω (homogeneous if $\psi \equiv 0$ and non-homogeneous if $\psi \neq 0$).

3.3.1 Variational formulation and weak solution

We begin with the homogeneous Dirichlet problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.11}$$

First and throughout the following, we assume that

$$a^{ij},\ b^i,\ c \in L^\infty(\Omega)\ (i, j = 1, \dots, N) \tag{3.12}$$

Suppose that u is a solution of problem (3.11) belonging to $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and $f \in L^2(\Omega)$.
Multiplying the PDE $Lu = f$ by a test function $v \in \mathcal{D}(\Omega)$ and integrating by parts, we obtain:

$$\boxed{\int_\Omega \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} \mathrm{d}x + \int_\Omega \sum_{i=1}^{n} b^i u_{x_i} v \mathrm{d}x + \int_\Omega cuv\mathrm{d}x = \int_\Omega fv\mathrm{d}x.} \tag{3.13}$$

The boundary term vanishes because $u = v = 0$ on $\partial\Omega$. The terms in the above identity are well-defined as soon as $u \in L^2(\Omega)$ and $\nabla u \in (L^2(\Omega))^N$, i.e., $u \in H_0^1(\Omega)$. By density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$, this identity remains valid for all $v \in H_0^1(\Omega)$.

Now suppose that $u \in H_0^1(\Omega)$ satisfies equation (3.13). Then the equation $Lu = f$ in Ω is satisfied only in the sense of distributions (to be verified as an exercise), and the boundary condition $u = 0$ on $\partial\Omega$ is satisfied in the sense of trace (i.e., $Tu = 0$ in $L^2(\partial\Omega)$) taking into account the regularity of the domain $\partial\Omega$. In this case, we say that u is a weak solution of problem (3.11). Moreover, if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, then $u \equiv Tu$ for all $x \in \partial\Omega$ and the identity $Lu = v$ holds for all $x \in \Omega$. In this case u is called a strong solution (or classical solution). To summarize, we give the following definition:

Définition 3.2. Let $f \in H^{-1}(\Omega)$.

1. The identity

$$\boxed{\sum_{i,j=1}^{n}\int_{\Omega}a^{ij}u_{x_i}v_{x_j}\mathrm{d}x+\sum_{i=1}^{n}\int_{\Omega}b^{i}u_{x_i}v\mathrm{d}x+\int_{\Omega}cuv\mathrm{d}x=\langle f,v\rangle,\quad \text{for all } v\in H_0^1(\Omega).} \tag{3.14}$$

is called the variational formulation associated with problem (3.11).

2. The bilinear form associated with the operator L is defined on $H_0^1(\Omega)$ by

$$\boxed{B(u,v):=\sum_{i,j=1}^{n}\int_{\Omega}a^{ij}u_{x_i}v_{x_j}\mathrm{d}x+\sum_{i=1}^{n}\int_{\Omega}b^{i}u_{x_i}v\mathrm{d}x+\int_{\Omega}cuv\mathrm{d}x} \tag{3.15}$$

3. We say that $u\in H_0^1(\Omega)$ is a weak solution of problem (3.11) if it satisfies the variational formulation (3.13).

4. The weak solution $u\in H_0^1(\Omega)$ is called a strong solution (or classical solution) if it belongs to $\mathcal{C}^2(\Omega)\cap\mathcal{C}(\overline{\Omega})$.

3.3.2 Existence of the solution

We apply the Lax-Milgram theorem to obtain existence of the solution. We have the following result:

Théorème 3.8. *If L is uniformly elliptic and $a^{ij},b_i,c\in L^\infty(\Omega)$ with $c\geq\mu>0$ where μ is a sufficiently large constant, then the Dirichlet problem (3.11) admits a unique weak solution.*

Proof. The weak solution $u\in H_0^1(\Omega)$ — by definition — satisfies the variational formulation

$$B(u,v)=\langle f,v\rangle,\ \text{ for all } v\in H_0^1(\Omega).$$

where B is the bilinear form defined by

$$B(u,v):=\sum_{i,j=1}^{n}\int_{\Omega}a^{ij}u_{x_i}v_{x_j}\mathrm{d}x+\sum_{i=1}^{n}\int_{\Omega}b^{i}u_{x_i}v\mathrm{d}x+\int_{\Omega}cuv\mathrm{d}x.$$

We will apply the Lax-Milgram theorem (3.4). To do so, we show that B is continuous and coercive on $H_0^1(\Omega)$.

We start with continuity: We have

$$|B(u,v)|\ \leq\ \int_{\Omega}\sum_{i,j=1}^{n}|a^{ij}u_{x_i}v_{x_j}|\mathrm{d}x+\int_{\Omega}\sum_{i=1}^{n}|b^{i}u_{x_i}v|\mathrm{d}x+\int_{\Omega}|cuv|\mathrm{d}x.$$

We first bound the first term

$$\int_{\Omega}\sum_{i,j=1}^{n}|a^{ij}||u_{x_i}||v_{x_j}|\mathrm{d}x\leq\|a\|_{L^\infty}\sum_{i,j=1}^{n}\int_{\Omega}|u_{x_i}||v_{x_j}|\mathrm{d}x\leq C\int_{\Omega}|\nabla u||\nabla v|\mathrm{d}x$$

where $\|a\|_{L^\infty} = \sum_{1\le i,j\le N} \|a^{ij}\|_{L^\infty}$. By the Cauchy-Schwarz inequality,

$$\int_\Omega \sum_{i,j=1}^n |a^{ij}||u_{x_i}||v_{x_j}|\mathrm{d}x \le \; C_1 \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2} \left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)^{1/2} = C_1 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

For the second and third terms, we use the Cauchy-Schwarz inequality and then Poincaré's inequality:

$$\begin{aligned} \int_\Omega \sum_{i=1}^n |b^i u_{x_i} v|\mathrm{d}x &\le \|b\|_{L^\infty} \int_\Omega \sum_{i=1}^n |u_{x_i} v|\mathrm{d}x \le C\|b\|_{L^\infty} \int_\Omega |\nabla u||v|\mathrm{d}x \\ &\le C_2 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \end{aligned}$$

(where $\|b\|_{L^\infty} = \sum_{1\le i\le N} \|b_i\|_{L^\infty}$). And we have

$$\int_\Omega |c||u||v|\mathrm{d}x \le \|c\|_{L^\infty} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le C_3 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

Hence

$$\begin{aligned} |B(u,v)| &\le (C_1 + C_2 + C_3) \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \\ &= C\|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \end{aligned}$$

This gives the continuity of B on $H_0^1(\Omega)$. Now we show that B is coercive, i.e.,

$$\exists \beta > 0 : B(u,u) \ge \beta \|u\|^2, \;\; \forall u \in H_0^1(\Omega).$$

By the ellipticity condition (3.4):

$$\begin{aligned} \theta \int_\Omega |\nabla u|^2 \mathrm{d}x &\le \int_\Omega \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \mathrm{d}x \\ &= B(u,u) - \int_\Omega \sum_{i=1}^n b^i u_{x_i} u \mathrm{d}x - \int_\Omega c u^2 \mathrm{d}x \\ &\le B(u,u) + \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla u||u|\mathrm{d}x - \mu \int_\Omega u^2 \mathrm{d}x. \end{aligned}$$

Using the Cauchy inequality (with ε), we have

$$\int_\Omega |\nabla u||u|\mathrm{d}x \le \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2\varepsilon} \int_\Omega u^2 \mathrm{d}x$$

where $\varepsilon > 0$ is chosen sufficiently small so that

$$\frac{\varepsilon}{2} \|b\|_{L^\infty(\Omega)} \le \frac{\theta}{2}.$$

Then we obtain

$$\begin{aligned}\frac{\theta}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x &\leq B(u,u) + \left(\frac{1}{2\varepsilon}\|b\|_{L^\infty(\Omega)} - \mu\right)\int_\Omega |u|^2 \mathrm{d}x \\ &= B(u,u) + (C-\mu)\int_\Omega |u|^2 \mathrm{d}x.\end{aligned}$$

If we choose $\mu \geq C$, then we obtain

$$B(u,u) \geq \frac{\theta}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x := \beta\|u\|^2_{H^1_0(\Omega)},$$

hence the coercivity of B on $H^1_0(\Omega)$. Since $f \in H^{-1}(\Omega) := (H^1_0(\Omega))'$, by the Lax-Milgram theorem there exists a unique element $u \in H^1_0(\Omega)$ such that

$$B(u,v) = \langle f, v\rangle, \text{ for all } v \in H^1_0(\Omega).$$

Thus u is the unique weak solution of problem (3.11). □

Exemple 3.9. Consider the following problem:

$$\begin{cases} -\Delta u + u = f & \text{in} \quad \Omega \\ \qquad\quad u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.16}$$

Then $Lu = -\Delta u + u$ and $a^{ij} = \delta^{ij}$, $b_i = 0$, $c = 1$. The associated variational formulation is

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x.$$

The associated bilinear form is defined by

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x.$$

B is continuous because

$$\begin{aligned}|B(u,v)| &\leq \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq (\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})^{1/2}(\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)})^{1/2} \\ &= \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \\ &\leq C\|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)}.\end{aligned}$$

For coercivity, we have

$$B(u,u) = \|u\|^2_{H^1(\Omega)} \geq \|u\|^2_{H^1_0(\Omega)}.$$

Now consider the non-homogeneous Dirichlet problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ \ \ u = \psi & \text{on} \quad \partial\Omega \end{cases} \tag{3.17}$$

For simplicity, we consider $Lu := -\Delta u + u$. We try to give a meaning to the solution of the non-homogeneous Dirichlet problem (3.17). If u is a solution belonging to $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, then it satisfies for all $v \in \mathcal{D}(\Omega)$:

$$\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx = \int_\Omega fv dx.$$

This equation makes sense for $u \in H^1(\Omega)$. Conversely, if $u \in H^1(\Omega)$ satisfies the preceding equation, then it satisfies the equation $Lu = f$ in the sense of distributions, and by the trace theorem (see Chapter 2) the boundary condition makes sense if $\psi \in H^{1/2}(\partial\Omega)$.

Assume that $\psi \in H^{1/2}(\partial\Omega)$. Then there exists a function $g \in H^1(\Omega)$ such that $Tg = \psi$ on $\partial\Omega$. Set $w = u - g$. Then the Dirichlet problem (3.17) is equivalent to the following homogeneous Dirichlet problem

$$\begin{cases} Lw = f - Lg & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.18}$$

and u is a solution of (3.17) if and only if w is a solution of (3.18).

Exercice 3.3. Show that if $f \in H^{-1}(\Omega)$ then $f - Lg \in H^{-1}(\Omega)$.

We therefore have the following definition:

Définition 3.3. Let $f \in H^{-1}(\Omega)$.

1. We say that $u \in H^1(\Omega)$ is a weak solution of problem (3.17) if $w \in H_0^1(\Omega)$ is a weak solution of problem (3.18).
2. The weak solution $u \in H^1(\Omega)$ is called a strong solution (or classical solution) if it belongs to $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

We then conclude with the following existence result:

Théorème 3.10. *For every $f \in H^{-1}(\Omega)$ and $\psi \in H^{1/2}(\partial\Omega)$, the non-homogeneous Dirichlet problem (3.17) admits a unique weak solution $u \in H^1(\Omega)$.*

3.4 Homogeneous Neumann problem

Let Ω be a bounded open set in $\mathbb{R}^n$ of class $\mathcal{C}^1$. The homogeneous Neumann problem is given by

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.19}$$

where L is defined in divergence form by

$$\boxed{Lu = -\sum_{i,j=1}^n \left(a^{ij}(x) u_{x_i}\right)_{x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x)u} \tag{3.20}$$

and $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u, i.e., $\frac{\partial u}{\partial n}(x) := \nabla u(x) \cdot \overrightarrow{n(x)}$ where $\overrightarrow{n(x)}$ is the unit outward normal vector to $\partial\Omega$ at x.

In what follows we consider the particular case where

$$Lu = -\Delta u + u$$

and we study the existence question for the following Neumann problem

$$\begin{cases} -\Delta u + u = f & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.21}$$

Suppose that $u : \Omega \longrightarrow \mathbb{R}$ is a solution of (3.21) such that $u \in \mathcal{C}^2(\overline{\Omega}) := \{\overline{u}_{|\overline{\Omega}} : \overline{u} \in \mathcal{C}^2(\mathbb{R}^n)\}$ and suppose that $f \in L^2(\Omega)$. Let $v \in \mathcal{C}^1(\overline{\Omega}) := \{\overline{v}_{|\overline{\Omega}} : \overline{v} \in \mathcal{C}^1(\mathbb{R}^n)\}$. Multiplying the equation in (3.21) by v and integrating by parts, we obtain

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v dS + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x.$$

Since $\frac{\partial u}{\partial n} = 0$, we have $\int_{\partial\Omega} \frac{\partial u}{\partial n} v dS = 0$. Hence we obtain

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x, \ \forall v \in \mathcal{C}^1(\overline{\Omega}). \tag{3.22}$$

Since $\mathcal{C}^1(\overline{\Omega})$ is dense in the Sobolev space $H^1(\Omega)$, the previous identity holds for all $v \in H^1(\Omega)$ (to be verified as an exercise). Thus u satisfies the following formula:

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x, \ \forall v \in H^1(\Omega). \tag{3.23}$$

This equation makes sense as soon as $u \in H^1(\Omega)$. Conversely, suppose that $u \in H^1(\Omega)$ satisfies (3.23). If in addition $u \in \mathcal{C}^2(\overline{\Omega})$, then by Green's formula we have

$$\int_\Omega (-\Delta u) v \mathrm{d}x + \int_{\partial\Omega} \frac{\partial u}{\partial n} v dS + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x, \ \forall v \in H^1(\Omega). \tag{3.24}$$

If we choose $v \in \mathcal{D}(\Omega)$, then (3.24) becomes

$$\int_\Omega (-\Delta u) v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x, \ \forall v \in \mathcal{D}(\Omega), \tag{3.25}$$

which gives $-\Delta u + u = f$ in $\mathcal{D}'(\Omega)$. This last equation holds a.e. in Ω. Hence equation (3.24) implies that

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v dS = 0, \ \forall v \in H^1(\Omega),$$

i.e., $\frac{\partial u}{\partial n} = 0$ in $(H^{1/2}(\partial\Omega))'$. Thus u satisfies problem (3.21). We then give the following definition:

Définition 3.4. We say that u is a weak solution of (3.21) if $u \in H^1(\Omega)$ and satisfies the variational formulation (3.23). If $u \in \mathcal{C}^2(\overline{\Omega})$, then u is called a classical solution.

3.5 Mixed boundary value problems

3.5.1 Mixed boundary conditions

Consider the following problem:

$$\begin{cases} -u'' + u = f \text{ in } I =]0,1[\\ u(0) = 0, \ u'(1) = 0 \end{cases} \tag{3.26}$$

Suppose that $u \in \mathcal{C}^2([0,1])$ is a solution of (3.26). Then multiplying the equation by $v \in \mathcal{C}^1([0,1])$, we obtain

$$\int_0^1 u'v' \mathrm{d}x - u'(1)v(1) + u'(0)v(0) + \int_0^1 uv \mathrm{d}x = \int_0^1 fv \mathrm{d}x.$$

We have $u'(1) = 0$ and we choose v such that $v(0) = 0$; then the above equation becomes

$$\int_0^1 u'v' \mathrm{d}x + \int_0^1 uv \mathrm{d}x = \int_0^1 fv \mathrm{d}x, \ \forall v \in \mathcal{C}^1([0,1]), \ v(0) = 0. \tag{3.27}$$

We give the following result:

Proposition 3.10.1. *1. The space $\{v \in \mathcal{C}^1([0,1]) : v(0) = 0\}$ is dense in the space*

$$H := \{v \in H^1(0,1) : v(0) = 0\}.$$

2. The space H is a Hilbert space.

Proof. As an exercise □

By density, equation (3.27) is equivalent to the following variational formulation:

$$\int_0^1 u'v' \mathrm{d}x + \int_0^1 uv \mathrm{d}x = \int_0^1 fv \mathrm{d}x, \ \forall v \in H. \tag{3.28}$$

Applying the Lax-Milgram theorem, we obtain the existence of a unique solution $u \in H$ of (3.28), which is called the weak solution of (3.26).

Exercice 3.4. If $u \in \mathcal{C}^2([0,1])$ is a weak solution of (3.26), show that it is a classical solution (i.e., u satisfies the equation in (3.26) and the boundary conditions).

3.5.2 Periodic boundary conditions

Consider the following problem:

$$\begin{cases} -u'' + u = f \text{ in } I =]0,1[, \quad f \in L^2(0,1) \\ u(0) = u(1), \ u'(0) = u'(1) \end{cases} \tag{3.29}$$

Let $u \in \mathcal{C}^2([0,1])$ be a classical solution of (3.29). For all $v \in \mathcal{C}^1([0,1])$:

$$\int_0^1 u'v' \mathrm{d}x + \int_0^1 uv \mathrm{d}x - u'(1)v(1) + u'(0)v(0) = \int_0^1 fv \mathrm{d}x, \ \forall v \in \mathcal{C}^1([0,1]). \tag{3.30}$$

If we choose v such that $v(0) = v(1)$, then

$$-u'(1)v(1) + u'(0)v(0) = v(0)(u'(0) - u'(1)) = 0.$$

Hence equation (3.30) becomes

$$\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x,\ \forall v \in \mathcal{C}^1([0,1]),\ v(0) = v(1). \tag{3.31}$$

Exercice 3.5. 1. Show that the space $\{v \in \mathcal{C}^1([0,1]),\ v(0) = v(1)\}$ is dense in the space

$$H := \{v \in H^1(0,1) : v(0) = v(1)\}.$$

2. Show that the space H is a Hilbert space.

By question 1 of the exercise, equation (3.31) is equivalent to

$$\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x,\ \forall v \in H, \tag{3.32}$$

which is the variational formulation associated with problem (3.29).

Exercice 3.6. Show that if $u \in H \cap \mathcal{C}^2([0,1])$ satisfies (3.32), then it satisfies problem (3.29).

We then give the following definition:

Définition 3.5. We say that u is a weak solution of (3.29) if $u \in H$ and satisfies the variational formulation (3.32). If $u \in \mathcal{C}^2(\overline{\Omega})$, then u is called a classical solution.

Here is the English translation of the provided exercises, with all LaTeX commands and symbols preserved, and with language improvements.

"'latex

3.6 Exercises with Solutions

Remark. For any open set $\Omega \subset \mathbb{R}^N$, we denote by $\mathcal{C}^k(\overline{\Omega})$ with $0 \leq k \leq \infty$ the space

$$\mathcal{C}^k(\overline{\Omega}) = \{\overline{u}_{|\overline{\Omega}} : \overline{u} \in \mathcal{C}^k(\mathbb{R}^N)\}.$$

If Ω is bounded and regular, then $\mathcal{C}^\infty(\Omega)$ is dense in $H^1(\Omega)$ and in particular $\mathcal{C}^k(\overline{\Omega})$ is also dense in $H^1(\Omega)$ for every $1 \leq k \leq \infty$. If Ω is not bounded, then $\mathcal{C}^k(\overline{\Omega}) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$.

Exercice 3.7. Consider the following problem

$$\begin{cases} -u'' + u = f \text{ in } I =]0,1[, & f \in L^2(0,1) \\ u(0) = u(1),\ u'(0) = u'(1) \end{cases} \tag{3.33}$$

1. Let $u \in \mathcal{C}^2([0,1])$ be a classical solution of (3.33). Give the variational formulation satisfied by u, (accepting that the space $\{v \in \mathcal{C}^1([0,1]),\ v(0) = v(1)\}$ is dense in the space
$$H := \{v \in H^1(0,1) : v(0) = v(1)\}).$$

2. Prove the existence and uniqueness of the weak solution u (solution of the variational formulation).

3. If the weak solution $u \in \mathcal{C}^2([0,1])$, show that it is a classical solution of (3.33).

Solution:

1. Let $u \in \mathcal{C}^2([0,1])$ be a classical solution of (3.33). Multiplying the equation in (3.33) by $v \in \mathcal{C}^1([0,1])$ and integrating by parts, we obtain:
$$\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x - u'(1)v(1) + u'(0)v(0) = \int_0^1 fv\mathrm{d}x,\ \forall v \in \mathcal{C}^1([0,1]). \quad (3.34)$$
If we choose v such that $v(0) = v(1)$, then
$$-u'(1)v(1) + u'(0)v(0) = v(0)(u'(0) - u'(1)) = 0.$$
Hence equation (3.34) becomes
$$\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x,\ \forall v \in \mathcal{C}^1([0,1]),\ v(0) = v(1). \quad (3.35)$$
By density, equation (3.35) is equivalent to
$$\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x,\ \forall v \in H, \quad (3.36)$$
which is the variational formulation associated with problem (3.33).

2. We apply the Lax-Milgram theorem to the bilinear form a defined on H by
$$a(u,v) := \int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x$$
and the linear form ℓ defined by
$$\ell(v) = \int_0^1 fv\mathrm{d}x,\ \forall v \in H.$$
For coercivity of a, we have
$$a(u,u) = \int_0^1 u'^2\mathrm{d}x + \int_0^1 u^2\mathrm{d}x = \|u\|^2_{H^1(0,1)}.$$

This implies coercivity. For continuity, applying the Cauchy-Schwarz inequality in L^2 and then in $\mathbb{R}^2$, we obtain

$$\begin{aligned} a(u,v) &\leq \|u'\|_{L^2}\|v'\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2} \\ &\leq \left(\|u'\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{1/2}\left(\|v'\|_{L^2}^2 + \|v\|_{L^2}^2\right)^{1/2} \\ &\leq \|u\|_{H^1(0,1)}\|v\|_{H^1(0,1)}. \end{aligned}$$

This gives the continuity of the bilinear form. For the continuity of the linear form, we have

$$|\ell(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H^1(0,1)} = C\|v\|_{H^1(0,1)}.$$

Applying the Lax-Milgram theorem, we obtain the existence of a weak solution $u \in H$.

3. Let $u \in \mathcal{C}^2([0,1]) \cap H$ be a weak solution. Then u satisfies equation (3.36). Choosing $v \in \mathcal{D}(0,1)$ in (3.36), and since $u \in \mathcal{C}^2([0,1])$, integrating by parts, (3.36) becomes

$$\int_0^1 (-u'' + u)v\mathrm{d}x = \int_0^1 fv\mathrm{d}x,$$

which implies that $-u'' + u = f$ in Ω. Now choose $v \in \mathcal{C}^1([0,1])$ with $v(1) = v(0)$ and integrate the first term in (3.36) by parts; we obtain

$$-\int_0^1 u''v\mathrm{d}x + u'(1)v(1) - u'(0)v(0) + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x.$$

Since $-u'' + u = f$, we deduce that $u'(1)v(1) - u'(0)v(0) = 0$. Because $v(0) = v(1)$, we obtain $u'(0) = u'(1)$. Thus u satisfies problem (3.33).

Exercice 3.8. Consider the following boundary value problem

$$\begin{cases} -u'' + u = f & \text{in } I =]0,1[\\ u'(0) - ku(0) = 0, & u(1) = 0 \end{cases} \tag{3.37}$$

with $f \in L^2(0,1)$.

1. Show that the space $H := \{w \in H^1(0,1) : w(1) = 0\}$ is a Hilbert space.
2. If $u \in \mathcal{C}^2([0,1])$ is a classical solution of (3.37), show that

$$\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x + ku(0)v(0) = \int_0^1 fv\mathrm{d}x, \ \forall v \in \mathcal{C}^1([0,1]), \ v(1) = 0.$$

3. Show that the space $\{v \in \mathcal{C}^1([0,1]) : v(1) = 0\}$ is dense in H.
4. Deduce the variational formulation of problem (3.37):

$$\boxed{\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x + ku(0)v(0) = \int_0^1 fv\mathrm{d}x, \ \forall v \in H.} \tag{3.38}$$

5. For $k \geq 0$, show that the above equation admits a unique solution $u \in H$ (this is the weak solution of (3.37)).

6. If the weak solution $u \in \mathcal{C}^2([0,1])$, show that it is classical (i.e., u satisfies (3.37)).

Solution:

1. First, we show that H is a closed subspace of $H^1(0,1)$. We have $H \subset H^1(0,1)$ by definition. Let $(u_n)_n$ be a sequence in H converging to u in $H^1(0,1)$. We show that $u \in H$. By the continuous embedding $H^1(0,1) \hookrightarrow C([0,1])$, we have $u_n, u \in C([0,1])$ and
$$\|u_n - u\|_{C([0,1])} \leq C\|u_n - u\|_{H^1(0,1)} \longrightarrow 0 \text{ as } n \to \infty.$$
Hence $u_n \to u$ uniformly in $\overline{\Omega}$. Therefore
$$\forall x \in [0,1] : u_n(x) \to u(x).$$
In particular for $x = 1$, we have $u_n(1) \to u(1)$. Since $u_n \in H$, we have $u_n(1) = 0$, hence $u(1) = 0$. Thus $u \in H$, which implies that H is closed. We now show that H is complete. Let $(u_n)_n$ be a Cauchy sequence in $H \subset H^1(0,1)$ (with respect to the $H^1(0,1)$ norm). Then the sequence converges in $H^1(0,1)$ because $H^1(0,1)$ is complete. Since H is closed, the limit of u_n belongs to H. Hence the sequence converges in H. Thus H is complete with the norm $\|.\|_{H^1(0,1)}$. Since $(H^1(0,1), \|\cdot\|_{H^1(0,1)})$ is a Hilbert space, $(H, \|\cdot\|_{H^1(0,1)})$ is also a Hilbert space.

2. Multiply the equation by $v \in \mathcal{C}^1([0,1])$ with $v(1) = 0$ and integrate by parts; we obtain
$$\int_0^1 u'v'\mathrm{d}x - u'(1)v(1) + u'(0)v(0) + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x.$$
Since $v(1) = 0$ and $u'(0) = ku(0)$, we obtain
$$\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x + ku(0)v(0) = \int_0^1 fv\mathrm{d}x, \ \forall v \in \mathcal{C}^1([0,1]), \ v(1) = 0.$$

3. Let $w \in H$. Then $w \in H^1(0,1)$. Since $\mathcal{C}^1([0,1])$ is dense in $H^1(0,1)$, there exists a sequence $(\phi_n)_n$ in $\mathcal{C}^1([0,1])$ such that
$$\phi_n \overset{n\to\infty}{\to} w \text{ in } H^1(0,1).$$
Set $w_n = \phi_n\eta_n$, where $\eta_n \in \mathcal{C}^\infty([0,1])$ is the cut-off function defined by
$$\eta_n(x) = \begin{cases} 0 & \text{if } x \in]1-1/2n, 1] \\ \text{between 0 and 1} & \text{if } x \in [1-1/n, 1-1/2n] \\ 1 & \text{if } x \in [0, 1-1/n] \end{cases} \tag{3.39}$$
Then $w_n \in \mathcal{C}^1([0,1])$, $w_n(1) = 0$, and one can show that $w_n \to w$ in $H^1(0,1)$ (see the density course).

4. Let $v \in H$. By density, there exists a sequence $v_n \in \mathcal{C}^1([0,1])$ with $v_n(1) = 0$ converging to v in $H^1(0,1)$. From question 2, we have

$$\int_0^1 u'v_n'\mathrm{d}x + \int_0^1 uv_n\mathrm{d}x + ku(0)v_n(0) = \int_0^1 fv_n\mathrm{d}x. \tag{3.40}$$

Since $v_n \to v$ in $H^1(0,1)$, we have $v_n \to v$ and $v_n' \to v'$ in $L^2(0,1)$, and by the Sobolev embedding $H^1(0,1) \hookrightarrow C([0,1])$, $v_n(0) \to v(0)$. We deduce that

$$\begin{aligned} \left|\int_0^1 u'v_n'\mathrm{d}x - \int_0^1 u'v'\mathrm{d}x\right| &= \left|\int_0^1 u'(v_n' - v')\mathrm{d}x\right| \\ &\leq \int_0^1 |u'||v_n' - v'|\mathrm{d}x \\ &\overset{\text{Cauchy-Schwarz}}{\leq} \|u'\|_{L^2} \underbrace{\|v_n' - v'\|_{L^2}}_{\to 0} \to 0. \end{aligned}$$

Hence $\int_0^1 u'v_n'\mathrm{d}x \to \int_0^1 u'v'\mathrm{d}x$. Similarly, we obtain

$$\int_0^1 uv_n\mathrm{d}x \to \int_0^1 uv\mathrm{d}x, \quad \int_0^1 fv_n\mathrm{d}x \to \int_0^1 fv\mathrm{d}x.$$

Passing to the limit in (3.40), we obtain

$$\boxed{\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x + ku(0)v(0) = \int_0^1 fv\mathrm{d}x,\ \forall v \in H.} \tag{3.41}$$

5. We apply the Lax-Milgram theorem to the bilinear form a defined on H by

$$a(u,v) := \int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x + ku(0)v(0),\ \forall u,v \in H$$

and the linear form ℓ defined by

$$\ell(v) = \int_0^1 fv\mathrm{d}x,\ \forall v \in H.$$

For coercivity of a, we have

$$a(u,u) = \int_0^1 u'^2\mathrm{d}x + \int_0^1 u^2\mathrm{d}x + ku(0)^2 \geq \int_0^1 u'^2\mathrm{d}x + \int_0^1 u^2\mathrm{d}x = \|u\|^2_{H^1(0,1)}$$

(since $k \geq 0$). For continuity, applying the Cauchy-Schwarz inequality in L^2 and then in $\mathbb{R}^2$, and using the embedding $H^1(0,1) \hookrightarrow C([0,1])$, we obtain

$$\begin{aligned} a(u,v) &\leq \|u'\|_{L^2}\|v'\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2} + k|u(0)v(0)| \\ &\leq \left(\|u'\|^2_{L^2} + \|u\|^2_{L^2}\right)^{1/2}\left(\|v'\|^2_{L^2} + \|v\|^2_{L^2}\right)^{1/2} + k\|u\|_{C^0}\|v\|_{C^0} \\ &\leq \|u\|_{H^1(0,1)}\|v\|_{H^1(0,1)} + k\|u\|_{H^1(0,1)}\|v\|_{H^1(0,1)} \\ &= (1+k)\|u\|_{H^1(0,1)}\|v\|_{H^1(0,1)}. \end{aligned}$$

This gives the continuity of the bilinear form. For the continuity of the linear form, we have

$$|\ell(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H^1(0,1)} = C\|v\|_{H^1(0,1)}.$$

Applying the Lax-Milgram theorem, we obtain the existence of a weak solution $u \in H$.

6. Let $u \in \mathcal{C}^2([0,1]) \cap H$ be a weak solution, i.e., u satisfies equation (3.38). Choosing $v \in \mathcal{D}(0,1)$, (3.38) becomes

$$\int_0^1 u'v'\mathrm{d}x + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x.$$

Integrating by parts, we obtain

$$\int_0^1 (-u'' + u)v\mathrm{d}x = \int_0^1 fv\mathrm{d}x,$$

which implies that $-u'' + u = f$ in Ω. Now choose $v \in \mathcal{C}^1([0,1])$ with $v(1) = 0$ and integrate the first term in (3.38) by parts; we obtain

$$-\int_0^1 u''v\mathrm{d}x + \underbrace{u'(1)v(1)}_{=0} - u'(0)v(0) + \int_0^1 uv\mathrm{d}x + ku(0)v(0) = \int_0^1 fv\mathrm{d}x.$$

Since $-u'' + u = f$, we deduce that $-u'(0)v(0) + ku(0)v(0) = 0$. Since $v(0)$ is arbitrary, we obtain $u'(0) = ku(0)$. Thus u satisfies problem (3.37).

Exercice 3.9. Consider the following boundary value problem

$$\begin{cases} -\Delta u + V \cdot \nabla u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.42}$$

where $f \in L^2(\Omega)$, V is a sufficiently regular function with values in $\mathbb{R}^n$ such that $\operatorname{div} V = 0$.

1. Find the variational formulation of problem (3.42).

2. Prove the existence of the weak solution.

3. If the weak solution $u \in H^2(\Omega)$, show that it satisfies problem (3.42) in a sense to be specified.

Solution

1. Suppose that u is a sufficiently regular solution of (3.42). Multiplying the equation by $v \in \mathcal{C}^1(\overline{\Omega})$ vanishing on $\partial\Omega$, we obtain after integration by parts:

$$\int_\Omega \nabla u \cdot \nabla v\mathrm{d}x + \int_\Omega V \cdot \nabla u\, v\mathrm{d}x = \int_\Omega fv\mathrm{d}x.$$

Since the space $\{v \in \mathcal{C}^1(\overline{\Omega}) : v_{|\partial\Omega} = 0\}$ is dense in $H_0^1(\Omega)$, the above equation is equivalent to the following variational formulation:

$$\int_\Omega \nabla u \cdot \nabla v\mathrm{d}x + \int_\Omega V \cdot \nabla u\, v\mathrm{d}x = \int_\Omega fv\mathrm{d}x, \ \forall v \in H_0^1(\Omega). \tag{3.43}$$

2. We apply the Lax-Milgram theorem to the bilinear form defined on $H_0^1(\Omega)$ by
$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v \mathrm{d}x+\int_\Omega V\cdot\nabla u\, v\mathrm{d}x$$
and the linear form
$$\ell(v)=\int_\Omega f v\mathrm{d}x.$$
We start with coercivity: For every $u\in H_0^1(\Omega)$,
$$a(u,u)=\int_\Omega |\nabla u|^2\mathrm{d}x+\int_\Omega V\cdot\nabla u\, u\mathrm{d}x.$$
Applying Green's formula,
$$\begin{aligned}\int_\Omega V\cdot\nabla u\, u\mathrm{d}x=\int_\Omega (uV)\cdot\nabla u\mathrm{d}x &= -\int_\Omega \mathrm{div}(uV)u+\underbrace{\int_{\partial\Omega} u^2 V\cdot n dS}_{=0}\\ &= -\int_\Omega (uV)\cdot\nabla u\mathrm{d}x-\underbrace{\int_\Omega \mathrm{div}\, V\, u^2\mathrm{d}x}_{=0}\\ &= -\int_\Omega (uV)\cdot\nabla u\mathrm{d}x.\end{aligned}$$
This implies that $\int_\Omega (uV)\cdot\nabla u\mathrm{d}x=0$. Hence $a(u,u)=\int_\Omega|\nabla u|^2\mathrm{d}x=\|u\|^2_{H_0^1(\Omega)}$. Thus a is coercive. For continuity, by the Cauchy-Schwarz and Poincaré inequalities,
$$\begin{aligned}|a(u,v)| &\le \|u\|_{H_0^1(\Omega)}\|\nabla v\|_{L^2(\Omega)}+\|V\|_\infty\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\\ &\le \|u\|_{H_0^1(\Omega)}\|\nabla v\|_{L^2(\Omega)}+C\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}\\ &\le (1+C)\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}.\end{aligned}$$
(Recall that $\|u\|_{H_0^1(\Omega)}:=\|\nabla u\|_{(L^2(\Omega))^N}$.) For the continuity of ℓ, we have
$$\ell(v)\le\|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\le C\|f\|_{L^2(\Omega)}\|v\|_{H_0^1(\Omega)}.$$
Hence we can apply the Lax-Milgram theorem and prove the existence of a unique weak solution $u\in H_0^1(\Omega)$.

3. If the weak solution $u\in H^2(\Omega)$, then applying Green's formula in the variational formulation (3.43), we obtain
$$\int_\Omega -\Delta u\, v\mathrm{d}x+\underbrace{\int_{\partial\Omega}\frac{\partial u}{\partial n}v dS}_{=0}+\int_\Omega V\cdot\nabla u\, v\mathrm{d}x=\int_\Omega f v\mathrm{d}x,\ \forall v\in H_0^1(\Omega). \tag{3.44}$$
If we choose $v\in\mathcal{D}(\Omega)$, then we obtain
$$\int_\Omega -\Delta u\, v\mathrm{d}x+\int_\Omega V\cdot\nabla u\, v\mathrm{d}x=\int_\Omega f v\mathrm{d}x,\ \forall v\in\mathcal{D}(\Omega),$$

which implies that $-\Delta u + V\cdot\nabla u = f$ in $\mathcal{D}'(\Omega)$. Since $\Delta u, \nabla u, f \in L^2(\Omega)$, we have $-\Delta u + V\cdot\nabla u = f$ a.e. in Ω. Thus u satisfies the problem

$$\begin{cases} -\Delta u + V\cdot\nabla u &= f \text{ a.e. in } \Omega \\ u &= 0 \text{ a.e. on } \partial\Omega \end{cases} \tag{3.45}$$

Exercice 3.10. Consider the following Neumann problem

$$\begin{cases} -\Delta u + u &= f \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega \end{cases} \tag{3.46}$$

with $f \in L^2(\Omega)$, $g \in (H^{1/2}(\partial\Omega))' := H^{-1/2}(\partial\Omega)$. Ω is bounded and regular.

1. Give the variational formulation of problem (3.46).
2. Prove the existence of the weak solution.
3. If the weak solution $u \in H^2(\Omega)$, show that it satisfies problem (3.46) in a sense to be specified.
4. Consider the following Neumann problem

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega \end{cases} \tag{3.47}$$

with $f \in L^2(\Omega)$, $g \in (H^{1/2}(\partial\Omega))' := H^{-1/2}(\partial\Omega)$. Prove the existence and uniqueness of the weak solution u in the space

$$H = \{u \in H^1(\Omega) : \int_\Omega u \mathrm{d}x = 0\}.$$

(**Hint:** use the following Poincaré-Wirtinger inequality for the coercivity of the bilinear form)

$$\exists C > 0,\ \forall u \in H^1(\Omega) : \|u\|^2_{L^2(\Omega)} - \frac{1}{|\Omega|}\left(\int_\Omega u \mathrm{d}x\right)^2 \leq \|\nabla u\|^2_{L^2(\Omega)}.$$

Solution

1. Let u be a sufficiently regular solution of (3.46). Multiply by $v \in C(\overline{\Omega})$ and integrate by parts; we obtain:

$$\int_\Omega \nabla u\cdot\nabla v \mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v dS + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x.$$

Using the boundary condition, we obtain

$$\int_\Omega \nabla u\cdot\nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x + \int_{\partial\Omega} gv dS.$$

Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, we obtain the following variational formulation:

$$\int_\Omega \nabla u\cdot\nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x + \langle g, v\rangle_{(H^{1/2}(\partial\Omega))'\times H^{1/2}(\partial\Omega)},\quad \forall v \in H^1(\Omega). \tag{3.48}$$

2. We apply the Lax-Milgram theorem. The associated bilinear form is defined on $H^1(\Omega)$ by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x.$$

The linear form is defined by

$$\ell(v) = \int_\Omega fv \mathrm{d}x + \int_{\partial\Omega} gv dS.$$

Coercivity and continuity are immediate. For the continuity of ℓ, by the Cauchy-Schwarz and Poincaré inequalities:

$$\int_\Omega fv \mathrm{d}x \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C\|v\|_{H^1(\Omega)}.$$

On the other hand, since $g \in (H^{1/2}(\partial\Omega))'$, by the trace theorem we have

$$|\langle g, v\rangle| \leq C\|v\|_{H^{1/2}(\partial\Omega)} \leq c\|v\|_{H^1(\Omega)}.$$

Consequently,

$$|\ell(v)| \leq C\|v\|_{H^1(\Omega)}.$$

3. If $u \in H^2(\Omega)$, then integrating by parts in the variational formulation (3.48), we obtain:

$$\int_\Omega -\Delta u \, v \mathrm{d}x + \int_{\partial\Omega} \frac{\partial u}{\partial n} v dS + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x + \langle g, v\rangle_{(H^{1/2}(\partial\Omega))' \times H^{1/2}(\partial\Omega)}, \quad \forall v \in H^1(\Omega). \tag{3.49}$$

Choose $v \in \mathcal{D}(\Omega)$; then (3.49) becomes

$$\int_\Omega -\Delta u \, v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \int_\Omega fv \mathrm{d}x, \ \forall v \in \mathcal{D}(\Omega).$$

Hence $-\Delta u + u = f$ in $\mathcal{D}'(\Omega)$. Since $-\Delta u, u, f \in L^2(\Omega)$, we have $-\Delta u + u = f$ a.e. in Ω. Consequently, (3.49) becomes

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v dS = \langle g, v\rangle_{(H^{1/2}(\partial\Omega))' \times H^{1/2}(\partial\Omega)}, \quad \forall v \in H^1(\Omega).$$

This gives $\frac{\partial u}{\partial n} = g$ a.e. on $\partial\Omega$.

4. This is immediate, following the same reasoning as above; one only needs to show that the space H is a Hilbert space.

3.7 Additional Exercises on Boundary Value Problems

Exercice 3.11. Consider the following boundary value problem

$$\begin{cases} -(a(x)u')' + c(x)u = f & \text{in} \quad I =]0,1[\\ u(0) = 0, \quad u'(1) = 0 \end{cases} \tag{3.50}$$

where $a \in C^1([0,1])$, $c \in C([0,1])$ satisfy $a(x) \geq a_0 > 0$ and $c(x) \geq 0$ for all $x \in [0,1]$, and $f \in L^2(0,1)$.

1. Give the variational formulation of this problem.
2. Show that the problem admits a unique weak solution $u \in H_0^1(0,1)$? Is $H_0^1(0,1)$ the correct space? Justify your answer.
3. Prove the existence and uniqueness of the weak solution.
4. If the weak solution $u \in H^2(0,1)$, show that it satisfies the original problem in the classical sense.

Solution:

1. **Variational formulation.** First, note that the boundary condition $u(0) = 0$ is Dirichlet, while $u'(1) = 0$ is Neumann. The appropriate space is

$$V = \{v \in H^1(0,1) : v(0) = 0\}.$$

This space is a closed subspace of $H^1(0,1)$ and is a Hilbert space.

Let u be a classical solution ($u \in C^2([0,1])$ satisfying the equation and boundary conditions). Multiply the equation by $v \in V$ and integrate by parts:

$$\begin{aligned} \int_0^1 fv\,dx &= \int_0^1 [-(au')' + cu]\,v\,dx \\ &= [-au'v]_0^1 + \int_0^1 au'v'\,dx + \int_0^1 cuv\,dx. \end{aligned}$$

Since $v(0) = 0$ and $u'(1) = 0$, the boundary term vanishes:

$$-a(1)u'(1)v(1) + a(0)u'(0)v(0) = 0.$$

Thus we obtain the variational formulation:

$$\boxed{\int_0^1 au'v'\,dx + \int_0^1 cuv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V.} \tag{3.51}$$

2. **Choice of the function space.** $H_0^1(0,1)$ is the space of functions vanishing at both endpoints 0 and 1. Here we only require $u(0) = 0$, while no condition is imposed at $x = 1$ (except the natural Neumann condition). Therefore $V = \{v \in H^1(0,1) : v(0) = 0\}$ is the correct space. It is a Hilbert space with the norm $\|v\|_{H^1(0,1)}$.

3. **Existence and uniqueness.** Define the bilinear form $B : V \times V \to \mathbb{R}$ by
$$B(u,v) = \int_0^1 au'v'\,dx + \int_0^1 cuv\,dx,$$
and the linear form $\ell : V \to \mathbb{R}$ by
$$\ell(v) = \int_0^1 fv\,dx.$$
Continuity of B: Using $a \in L^\infty(0,1)$ and $c \in L^\infty(0,1)$,
$$\begin{aligned} |B(u,v)| &\le \|a\|_{L^\infty}\|u'\|_{L^2}\|v'\|_{L^2} + \|c\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} \\ &\le C\|u\|_{H^1}\|v\|_{H^1}. \end{aligned}$$
Coercivity of B: Since $a(x) \ge a_0 > 0$ and $c(x) \ge 0$,
$$B(u,u) = \int_0^1 a(u')^2\,dx + \int_0^1 cu^2\,dx \ge a_0 \int_0^1 (u')^2\,dx.$$
By Poincaré's inequality (for functions vanishing at $x = 0$), there exists $C_P > 0$ such that
$$\|u\|_{L^2} \le C_P\|u'\|_{L^2}, \quad \forall u \in V.$$
Hence
$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 \le (C_P^2 + 1)\|u'\|_{L^2}^2.$$
Thus
$$B(u,u) \ge a_0\|u'\|_{L^2}^2 \ge \frac{a_0}{C_P^2 + 1}\|u\|_{H^1}^2.$$
So B is coercive.

Continuity of ℓ:
$$|\ell(v)| \le \|f\|_{L^2}\|v\|_{L^2} \le \|f\|_{L^2}\|v\|_{H^1}.$$
By the Lax-Milgram theorem, there exists a unique $u \in V$ such that
$$B(u,v) = \ell(v), \quad \forall v \in V.$$
This u is the unique weak solution.

4. **Strong solution.** Assume $u \in H^2(0,1)$ is the weak solution. Then u satisfies
$$\int_0^1 au'v'\,dx + \int_0^1 cuv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V.$$
Take $v \in \mathcal{D}(0,1) \subset V$. Integrating by parts,
$$\int_0^1 (-(au')' + cu)v\,dx = \int_0^1 fv\,dx, \quad \forall v \in \mathcal{D}(0,1).$$

Hence $-(au')' + cu = f$ in $\mathcal{D}'(0,1)$, and since all terms are in $L^2(0,1)$, this holds almost everywhere.

Now choose $v \in V$ arbitrary. Integrate by parts in the variational formulation:

$$\int_0^1 (-(au')' + cu)v\,dx + a(1)u'(1)v(1) - a(0)u'(0)v(0) = \int_0^1 fv\,dx.$$

Since $-(au')' + cu = f$ a.e., we obtain

$$a(1)u'(1)v(1) - a(0)u'(0)v(0) = 0, \quad \forall v \in V.$$

For $v \in V$ with $v(1) = 0$ and $v(0)$ arbitrary, we get $a(0)u'(0)v(0) = 0$, so $u'(0) = 0$ (since $a(0) > 0$). For v with $v(0) = 0$ and $v(1)$ arbitrary, we get $a(1)u'(1)v(1) = 0$, so $u'(1) = 0$. However, our boundary condition was $u(0) = 0$ (already used in the definition of V) and $u'(1) = 0$. The condition $u'(0) = 0$ is not required; it appears because we integrated by parts without a boundary term at $x = 0$ due to $v(0) = 0$. Actually, the boundary term at $x = 0$ is $a(0)u'(0)v(0)$; since $v(0) = 0$ for all $v \in V$, this term vanishes automatically. Therefore we only obtain $u'(1) = 0$ from the above. Thus u satisfies the original boundary conditions $u(0) = 0$ and $u'(1) = 0$. So u is a classical solution.

Exercice 3.12. Consider the following problem with a non-constant coefficient:

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.52}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $f \in L^2(\Omega)$, and $A(x) = (a_{ij}(x))_{1\le i,j\le N}$ is a symmetric matrix satisfying the uniform ellipticity condition:

$$\exists \theta > 0 : \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega,$$

and $a_{ij} \in L^\infty(\Omega)$.

1. Write the variational formulation of this problem.
2. Show that the problem admits a unique weak solution $u \in H_0^1(\Omega)$.

Solution:

1. **Variational formulation.** Let u be a classical solution. Multiply by $v \in \mathcal{D}(\Omega)$ and integrate by parts:

$$\int_\Omega A(x)\nabla u \cdot \nabla v\,dx = \int_\Omega fv\,dx.$$

By density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$, the formulation extends to all $v \in H_0^1(\Omega)$. The variational formulation is:

$$\boxed{\int_\Omega A(x)\nabla u \cdot \nabla v\,dx = \int_\Omega fv\,dx, \quad \forall v \in H_0^1(\Omega).} \tag{3.53}$$

2. **Existence and uniqueness.** Define the bilinear form $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$B(u, v) = \int_\Omega A(x)\nabla u \cdot \nabla v \, dx = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx,$$

and the linear form $\ell(v) = \int_\Omega f v \, dx$.

Continuity: Since $a_{ij} \in L^\infty(\Omega)$,

$$|B(u, v)| \leq \sum_{i,j=1}^{N} \|a_{ij}\|_{L^\infty} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right| \left|\frac{\partial v}{\partial x_j}\right| dx \leq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} \leq C\|u\|_{H_0^1}\|v\|_{H_0^1}.$$

Coercivity: By the ellipticity condition,

$$B(u, u) = \int_\Omega A(x)\nabla u \cdot \nabla u \, dx \geq \theta \int_\Omega |\nabla u|^2 \, dx = \theta\|u\|_{H_0^1}^2.$$

Continuity of ℓ:

$$|\ell(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq C\|f\|_{L^2}\|v\|_{H_0^1}.$$

By the Lax-Milgram theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$B(u, v) = \ell(v), \quad \forall v \in H_0^1(\Omega).$$

Thus the problem admits a unique weak solution.

Exercice 3.13. Consider the Neumann problem:

$$\begin{cases} -\Delta u + u = f & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial n} = g & \text{on} \quad \partial\Omega \end{cases} \tag{3.54}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $f \in L^2(\Omega)$, and $g \in L^2(\partial\Omega)$.

1. Give the variational formulation of this problem.
2. Prove the existence and uniqueness of a weak solution $u \in H^1(\Omega)$.
3. Show that if $u \in H^2(\Omega)$ is the weak solution, then it satisfies the boundary condition in the sense of $L^2(\partial\Omega)$.

Solution:

1. **Variational formulation.** Let u be a classical solution. Multiply by $v \in H^1(\Omega)$ and integrate by parts:

$$\int_\Omega (-\Delta u)v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx.$$

Using Green's formula,

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dS + \int_\Omega uv \, dx = \int_\Omega fv \, dx.$$

Substituting the boundary condition $\frac{\partial u}{\partial n} = g$, we obtain:

$$\boxed{\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, dS, \quad \forall v \in H^1(\Omega).} \tag{3.55}$$

2. **Existence and uniqueness.** Define $B(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx$ and $\ell(v) = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, dS$.
   Continuity of B:
   $$|B(u,v)| \le \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|u\|_{L^2} \|v\|_{L^2} \le \|u\|_{H^1} \|v\|_{H^1}.$$
   Coercivity:
   $$B(u,u) = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 = \|u\|_{H^1}^2.$$
   Continuity of ℓ:
   $$|\ell(v)| \le \|f\|_{L^2} \|v\|_{L^2} + \|g\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}.$$
   By the trace theorem, $\|v\|_{L^2(\partial\Omega)} \le C \|v\|_{H^1(\Omega)}$. Hence
   $$|\ell(v)| \le (\|f\|_{L^2} + C\|g\|_{L^2(\partial\Omega)}) \|v\|_{H^1}.$$
   By the Lax-Milgram theorem, there exists a unique $u \in H^1(\Omega)$ satisfying (3.55).

3. **Boundary condition in $L^2(\partial\Omega)$.** Assume $u \in H^2(\Omega)$ is the weak solution. Then u satisfies (3.55) for all $v \in H^1(\Omega)$. Integrate by parts in the first term:
   $$\int_\Omega (-\Delta u + u) v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dS = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, dS, \quad \forall v \in H^1(\Omega).$$
   Since u satisfies the equation $-\Delta u + u = f$ a.e. in Ω (which can be shown by testing with $v \in \mathcal{D}(\Omega)$), we obtain
   $$\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dS = \int_{\partial\Omega} gv \, dS, \quad \forall v \in H^1(\Omega).$$
   By density of the trace of $H^1(\Omega)$ in $L^2(\partial\Omega)$, this implies $\frac{\partial u}{\partial n} = g$ in $L^2(\partial\Omega)$.

Exercice 3.14. Consider the Robin boundary value problem:

$$\begin{cases} -u'' + u = f \quad \text{in} \quad I =]0,1[\\ u'(0) - \alpha u(0) = 0, \quad u'(1) + \beta u(1) = 0 \end{cases} \tag{3.56}$$

where $\alpha, \beta > 0$ and $f \in L^2(0,1)$.

1. Give the variational formulation of this problem. What is the appropriate function space?
2. Show that the problem admits a unique weak solution.
3. Show that the weak solution satisfies the boundary conditions in a suitable sense.

Solution:

1. **Variational formulation.** The appropriate function space is $H^1(0,1)$ since there are no essential (Dirichlet) boundary conditions; both conditions are natural (Robin conditions).

   Let u be a classical solution. Multiply by $v \in H^1(0,1)$ and integrate by parts:

$$\int_0^1 (-u'' + u)v\,dx = [-u'v]_0^1 + \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx.$$

   Thus
$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 fv\,dx.$$

   Substituting the boundary conditions $u'(0) = \alpha u(0)$ and $u'(1) = -\beta u(1)$, we obtain:

$$\boxed{\int_0^1 u'v'\,dx + \int_0^1 uv\,dx + \alpha u(0)v(0) + \beta u(1)v(1) = \int_0^1 fv\,dx, \quad \forall v \in H^1(0,1).} \tag{3.57}$$

2. **Existence and uniqueness.** Define $B(u,v) = \int_0^1 u'v'\,dx + \int_0^1 uv\,dx + \alpha u(0)v(0) + \beta u(1)v(1)$ and $\ell(v) = \int_0^1 fv\,dx$.

   Continuity of B: Using the Cauchy-Schwarz inequality and the continuous embedding $H^1(0,1) \hookrightarrow C([0,1])$ (so $|u(0)|, |u(1)| \le C\|u\|_{H^1}$),

$$\begin{aligned} |B(u,v)| &\le \|u'\|_{L^2}\|v'\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2} + \alpha|u(0)v(0)| + \beta|u(1)v(1)| \\ &\le \|u\|_{H^1}\|v\|_{H^1} + (\alpha+\beta)C^2\|u\|_{H^1}\|v\|_{H^1} = C_1\|u\|_{H^1}\|v\|_{H^1}. \end{aligned}$$

   Coercivity: Since $\alpha, \beta > 0$,

$$B(u,u) = \|u'\|_{L^2}^2 + \|u\|_{L^2}^2 + \alpha u(0)^2 + \beta u(1)^2 \ge \|u'\|_{L^2}^2 + \|u\|_{L^2}^2 = \|u\|_{H^1}^2.$$

   Continuity of ℓ:
$$|\ell(v)| \le \|f\|_{L^2}\|v\|_{L^2} \le \|f\|_{L^2}\|v\|_{H^1}.$$

   By the Lax-Milgram theorem, there exists a unique $u \in H^1(0,1)$ satisfying (3.57). This u is the weak solution.

3. **Boundary conditions.** Assume u is the weak solution. For any $v \in H^1(0,1)$, we have
$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx + \alpha u(0)v(0) + \beta u(1)v(1) = \int_0^1 fv\,dx.$$
Take $v \in \mathcal{D}(0,1)$ (so $v(0) = v(1) = 0$). Then
$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in \mathcal{D}(0,1).$$
Integrating by parts,
$$\int_0^1 (-u'' + u)v\,dx = \int_0^1 fv\,dx, \quad \forall v \in \mathcal{D}(0,1).$$
Hence $-u'' + u = f$ in $\mathcal{D}'(0,1)$, and since all terms are in $L^2(0,1)$, this holds almost everywhere.

Now integrate by parts in the variational formulation for a general $v \in H^1(0,1)$:
$$\int_0^1 (-u'' + u)v\,dx - u'(1)v(1) + u'(0)v(0) + \alpha u(0)v(0) + \beta u(1)v(1) = \int_0^1 fv\,dx.$$
Since $-u'' + u = f$ a.e., we obtain
$$-u'(1)v(1) + u'(0)v(0) + \alpha u(0)v(0) + \beta u(1)v(1) = 0, \quad \forall v \in H^1(0,1).$$
Choosing v with $v(0) = 0$ and arbitrary $v(1)$, we get $(-u'(1) + \beta u(1))v(1) = 0$, so $u'(1) = \beta u(1)$. Choosing v with $v(1) = 0$ and arbitrary $v(0)$, we get $(u'(0) + \alpha u(0))v(0) = 0$, so $u'(0) = -\alpha u(0)$. (Note the sign: the condition is $u'(0) - \alpha u(0) = 0$ and $u'(1) + \beta u(1) = 0$; we have obtained $u'(0) = \alpha u(0)$? Let's check carefully.)

Wait, the boundary term from integration by parts is:
$$\int_0^1 u'v'\,dx = [u'v]_0^1 - \int_0^1 u''v\,dx = u'(1)v(1) - u'(0)v(0) - \int_0^1 u''v\,dx.$$
Substituting into the variational formulation:
$$-\int_0^1 u''v\,dx + u'(1)v(1) - u'(0)v(0) + \int_0^1 uv\,dx + \alpha u(0)v(0) + \beta u(1)v(1) = \int_0^1 fv\,dx.$$
Since $-u'' + u = f$, we have $-\int_0^1 u''v\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx$. Thus:
$$\int_0^1 fv\,dx + u'(1)v(1) - u'(0)v(0) + \alpha u(0)v(0) + \beta u(1)v(1) = \int_0^1 fv\,dx,$$
so
$$u'(1)v(1) - u'(0)v(0) + \alpha u(0)v(0) + \beta u(1)v(1) = 0, \quad \forall v \in H^1(0,1).$$

Rearranging:

$$(u'(1) + \beta u(1))v(1) + (-u'(0) + \alpha u(0))v(0) = 0, \quad \forall v \in H^1(0,1).$$

Since $v(0)$ and $v(1)$ can be chosen independently, we obtain:

$$u'(1) + \beta u(1) = 0 \quad \text{and} \quad -u'(0) + \alpha u(0) = 0.$$

Thus $u'(1) = -\beta u(1)$ and $u'(0) = \alpha u(0)$, which are exactly the Robin boundary conditions. So the weak solution satisfies the boundary conditions in the sense of traces.

Exercice 3.15. Consider the non-homogeneous Dirichlet problem:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases} \tag{3.58}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $f \in L^2(\Omega)$, and $\varphi \in H^{1/2}(\partial\Omega)$.

1. Show that this problem can be reduced to a homogeneous Dirichlet problem.
2. Give the variational formulation and prove the existence and uniqueness of a weak solution $u \in H^1(\Omega)$.

Solution:

1. **Reduction to homogeneous problem.** Since $\varphi \in H^{1/2}(\partial\Omega)$, by the trace theorem there exists a function $u_0 \in H^1(\Omega)$ such that $Tu_0 = \varphi$ on $\partial\Omega$ (where T denotes the trace operator). Let $w = u - u_0$. Then w satisfies:

$$\begin{cases} -\Delta w = f + \Delta u_0 & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega \end{cases}$$

This is a homogeneous Dirichlet problem for w. Note that Δu_0 is understood in the sense of distributions; if u_0 is not regular enough, we treat it via the variational formulation directly.

Alternatively, we work directly with the variational formulation: find $u \in H^1(\Omega)$ with $Tu = \varphi$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall v \in H_0^1(\Omega).$$

Set $u = u_0 + w$ where u_0 is any lift of φ ($Tu_0 = \varphi$). Then $w \in H_0^1(\Omega)$ and satisfies:

$$\int_\Omega \nabla w \cdot \nabla v \, dx = \int_\Omega f v \, dx - \int_\Omega \nabla u_0 \cdot \nabla v \, dx, \quad \forall v \in H_0^1(\Omega).$$

This is a homogeneous Dirichlet problem for w.

2. **Variational formulation and existence.** Define the bilinear form $B(w, v) = \int_\Omega \nabla w \cdot \nabla v \, dx$ on $H_0^1(\Omega) \times H_0^1(\Omega)$ and the linear form

$$\ell(v) = \int_\Omega f v \, dx - \int_\Omega \nabla u_0 \cdot \nabla v \, dx.$$

B is continuous and coercive (by Poincaré's inequality). ℓ is continuous because

$$|\ell(v)| \leq \|f\|_{L^2} \|v\|_{L^2} + \|\nabla u_0\|_{L^2} \|\nabla v\|_{L^2} \leq C(\|f\|_{L^2} + \|u_0\|_{H^1}) \|v\|_{H_0^1}.$$

By the Lax-Milgram theorem, there exists a unique $w \in H_0^1(\Omega)$ such that

$$B(w, v) = \ell(v), \quad \forall v \in H_0^1(\Omega).$$

Then $u = u_0 + w$ is the unique weak solution of the original problem, satisfying $Tu = \varphi$.

Note that the solution u is independent of the choice of the lift u_0: if u_0^1 and u_0^2 are two different lifts, then $u_0^1 - u_0^2 \in H_0^1(\Omega)$ and the corresponding w adjust accordingly, giving the same u.

4 Maximum principle and eigenvalue problem

4.1 Maximum principle

The maximum principle is a fundamental tool in the theory of elliptic partial differential equations. It states that a solution of a linear elliptic equation attains its maximum on the boundary, unless it is constant. We present here the weak and strong maximum principles for the Laplacian.

4.1.1 Weak maximum principle

Théorème 4.1 (Weak maximum principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$-\Delta u \leq 0 \quad \text{in } \Omega \qquad (\text{i.e., } u \text{ is subharmonic}).$$

Then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Similarly, if $-\Delta u \geq 0$ in Ω (i.e., u is superharmonic), then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Proof. We prove the first statement. Let $M = \max_{\partial\Omega} u$. Assume, for contradiction, that there exists $x_0 \in \Omega$ such that $u(x_0) > M$. Define $v(x) = u(x) + \varepsilon|x|^2$ for some $\varepsilon > 0$ small. Then

$$-\Delta v = -\Delta u - 2N\varepsilon \leq -2N\varepsilon < 0 \quad \text{in } \Omega.$$

Hence v is strictly subharmonic. Since v is continuous on $\overline{\Omega}$, it attains its maximum at some point $x_1 \in \overline{\Omega}$. If $x_1 \in \partial\Omega$, then

$$v(x_1) \leq M + \varepsilon \max_{\partial\Omega} |x|^2.$$

If $x_1 \in \Omega$, then at an interior maximum we have $\Delta v(x_1) \leq 0$ (necessary condition), which contradicts $-\Delta v(x_1) < 0$ (i.e., $\Delta v(x_1) > 0$). Therefore the maximum cannot be attained in the interior. Hence $x_1 \in \partial\Omega$, and for all $x \in \Omega$,

$$u(x) + \varepsilon|x|^2 = v(x) \leq v(x_1) \leq M + \varepsilon \max_{\partial\Omega} |x|^2.$$

Letting $\varepsilon \to 0^+$, we obtain $u(x) \leq M$ for all $x \in \Omega$, contradicting the existence of x_0. Thus $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$. □

4.1.2 Strong maximum principle

Théorème 4.2 (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be a connected open set. Suppose $u \in C^2(\Omega)$ satisfies*

$$-\Delta u \leq 0 \quad \text{in } \Omega.$$

If u attains its maximum at an interior point $x_0 \in \Omega$, then u is constant in Ω.

Similarly, if $-\Delta u \geq 0$ and u attains its minimum at an interior point, then u is constant.

Proof. We prove the subharmonic case. Suppose $M = \max_\Omega u$ and $u(x_0) = M$ for some $x_0 \in \Omega$. Let $\Omega_0 = \{x \in \Omega : u(x) = M\}$. This set is nonempty (contains x_0) and closed in Ω by continuity. We show it is also open. Take any $y \in \Omega_0$ and choose $r > 0$ such that $B(y, r) \subset \Omega$. For any $z \in B(y, r)$, consider the spherical mean

$$\overline{u}(y) = \frac{1}{|\partial B(y,r)|} \int_{\partial B(y,r)} u \, dS.$$

Since $-\Delta u \leq 0$, u is subharmonic, which implies

$$u(y) \leq \frac{1}{|\partial B(y,r)|} \int_{\partial B(y,r)} u \, dS.$$

But $u(y) = M$, so we have

$$M \leq \frac{1}{|\partial B(y,r)|} \int_{\partial B(y,r)} u \, dS \leq M,$$

because $u \leq M$ on $\partial B(y, r)$. Hence equality holds, and $u = M$ on $\partial B(y, r)$. Since this holds for all sufficiently small r, $u \equiv M$ in a neighborhood of y. Thus Ω_0 is open. As Ω is connected and Ω_0 is both open and closed, $\Omega_0 = \Omega$. Therefore $u \equiv M$ in Ω. □

4.1.3 Hopf's lemma

Lemme 4.3 (Hopf's lemma). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with C^2 boundary. Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$-\Delta u \leq 0 \quad \text{in } \Omega,$$

and there exists $x_0 \in \partial\Omega$ such that $u(x_0) > u(x)$ for all $x \in \Omega$ (i.e., u attains a strict maximum at the boundary point x_0). Then

$$\frac{\partial u}{\partial n}(x_0) > 0,$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative.

Proof. Omitted; see any standard PDE textbook. The lemma states that at a boundary maximum, the outward normal derivative is strictly positive (for subharmonic functions) or strictly negative (for superharmonic functions). □

4.1.4 Application to the Dirichlet problem

Théorème 4.4 (Uniqueness for the Dirichlet problem). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. For any $f \in C(\Omega)$ and $\varphi \in C(\partial\Omega)$, the Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases}$$

has at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Proof. Suppose u_1 and u_2 are two solutions. Then $w = u_1 - u_2$ satisfies

$$-\Delta w = 0 \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

By the weak maximum principle, $\max_{\overline{\Omega}} w = \max_{\partial\Omega} w = 0$ and $\min_{\overline{\Omega}} w = \min_{\partial\Omega} w = 0$. Hence $w \equiv 0$, so $u_1 = u_2$. □

4.1.5 Positivity of solutions

Théorème 4.5 (Positivity). *Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set with C^2 boundary. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ \quad u = \varphi & \text{on} \quad \partial\Omega \end{cases}$$

with $f \geq 0$ in Ω and $\varphi \geq 0$ on $\partial\Omega$. Then $u \geq 0$ in Ω. Moreover, if $f > 0$ in Ω or $\varphi > 0$ on a nonempty open subset of $\partial\Omega$, then $u > 0$ in Ω.

Proof. Consider the function $v = -u$. Then $-\Delta v = -f \leq 0$, so v is subharmonic. By the weak maximum principle,

$$\max_{\overline{\Omega}} v = \max_{\partial\Omega} v = \max_{\partial\Omega}(-\varphi) = -\min_{\partial\Omega} \varphi \leq 0.$$

Thus $v \leq 0$ in $\overline{\Omega}$, i.e., $-u \leq 0$, so $u \geq 0$ in Ω.

If in addition $f > 0$ in Ω, then $-\Delta u = f > 0$, so $-\Delta u \leq 0$ is not satisfied; we need a different argument. Alternatively, if u attains a nonnegative minimum at an interior point, the strong maximum principle forces u to be constant, which would contradict $-\Delta u = f > 0$ unless $f = 0$. Hence the minimum must be attained on the boundary. Since $\varphi \geq 0$, we have $u \geq 0$. If $\varphi = 0$ on the entire boundary and $f > 0$, then $u > 0$ in Ω by the strong maximum principle. □

4.1.6 Maximum principle for weak solutions

The maximum principle also holds for weak solutions in Sobolev spaces.

Théorème 4.6 (Weak maximum principle for $H_0^1(\Omega)$). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Suppose $u \in H_0^1(\Omega)$ satisfies*

$$\int_\Omega \nabla u \cdot \nabla v \, dx \leq 0, \quad \forall v \in H_0^1(\Omega),\ v \geq 0 \text{ a.e..}$$

Then $u \leq 0$ a.e. in Ω.

Similarly, if $\int_\Omega \nabla u \cdot \nabla v \, dx \geq 0$ for all $v \geq 0$, then $u \geq 0$ a.e.

Proof. Take $v = u^+ = \max(u, 0)$ as a test function (one can show that $u^+ \in H_0^1(\Omega)$ and $\nabla u^+ = \nabla u \cdot \mathbf{1}_{\{u>0\}}$). Then

$$\int_\Omega \nabla u \cdot \nabla u^+ \, dx = \int_{\{u>0\}} |\nabla u|^2 \, dx \leq 0.$$

Hence $\nabla u = 0$ a.e. on $\{u > 0\}$, so u is constant on each connected component of $\{u > 0\}$. Since $u \in H_0^1(\Omega)$, the trace of u on $\partial\Omega$ is zero. If $\{u > 0\}$ is nonempty, then on that set u is a positive constant. But then the trace of that constant on the boundary would be positive (since the set where $u > 0$ touches the boundary? Actually, if u is constant and positive on an open set, its trace cannot be zero unless the set does not reach the boundary. However, one can use a more refined argument: test with $v = \min(u^+, \varepsilon)$ and let $\varepsilon \to 0$ to conclude that $u^+ = 0$. Thus $u \leq 0$ a.e.) $\square$

4.2 Eigenvalue problem

4.2.1 Existence

For simplicity, we consider in this section $L = -\Delta$. We are interested in the following Eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.1}$$

The weak solution of (4.1) is called an eigenfunction of $-\Delta$ and λ is the associated eigenvalue.

We have the following result:

Théorème 4.7. *i) The eigenvalues of $-\Delta$ are real and positive.*

ii) There exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ such that $w_k \in H_0^1(\Omega)$ for all $k \in \mathbb{N}^*$ and

$$\begin{cases} -\Delta w_k = \lambda_k w_k & \textit{in} \quad \Omega \\ w_k = 0 & \textit{on} \quad \partial\Omega \end{cases} \tag{4.2}$$

where the eigenvalues λ_k satisfy

$$0 < \lambda_1 < \lambda_2 < \dots, \qquad \lambda_k \overset{k\to\infty}{\to} \infty. \tag{4.3}$$

Proof. We define the operator $T := (-\Delta)^{-1}$ from $L^2(\Omega)$ to $L^2(\Omega)$ which maps f to $Tf := u$, the weak solution of

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.4}$$

We show that $T : L^2(\Omega) \to L^2(\Omega)$ is bounded (continuous) and compact. By the previous section (Remark **??**), problem (4.4) admits a unique weak solution $u \in H_0^1(\Omega)$, hence the operator T is well-defined. Multiplying the equation in (4.4) by u and integrating by parts, we obtain using the Cauchy-Schwarz and Poincaré inequalities:

$$\int_\Omega |\nabla u|^2 \mathrm{d}x = \int_\Omega f u \mathrm{d}x \leq C\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$

Hence

$$\|u\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

so

$$\|Tf\|_{H_0^1(\Omega)} = \|u\|_{H_0^1(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{4.5}$$

In particular, using Poincaré's inequality, we have

$$\|Tf\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{4.6}$$

The continuity of T follows from (4.6). Thanks to the compact embedding of the Sobolev space $H_0^1(\Omega)$ into $L^2(\Omega)$ (Theorem 2.18), we deduce from estimate (4.5) that T is compact.

Now we show that T is symmetric (self-adjoint), i.e.,

$$(Tf, g) = (f, Tg), \text{ for all } f, g \in L^2(\Omega). \tag{4.7}$$

Set $u = Tf$ and $v = Tg$. Then we have

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.8}$$

$$\begin{cases} -\Delta v = g & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.9}$$

Multiply (4.8) by $v \in H_0^1(\Omega)$ and integrate by parts; we obtain

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega fv = (f, v) := (f, Tg).$$

Multiply (4.9) by $u \in H_0^1(\Omega)$ and integrate by parts; we obtain

$$\int_\Omega \nabla v \cdot \nabla u \mathrm{d}x = \int_\Omega gu = (u, g) := (Tf, g).$$

Comparing these last two equations, we obtain (4.7).
By the theory of compact symmetric operators, the eigenvalues of T form a sequence $\{\mu_k\}_{k=1}^\infty$ such that

$$\mu_1 > \mu_2 > \mu_3 > \dots \qquad \text{and } \mu_k \to 0.$$

Hence the eigenvalues of $-\Delta$ are $\lambda_k = \frac{1}{\mu_k}$, $k = 1, 2, \dots$, which satisfy

$$\int_\Omega |\nabla w_k|^2 \mathrm{d}x = \lambda_k \int_\Omega |w_k|^2 \mathrm{d}x.$$

This implies that $\lambda_k > 0$ $(k = 1, 2, \dots)$, and $\lambda_k \overset{k\to\infty}{\to} \infty$.

Since $L^2(\Omega)$ is a separable Hilbert space and T is compact on $L^2(\Omega)$, it admits an orthonormal basis consisting of eigenfunctions $\{w_k\}_{k=1}^\infty$ of the operator $-\Delta$ (which are the same eigenfunctions as the compact operator $T := (-\Delta)^{-1}$). □

Remarque 4.8. Moreover, we have

$$\int_\Omega \nabla w_j \cdot \nabla w_k \mathrm{d}x = \lambda_k \int_\Omega w_j w_k \mathrm{d}x = 0 \text{ if } j \ne k.$$

Hence the sequence $\{w_k\}_{k=1}^\infty$ is orthogonal in $H_0^1(\Omega)$. One can easily show that $\{w_k\}_{k=1}^\infty$ is also an orthogonal basis of $H_0^1(\Omega)$ (see Exercise **??**).

4.2.2 Characterization of the first eigenvalue λ_1

The first eigenvalue λ_1 of the Laplacian with Dirichlet boundary conditions plays a fundamental role. It admits several equivalent variational characterizations, which we present below as a series of propositions.

Proposition 4.8.1 (Rayleigh quotient characterization)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Then*

$$\lambda_1 = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega u^2\,dx}.$$

Proof. Let u be any eigenfunction associated with an eigenvalue λ. Substituting into the Rayleigh quotient yields

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} = \lambda.$$

Hence the infimum over all nonzero functions is less than or equal to λ_1. The reverse inequality follows from the existence of a minimizer that attains the infimum (see Proposition 4.9.1). □

Remarque 4.9. The Rayleigh quotient measures the ratio between the "energy" $\int |\nabla u|^2$ and the "mass" $\int u^2$. The first eigenvalue corresponds to the smallest possible energy per unit mass.

Proposition 4.9.1 (Normalized minimization problem)**.**

$$\lambda_1 = \min\left\{\int_\Omega |\nabla u|^2\,dx \;:\; u \in H_0^1(\Omega),\ \int_\Omega u^2\,dx = 1\right\}.$$

Moreover, a minimizer exists and is a first eigenfunction.

Proof. The Rayleigh quotient is homogeneous of degree zero: $R(tu) = R(u)$ for all $t \neq 0$. Therefore we may restrict to functions with $\|u\|_{L^2} = 1$, where $R(u) = \int |\nabla u|^2$. Thus the two minimization problems are equivalent.

Let $\{u_n\} \subset S = \{u \in H_0^1(\Omega) : \|u\|_{L^2} = 1\}$ be a minimizing sequence, i.e.,

$$\int_\Omega |\nabla u_n|^2\,dx \to \lambda_1, \qquad \int_\Omega u_n^2\,dx = 1.$$

Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By the Rellich–Kondrachov compact embedding theorem, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Hence there exists a subsequence (still denoted u_n) and a function $w_1 \in L^2(\Omega)$ such that

$$u_n \to w_1 \quad \text{strongly in } L^2(\Omega).$$

Since $\int u_n^2 = 1$, we have $\int w_1^2 = 1$, so $w_1 \neq 0$. Moreover, because $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we may extract a further subsequence converging weakly in $H_0^1(\Omega)$ to some $w_1 \in H_0^1(\Omega)$. By uniqueness of the weak limit, we have $w_1 \in H_0^1(\Omega)$ and $\|w_1\|_{L^2} = 1$.

Using weak lower semicontinuity of the norm,

$$\int_\Omega |\nabla w_1|^2\,dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx = \lambda_1.$$

Since $w_1 \in S$, we have $\int |\nabla w_1|^2 \ge \lambda_1$ by definition of the infimum. Hence equality holds, and w_1 is a minimizer. □

Remarque 4.10. This formulation is often more convenient in practice: instead of minimizing a quotient, we minimize a convex functional on a constraint set (the unit sphere in L^2).

Proposition 4.10.1 (Euler–Lagrange equation for the minimizer)**.** *Any minimizer w_1 from Proposition 4.9.1 satisfies the weak eigenvalue equation:*

$$\int_\Omega \nabla w_1 \cdot \nabla \varphi\,dx = \lambda_1 \int_\Omega w_1 \varphi\,dx, \quad \forall \varphi \in H_0^1(\Omega).$$

Thus w_1 is an eigenfunction associated with λ_1.

Proof. Let w_1 be a minimizer with $\|w_1\|_{L^2} = 1$. For any test function $\varphi \in C_c^\infty(\Omega)$, define

$$j(t) = \int_\Omega |\nabla(w_1 + t\varphi)|^2\,dx - \lambda_1 \int_\Omega (w_1 + t\varphi)^2\,dx, \quad t \in \mathbb{R}.$$

Since w_1 minimizes $\int |\nabla u|^2$ under the constraint $\int u^2 = 1$, we have $j(t) \ge 0$ for all sufficiently small t, with $j(0) = 0$. Differentiating at $t = 0$ yields

$$j'(0) = 2\int_\Omega \nabla w_1 \cdot \nabla \varphi\,dx - 2\lambda_1 \int_\Omega w_1 \varphi\,dx = 0.$$

By density, this identity holds for all $\varphi \in H_0^1(\Omega)$. □

Remarque 4.11. This shows that eigenvalues arise naturally as critical points of an energy functional under a constraint. This is the starting point of the *variational approach* to partial differential equations.

Proposition 4.11.1 (Minimality of λ_1)**.** *Let λ be any eigenvalue of $-\Delta$ on Ω with Dirichlet condition. Then*

$$\lambda_1 \le \lambda.$$

Proof. Let u be an eigenfunction for λ, normalized so that $\|u\|_{L^2} = 1$. Then $\int_\Omega |\nabla u|^2 = \lambda$. By the characterization of λ_1 as the minimum of $\int |\nabla v|^2$ over $\|v\|_{L^2} = 1$, we obtain $\lambda_1 \le \lambda$. □

Remarque 4.12. This explains why λ_1 is called the *first* eigenvalue: it is the smallest one, and it governs many qualitative properties (stability, decay, etc.).

Proposition 4.12.1 (Positivity and simplicity of λ_1)**.** *(i) The first eigenfunction w_1 can be chosen to be strictly positive in Ω.*

(ii) The eigenspace associated with λ_1 is one-dimensional.

Proof. Positivity. Let w_1 be a minimizer. Then $|w_1| \in H_0^1(\Omega)$ and $|\nabla|w_1|| = |\nabla w_1|$ almost everywhere (by the chain rule for weak derivatives). Hence

$$\int_\Omega |\nabla|w_1||^2\,dx = \int_\Omega |\nabla w_1|^2\,dx = \lambda_1 \int_\Omega w_1^2\,dx = \lambda_1 \int_\Omega |w_1|^2\,dx.$$

Thus $|w_1|$ also achieves the minimum and satisfies the same eigenvalue equation. By the strong maximum principle (or Hopf's lemma), a nonnegative eigenfunction that is not identically zero must be strictly positive in Ω. Therefore we can take $w_1 \geq 0$ and, in fact, $w_1 > 0$ in Ω.

Simplicity. Suppose u and v are two eigenfunctions associated with λ_1. If they were linearly independent, one could find a nontrivial linear combination that changes sign (for example, $u - cv$ for some c). This would be a sign-changing eigenfunction for λ_1, contradicting the fact that the first eigenfunction does not change sign (by the maximum principle). Hence the eigenspace is one-dimensional. □

Remarque 4.13. The positivity of the first eigenfunction has important consequences: it allows the use of the maximum principle and justifies the simplicity of λ_1.

Proposition 4.13.1 (Poincaré inequality and optimal constant)**.** *For any $u \in H_0^1(\Omega)$, we have the optimal Poincaré inequality*

$$\int_\Omega u^2\,dx \leq \frac{1}{\lambda_1}\int_\Omega |\nabla u|^2\,dx,$$

where λ_1 is the first Dirichlet eigenvalue of $-\Delta$. The constant $1/\lambda_1$ is optimal and is achieved by the first eigenfunction w_1.

Proof. From the characterization of λ_1, for any $u \in H_0^1(\Omega)$ with $u \neq 0$,

$$\lambda_1 \leq \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \quad \Longrightarrow \quad \int_\Omega u^2 \leq \frac{1}{\lambda_1}\int_\Omega |\nabla u|^2.$$

For $u = w_1$, we have equality because $\int_\Omega |\nabla w_1|^2 = \lambda_1 \int_\Omega w_1^2$. Thus the constant is optimal. □

Remarque 4.14. This result establishes a precise link between spectral theory and functional inequalities. The constant $1/\lambda_1$ cannot be improved.

Proposition 4.14.1 (Comparison principle for eigenvalues)**.** *Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ be two bounded open sets. Denote by $\lambda_1(\Omega_1)$ and $\lambda_1(\Omega_2)$ the first eigenvalues of $-\Delta$ with Dirichlet boundary conditions on Ω_1 and Ω_2, respectively. Then*

$$\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1).$$

In other words, the first eigenvalue decreases as the domain expands.

Proof. Extend any function $u \in H_0^1(\Omega_1)$ by zero to Ω_2. This extension $\tilde{u}$ belongs to $H_0^1(\Omega_2)$. Then

$$\lambda_1(\Omega_2) = \min_{v \in H_0^1(\Omega_2)\setminus\{0\}} \frac{\int_{\Omega_2} |\nabla v|^2\, dx}{\int_{\Omega_2} v^2\, dx} \leq \frac{\int_{\Omega_2} |\nabla \tilde{u}|^2\, dx}{\int_{\Omega_2} \tilde{u}^2\, dx} = \frac{\int_{\Omega_1} |\nabla u|^2\, dx}{\int_{\Omega_1} u^2\, dx}$$

for any $u \in H_0^1(\Omega_1) \setminus \{0\}$. Taking the infimum over u gives $\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1)$. □

Remarque 4.15. This monotonicity property is very useful for estimating eigenvalues on complicated domains by comparing them with simpler ones.

Exemple 4.16 (The first eigenvalue in 1D)**.** Let $\Omega = (0,1) \subset \mathbb{R}$. The eigenvalue problem

$$-u'' = \lambda u, \quad u(0) = u(1) = 0$$

has solutions $u_k(x) = \sin(k\pi x)$ with eigenvalues $\lambda_k = k^2\pi^2$ for $k \in \mathbb{N}^*$. The first eigenvalue is

$$\lambda_1 = \pi^2,$$

and the corresponding eigenfunction is $w_1(x) = \sin(\pi x) > 0$ for $x \in (0,1)$.

The Rayleigh quotient characterization gives:

$$\pi^2 = \min_{u \in H_0^1(0,1)\setminus\{0\}} \frac{\int_0^1 (u')^2\, dx}{\int_0^1 u^2\, dx},$$

and the minimum is achieved by $u(x) = \sin(\pi x)$.

Exemple 4.17 (The first eigenvalue in 2D for a rectangle)**.** Let $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$. Using separation of variables, the eigenfunctions are

$$u_{k,\ell}(x,y) = \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{\ell\pi y}{b}\right), \quad k, \ell \in \mathbb{N}^*,$$

with eigenvalues

$$\lambda_{k,\ell} = \pi^2 \left(\frac{k^2}{a^2} + \frac{\ell^2}{b^2}\right).$$

The first eigenvalue is

$$\lambda_1 = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

and the corresponding eigenfunction is $w_1(x,y) = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) > 0$ in Ω.

Exercice 4.1.
1. Compute λ_1 for $\Omega = (0,L) \subset \mathbb{R}$ and verify the Rayleigh quotient characterization directly.
2. Show that the minimizer of the Rayleigh quotient is unique up to scalar multiplication.
3. Let $\Omega = B(0,R) \subset \mathbb{R}^N$ be a ball of radius R. Using the comparison principle, show that $\lambda_1(B(0,R)) = \frac{\mu_1}{R^2}$, where $\mu_1 = \lambda_1(B(0,1))$.

4.3 Exercises on the Maximum Principle and Eigenvalue Problems

Exercice 4.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$-\Delta u + c(x)u \geq 0 \quad \text{in } \Omega,$$

with $c(x) \geq 0$ in Ω. Prove that if u attains a nonpositive minimum at an interior point, then u is constant. Deduce the strong maximum principle for such operators.

Solution:

Suppose u attains a nonpositive minimum $m \leq 0$ at an interior point $x_0 \in \Omega$. Define $v = u - m \geq 0$ in Ω, with $v(x_0) = 0$. Then v satisfies

$$-\Delta v + c(x)v = -\Delta u + c(x)u - c(x)m \geq -c(x)m \geq 0,$$

since $m \leq 0$ and $c(x) \geq 0$. Thus $-\Delta v + c(x)v \geq 0$ in Ω and $v \geq 0$ with $v(x_0) = 0$. By the strong maximum principle (Hopf's lemma), $v \equiv 0$ in Ω. Hence $u \equiv m$ is constant. The classical strong maximum principle follows by taking $c \equiv 0$.

Exercice 4.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$-\Delta u = \lambda_1 u \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

where $\lambda_1 > 0$ is the first eigenvalue. Show that u cannot change sign in Ω unless $u \equiv 0$.

Solution:

Assume u changes sign. Then the positive part $u^+ = \max(u, 0)$ is nonnegative and vanishes on part of Ω. Since u satisfies the eigenvalue equation, the maximum principle implies that u cannot have an interior zero unless $u \equiv 0$. More rigorously, suppose there exists $x_0 \in \Omega$ with $u(x_0) > 0$ and another point with $u(x_1) < 0$. By continuity, there exists a nodal set where $u = 0$. Consider a connected component of $\{u > 0\}$. On this component, $-\Delta u = \lambda_1 u > 0$, so u is superharmonic? Actually, the strong maximum principle applied to $-u$ shows that if $u \geq 0$ and satisfies $-\Delta u = \lambda_1 u$ with $\lambda_1 > 0$, then either $u > 0$ or $u \equiv 0$. Therefore u cannot take both positive and negative values unless it is identically zero. Hence any nontrivial eigenfunction for λ_1 has constant sign.

Exercice 4.4. Let $\Omega = (0, \pi) \subset \mathbb{R}$. Solve explicitly the eigenvalue problem

$$-u'' = \lambda u, \quad u(0) = u(\pi) = 0.$$

Determine all eigenvalues and eigenfunctions. Verify directly that the first eigenfunction is positive and that $\lambda_1 = 1$.

Solution:

The general solution of $u'' + \lambda u = 0$ depends on λ:

- If $\lambda < 0$: $u(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$. The boundary conditions force $A = B = 0$.
- If $\lambda = 0$: $u(x) = Ax + B$. Boundary conditions give $u \equiv 0$.

- If $\lambda > 0$: $u(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. From $u(0) = 0$ we get $A = 0$. From $u(\pi) = 0$ we get $B\sin(\sqrt{\lambda}\pi) = 0$, so $\sqrt{\lambda}\pi = k\pi$ for $k \in \mathbb{N}^*$. Hence $\lambda_k = k^2$ and $u_k(x) = B\sin(kx)$.

  The eigenvalues are $\lambda_k = k^2$ with eigenfunctions $u_k(x) = \sin(kx)$. The first eigenvalue is $\lambda_1 = 1$ with eigenfunction $u_1(x) = \sin x > 0$ on $(0, \pi)$.

Exercice 4.5. Use the Rayleigh quotient to prove the Poincaré inequality on $\Omega = (0, \pi)$:

$$\int_0^\pi u^2\,dx \le \int_0^\pi (u')^2\,dx, \quad \forall u \in H_0^1(0,\pi),$$

and show that the constant 1 is optimal.

Solution:

By Proposition 4.8.1, $\lambda_1 = 1$ is the minimum of the Rayleigh quotient. Hence for any $u \in H_0^1(0,\pi) \setminus \{0\}$,

$$1 \le \frac{\int_0^\pi (u')^2\,dx}{\int_0^\pi u^2\,dx} \quad \Longrightarrow \quad \int_0^\pi u^2\,dx \le \int_0^\pi (u')^2\,dx.$$

For $u(x) = \sin x$, we have equality because $\int_0^\pi \sin^2 x\,dx = \pi/2$ and $\int_0^\pi \cos^2 x\,dx = \pi/2$. Thus the constant 1 is optimal.

Exercice 4.6. Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ be bounded domains. Using the variational characterization, prove that

$$\lambda_1(\Omega_2) \le \lambda_1(\Omega_1).$$

Solution:

Let $u_1 \in H_0^1(\Omega_1)$ be a first eigenfunction on Ω_1, normalized by $\|u_1\|_{L^2(\Omega_1)} = 1$. Extend u_1 by zero to Ω_2; call this extension $\tilde{u}_1 \in H_0^1(\Omega_2)$. Then

$$\lambda_1(\Omega_2) = \inf_{v \in H_0^1(\Omega_2)\setminus\{0\}} \frac{\int_{\Omega_2} |\nabla v|^2}{\int_{\Omega_2} v^2} \le \frac{\int_{\Omega_2} |\nabla \tilde{u}_1|^2}{\int_{\Omega_2} \tilde{u}_1^2} = \frac{\int_{\Omega_1} |\nabla u_1|^2}{\int_{\Omega_1} u_1^2} = \lambda_1(\Omega_1).$$

Hence $\lambda_1(\Omega_2) \le \lambda_1(\Omega_1)$.

Exercice 4.7. Consider the eigenvalue problem with a potential:

$$-\Delta u + V(x)u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

where $V \in L^\infty(\Omega)$ is a nonnegative function. Show that the first eigenvalue $\lambda_1(V)$ satisfies

$$\lambda_1(V) \ge \lambda_1(0),$$

where $\lambda_1(0)$ is the first eigenvalue of $-\Delta$ on Ω (with $V \equiv 0$).

Solution:

The Rayleigh quotient for this problem is

$$R_V(u) = \frac{\int_\Omega |\nabla u|^2\,dx + \int_\Omega V u^2\,dx}{\int_\Omega u^2\,dx}.$$

Since $V \geq 0$, we have $R_V(u) \geq R_0(u)$ for all $u \in H_0^1(\Omega) \setminus \{0\}$. Taking the infimum,

$$\lambda_1(V) = \inf_{u \neq 0} R_V(u) \geq \inf_{u \neq 0} R_0(u) = \lambda_1(0).$$

Thus adding a nonnegative potential increases the first eigenvalue.

Exercice 4.8. Let $\Omega = B(0, R) \subset \mathbb{R}^N$ be a ball of radius R. Using the comparison principle and scaling, show that

$$\lambda_1(\Omega) = \frac{\mu_1}{R^2},$$

where $\mu_1 = \lambda_1(B(0, 1))$.

Solution:

Consider the scaling transformation $x = Ry$ with $y \in B(0, 1)$. If $u(x)$ is an eigenfunction on $B(0, R)$ with eigenvalue λ, define $v(y) = u(Ry)$. Then

$$-\Delta_y v(y) = -R^2 \Delta_x u(Ry) = R^2 \lambda u(Ry) = (R^2 \lambda) v(y).$$

Thus v is an eigenfunction on $B(0, 1)$ with eigenvalue $R^2\lambda$. Conversely, if v is an eigenfunction on $B(0, 1)$ with eigenvalue μ, then $u(x) = v(x/R)$ is an eigenfunction on $B(0, R)$ with eigenvalue μ/R^2. Therefore

$$\lambda_1(B(0, R)) = \frac{\lambda_1(B(0, 1))}{R^2} = \frac{\mu_1}{R^2}.$$

Exercice 4.9. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Prove that the first eigenfunction of $-\Delta$ with Dirichlet condition is unique up to scalar multiplication. (Hint: use the fact that it has constant sign and apply the maximum principle.)

Solution:

Let u and v be two eigenfunctions for λ_1, both normalized in L^2. By Proposition 4.12.1, we may assume $u > 0$ and $v > 0$ in Ω (or replace them by their absolute values). Consider the function $w = u/v$ on Ω. Both u and v satisfy $-\Delta u = \lambda_1 u$ and $-\Delta v = \lambda_1 v$. A direct computation shows that

$$v^2 \nabla w = v \nabla u - u \nabla v.$$

Taking the divergence and using the eigenvalue equations yields $\operatorname{div}(v^2 \nabla w) = 0$ in Ω. Since w is constant on $\partial\Omega$ (both u and v vanish), by the maximum principle w is constant. Hence $u = cv$ for some constant c. Therefore the eigenspace is one-dimensional.

Exercice 4.10. Let $\Omega = (0, 1) \times (0, 1)$ be the unit square. Compute the first eigenvalue and eigenfunction explicitly. Verify that the eigenfunction is positive.

Solution:

Using separation of variables, look for solutions of the form $u(x, y) = X(x)Y(y)$. The equation $-\Delta u = \lambda u$ becomes

$$-\frac{X''}{X} - \frac{Y''}{Y} = \lambda.$$

Hence there exist constants α, β such that $-X'' = \alpha X$, $-Y'' = \beta Y$, with $\alpha + \beta = \lambda$. The Dirichlet conditions give $X(0) = X(1) = 0$ and $Y(0) = Y(1) = 0$. Thus

$$X_k(x) = \sin(k\pi x), \quad \alpha_k = k^2\pi^2, \qquad Y_\ell(y) = \sin(\ell\pi y), \quad \beta_\ell = \ell^2\pi^2,$$

for $k, \ell \in \mathbb{N}^*$. The eigenvalues are $\lambda_{k,\ell} = \pi^2(k^2 + \ell^2)$. The smallest eigenvalue is for $k = \ell = 1$:

$$\lambda_1 = 2\pi^2, \qquad u_1(x,y) = \sin(\pi x)\sin(\pi y) > 0 \text{ on } (0,1)^2.$$

Exercice 4.11. Use the maximum principle to prove that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

with $f \geq 0$ in Ω and $f \not\equiv 0$, then $u > 0$ in Ω.

Solution:

Since $f \geq 0$, we have $-\Delta u \geq 0$ in Ω. By the weak maximum principle, u attains its maximum on $\partial\Omega$, where $u = 0$. Hence $u \leq 0$ in Ω. Actually, this gives $u \leq 0$. Wait, we need to be careful: $-\Delta u \geq 0$ means $\Delta u \leq 0$, so u is superharmonic. The maximum principle for superharmonic functions says that the minimum is attained on the boundary. Since $u = 0$ on $\partial\Omega$, we have $u \geq 0$ in Ω. Moreover, if $u(x_0) = 0$ at some interior point, by the strong maximum principle, $u \equiv 0$, which would imply $f \equiv 0$, contradiction. Hence $u > 0$ in Ω.

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