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Master 1, EDP (SI) : 2025/2026 — B. Bougherara

Course : Distributions theory

Exam : Jan 11^{th}, 2025, Duration : 2h

Exercice 1.

1. Recall the definition of the space of test functions $\mathcal{D}(\Omega)$.
2. State precisely what it means for a sequence $(\varphi_n)_n$ to converge to 0 in $\mathcal{D}(\Omega)$.
3. Let $f(x) = \ln|x|$, $x \in \mathbb{R}$.
   - Show that f defines a distribution on $\mathbb{R}$, that is, $f \in \mathcal{D}'(\mathbb{R})$.
   - Compute the distributional derivative f'.

Exercice 2.

1. Let $T \in \mathcal{D}'(\mathbb{R}^N)$. Give the definition of the distributional derivative $\partial_i T$, $1 \leq i \leq N$.
2. Let $f \in \mathcal{C}^\infty(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$.
   - Define the product fT in the sense of distributions.
   - Let $T = \mathrm{pv}\left(\frac{1}{x}\right)$ (Cauchy principal value). Compute explicitly xT and x^2T in $\mathcal{D}'(\mathbb{R})$.
3. Let $(f_n)_n$ be the sequence of functions defined by

$$f_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}, \qquad x \in \mathbb{R}.$$

   - Show that $(f_n)_n$ converges almost everywhere to 0 on $\mathbb{R}$.
   - Compute the limit of $(f_n)_n$ in the sense of distributions.
   - Deduce the following distributional limit :

$$\lim_{n\to\infty} \frac{n^3 x}{\sqrt{\pi}} e^{-n^2x^2}.$$

Exercice 3.

1. Define the Schwartz space $\mathcal{S}(\mathbb{R})$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R})$.
2. Let $\varphi_n(x) = \frac{1}{n}e^{-x^2}$, $x \in \mathbb{R}$. Show that $\varphi_n \in \mathcal{S}(\mathbb{R})$ for all $n \in \mathbb{N}$.
3. Consider the function $g(x) = e^{x^2}$, $x \in \mathbb{R}$. Is g a tempered distribution ? Justify your answer. (You may use the fact that $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R})$.)
4. Recall the definition of the Fourier transform $\widehat{T}$ of a tempered distribution T. and prove that $\widehat{T'} = (ix)\,\widehat{T}$
5. Let $f \in \mathcal{D}(\mathbb{R})$ and $T \in \mathcal{S}'(\mathbb{R})$. Recall the definition of $T \star f$ and prove that $\widehat{T \star f} = \hat{f}\hat{T}$.

Correction

Solution Ex01

1. **Space of test functions**

   Let $\Omega \subset \mathbb{R}^N$ be an open set. The space of test functions $\mathcal{D}(\Omega)$ is defined by
   $$\boxed{\mathcal{D}(\Omega) = C_c^\infty(\Omega),}$$
   the space of infinitely differentiable functions with compact support contained in Ω.

2. **Convergence in $\mathcal{D}(\Omega)$**

   A sequence $(\varphi_n)_n$ converges to 0 in $\mathcal{D}(\Omega)$ if and only if there exists a compact set $K \subset \Omega$ such that
   $$\operatorname{supp}(\varphi_n) \subset K \quad \forall n,$$
   and for every multi-index α,
   $$\sup_{x\in K} \big|\partial^\alpha \varphi_n(x)\big| \longrightarrow 0 \quad \text{as } n \to \infty.$$

3. **The function $f(x) = \ln|x|$**

   - **Verification that f defines a distribution**

     The function $f(x) = \ln|x|$ is locally integrable on $\mathbb{R}$. Indeed, for any $R > 0$,
     $$\int_{-R}^{R} |\ln|x||\, dx = 2\int_0^R |\ln x| dx < +\infty,$$
     Therefore,
     $$f \in L^1_{\text{loc}}(\mathbb{R}).$$
     It follows that f defines a distribution $T_f \in \mathcal{D}'(\mathbb{R})$ by
     $$\boxed{\langle T_f, \varphi\rangle = \int_{\mathbb{R}} \ln|x|\, \varphi(x)\, dx, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}).}$$

   - **Distributional derivative of f**

     Let $\varphi \in \mathcal{D}(\mathbb{R})$. By definition,
     $$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{\mathbb{R}} \ln|x|\, \varphi'(x)\, dx = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \ln|x|\, \varphi'(x)\, dx$$
     Using integration by parts on $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$ and the compact support of φ, we obtain
     $$\langle f', \varphi\rangle = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{1}{x}\, \varphi(x)\, dx.$$
     Hence,
     $$\boxed{f' = \operatorname{pv}\left(\frac{1}{x}\right) \quad \text{in } \mathcal{D}'(\mathbb{R}).}$$

Solution Ex02

1. **Distributional derivative**

   Let $T \in \mathcal{D}'(\mathbb{R}^N)$ and $1 \le i \le N$. The distributional derivative $\partial_i T$ is the distribution defined by

$$\boxed{\langle \partial_i T, \varphi \rangle := -\langle T, \partial_i \varphi \rangle, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).}$$

2. **Multiplication of distributions**

   - Let $f \in \mathcal{C}^\infty(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$. The product fT is defined by

$$\boxed{\langle fT, \varphi \rangle := \langle T, f\varphi \rangle, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}).}$$

   - Let $T = \mathrm{pv}\left(\dfrac{1}{x}\right)$. Recall that

$$\left\langle \mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx.$$

     Computation of xT :

$$\langle xT, \varphi \rangle = \left\langle \mathrm{pv}\left(\frac{1}{x}\right), x\varphi \right\rangle = \int_{\mathbb{R}} \varphi(x)\, dx.$$

     Hence,

$$\boxed{x\,\mathrm{pv}\left(\frac{1}{x}\right) = 1} \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

     Computation of x^2T :

$$\langle x^2T, \varphi \rangle = \left\langle \mathrm{pv}\left(\frac{1}{x}\right), x^2\varphi \right\rangle = \int_{\mathbb{R}} x\varphi(x)\, dx.$$

     Therefore,

$$\boxed{x^2\,\mathrm{pv}\left(\frac{1}{x}\right) = x} \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

3. **Gaussian sequence and limits**

   Let

$$f_n(x) = \frac{n}{\sqrt{\pi}}\, e^{-n^2x^2}.$$

   - **Almost everywhere convergence**

     For $x \neq 0$, $n^2x^2 \to +\infty$, hence $e^{-n^2x^2} \to 0$ and

$$f_n(x) \to 0.$$

     At $x = 0$, $f_n(0) \to +\infty$, but this is a single point. Thus,

$$\boxed{f_n \to 0 \quad \text{a.e. on } \mathbb{R}.}$$

- **Limit in the sense of distributions**
  Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then
$$\langle f_n, \varphi \rangle = \int_{\mathbb{R}} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \varphi(x)\, dx.$$
  With the change of variables $y = nx$,
$$\langle f_n, \varphi \rangle = \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-y^2} \varphi\Big(\frac{y}{n}\Big)\, dy.$$
  Since $\varphi(y/n) \to \varphi(0)$ and $\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} dy = 1$, we obtain by Lebesgue theorem
$$\lim_{n\to+\infty} \langle f_n, \varphi \rangle = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \lim_{n\to+\infty} e^{-y^2} \varphi(y/n) dy = \varphi(0).$$
  Hence,
$$\boxed{f_n \longrightarrow \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}).}$$
- **Deduction**
  Observe that
$$\frac{n^3 x}{\sqrt{\pi}} e^{-n^2x^2} = -\frac{1}{2} \frac{d}{dx} \left(\frac{n}{\sqrt{\pi}} e^{-n^2x^2} \right) = -\frac{1}{2} f_n'(x).$$
  Since $f_n \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$, we have $f_n' \to \delta_0'$. Therefore,
$$\boxed{\frac{n^3 x}{\sqrt{\pi}} e^{-n^2x^2} \longrightarrow -\frac{1}{2}\, \delta_0' \quad \text{in } \mathcal{D}'(\mathbb{R}).}$$

Solution Ex 03

1. **Schwartz space and tempered distributions**
   The Schwartz space $\mathcal{S}(\mathbb{R})$ is defined as the set of all functions $\varphi \in C^\infty(\mathbb{R})$ such that for all integers $k, m \geq 0$,
$$\sup_{x\in\mathbb{R}} |x^k \partial^m \varphi(x)| < +\infty.$$
   The space of tempered distributions $\mathcal{S}'(\mathbb{R})$ is the topological dual of $\mathcal{S}(\mathbb{R})$, that is, the space of continuous linear forms on $\mathcal{S}(\mathbb{R})$.
2. **The sequence** $\varphi_n(x) = \frac{1}{n} e^{-x^2}$
   Since $e^{-x^2} \in \mathcal{C}^\infty(\mathbb{R})$, and we have for all $k, m \in \mathcal{N}$,
$$\sup_{x\in\mathbb{R}} |x^k \varphi_n^{(m)}(x)| = \frac{1}{n} \sup_{x\in\mathbb{R}} |x^k P_{2m}(x) e^{-x^2}(x)| < +\infty.$$
   where P_{2m} is a polynomial of degree $2m$. Hence $\varphi_n \in \mathcal{S}(\mathbb{R})$.

3. **The function** $g(x) = e^{x^2}$

   Assume that g defines a tempered distribution. Then the functional

$$\varphi \mapsto \int_{\mathbb{R}} e^{x^2} \varphi(x)\, dx$$

   must be continuous on $\mathcal{S}(\mathbb{R})$.

   Consider the sequence $\varphi_n(x) = \frac{1}{n} e^{-x^2}$. It is clear that $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R})$. However,

$$\langle g, \varphi_n \rangle = \int_{\mathbb{R}} e^{x^2} \frac{1}{n} e^{-x^2}\, dx = \frac{1}{n} \int_{\mathbb{R}} 1\, dx = +\infty,$$

   which is impossible.

   Hence g does *not* define a tempered distribution. Therefore,

$$\boxed{e^{x^2} \notin \mathcal{S}'(\mathbb{R}).}$$

4. **Fourier transform of a tempered distribution**

   - Let $T \in \mathcal{S}'(\mathbb{R})$. The Fourier transform $\widehat{T}$ is defined by

$$\boxed{\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}),}$$

     where

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} \varphi(x)\, dx.$$

   - Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$\langle \widehat{T'}, \varphi \rangle = \langle T', \widehat{\varphi} \rangle = -\langle T, (\widehat{\varphi})' \rangle.$$

     Since $(\widehat{\varphi})'(\xi) = \widehat{(-i\xi\varphi)}(\xi)$, we obtain

$$\langle \widehat{T'}, \varphi \rangle = \langle i\xi \widehat{T}, \varphi \rangle.$$

     Thus,

$$\boxed{\widehat{T'} = i\xi \widehat{T} \quad \text{in } \mathcal{S}'(\mathbb{R}).}$$

5. Let $f \in \mathcal{D}(\mathbb{R})$ and $T \in \mathcal{S}'(\mathbb{R})$.
   Step 1 : Definition of the convolution $T * f$**.** The convolution of the tempered distribution T with the function f is defined by

$$\boxed{\langle T * f, \varphi \rangle = \langle T_x, \langle f(y), \varphi(x+y) \rangle \rangle \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}),}$$

   Computation of $\widehat{T * f}$**.**

Let $\varphi \in \mathcal{S}(\mathbb{R})$. By definition of the Fourier transform,

$$\langle \widehat{T * f}, \varphi \rangle = \langle T * f, \widehat{\varphi} \rangle.$$

Using the definition of the convolution and Fourier, we obtain

$$\begin{aligned}
\langle \widehat{T * f}, \varphi \rangle &= \langle T_x, \langle f(y), \hat{\varphi}(x+y) \rangle \rangle = \langle T_x, \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} e^{-i(x+y)z} \varphi(z) dz dy \rangle \\
&= \langle T_x, \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-iyz} f(y) dy e^{-ixz} \varphi(z) dz \rangle \\
&= \langle T_x, \int_{\mathbb{R}} \hat{f}(z) \varphi(z) e^{-ixz} dz \rangle = \langle T_x, \widehat{\hat{f}\varphi}(x) \rangle \\
&= \langle \hat{f} \hat{T}, \varphi \rangle.
\end{aligned}$$