MOHAMED BOUDIAF UNIVERSITY OF M'SILA

DEPARTMENT OF COMPUTER SCIENCE

Advanced Graph Theory

Chapter 1: Basic Concepts

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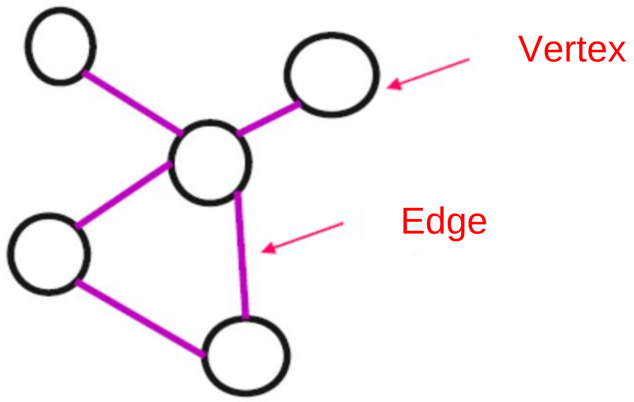
Definition of a Graph

- **A Graph is a collection of vertices connected by edges or arcs.**

- **We call a graph the pair G(X, U) such that:**
- **X = {x1, x2, ….., xn} is the set of vertices of the graph.**
- **U={u1, u2, …., un} is the set of arcs of the graph**
- **U⊂XxX**



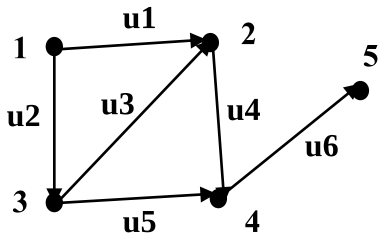
- **The term network:** is used to denote real systems (road network, electrical network, computer network, etc.).

- **The term Graph**: is used to denote a mathematical representation of a network.

But, generally:"**Network**" ≡ "**Graph**"

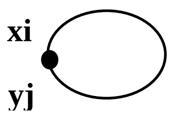
Properties of a graph:

- $n=|X|$ is called the order of the graph G.
- $m=|U|$ is the size of the graph (nb.arcs)
- a vertex x_i is represented by a point.
- An arc $U=(x_i, x_j)\in XxX$ is represented by an arrow or a line segment (depending on the type of graph, directed/undirected)



n=|X|=5 m=|U|=6

- If $x_i = x_j$ ==> u=(xi, xj) is represented by a loop (i.e. the two vertices are the same).



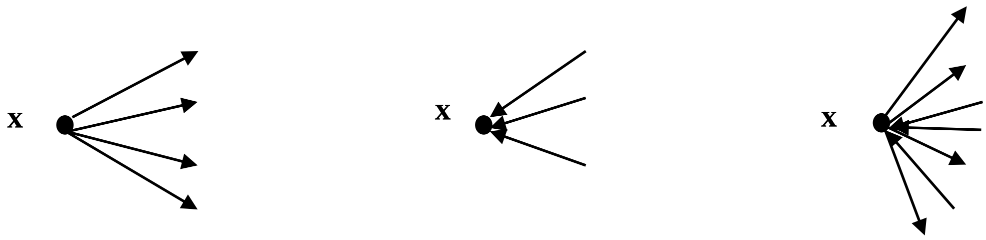
- The external half-degree of a vertex x is the number of arcs for which x is the initial endpoint or the number of outgoing arcs of x.

  We denote $d^+(x)$.
- The internal half-degree of the a vertex x is the number of arcs for which x is the terminal end or the number of arcs entering to x.

  We denote $d^-(x)$.

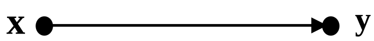
- The degree of the vertex x is the number of arcs for which x is the initial or the terminal end.

  We denote $d(x) = d^+(x) + d^-(x)$.

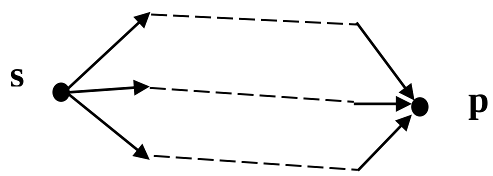


$d^+(x) = 4$ $d^-(x) = 3$ $d(x) = d^+(x) + d^-(x) = 7$

- If $(x, y) \in U$, then x is said to be the predecessor of y, y is the successor of x.



- $\Gamma^+(x)$: the set of successors of x.
- $\Gamma^-(x)$: the set of predecessors of x.
- A source vertex *s* is a vertex that has no predecessor ($\Gamma^-(s)=\phi$/ there are only outgoing arcs).
- A sink vertex *p* is a vertex that has no successor ($\Gamma^+(x) =\phi$) / there are only incoming arcs).
- An isolated vertex x is a vertex that has no neighbors (neither predecessor nor successor) (($\Gamma^+(x)$= ($\Gamma^-(x) =\phi$). So this is a source and sink vertex at the same time, it is also called an inaccessible vertex.



Example:

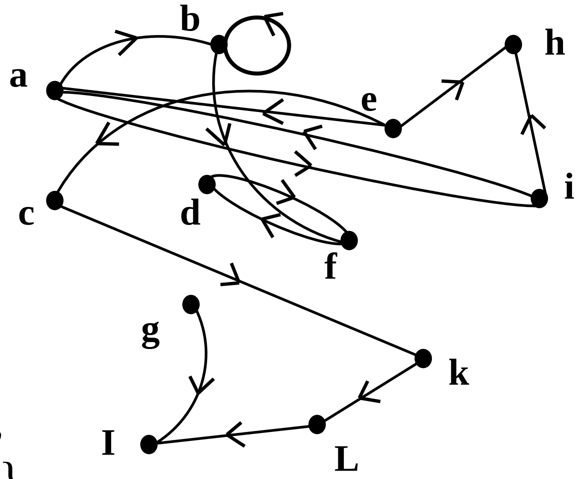
X={a, b, c, d, e, f, g, h, I, j, k, l}

U={(a, b), (a, i), (b, f), (f, d), (d, f),

(e, a), (e, h), (k, l), (e, c), (c, k), (b, b),

(g, j), (l, j)}

$\Gamma^{+}(f) = \{d\}, \Gamma^{+}(d) = \{f\}, \Gamma^{+}(b) = \{f, b\},$
$\Gamma^{+}(j) = \phi, \Gamma^{+}(e) = \{a, h, c\}, \Gamma^{-}(f) = \{b, d\},$
$\Gamma^{-}(e) = \phi, \Gamma^{-}(a) = \{i, e\}$



Examples of graphs:

- Graph representing a road network, where vertices represent intersections and arcs represent roads.
- Drinking Water Distribution Network AEP in a city.
- Computer network in a compagny.
- The international information network Internet.

Directed graph and undirected graph

Undirected graph

Lines: unoriented (*symmetrical*)

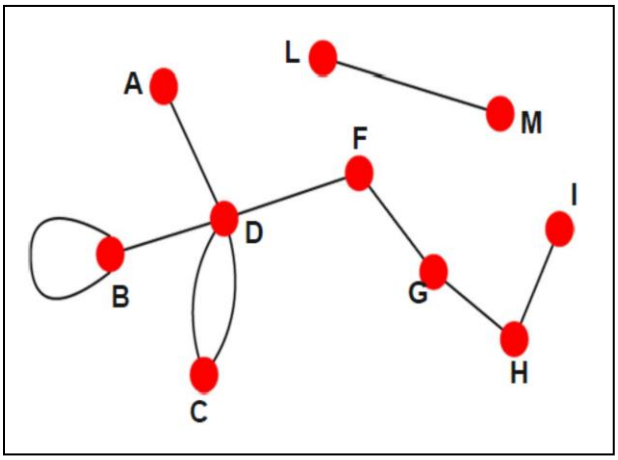
➔edges

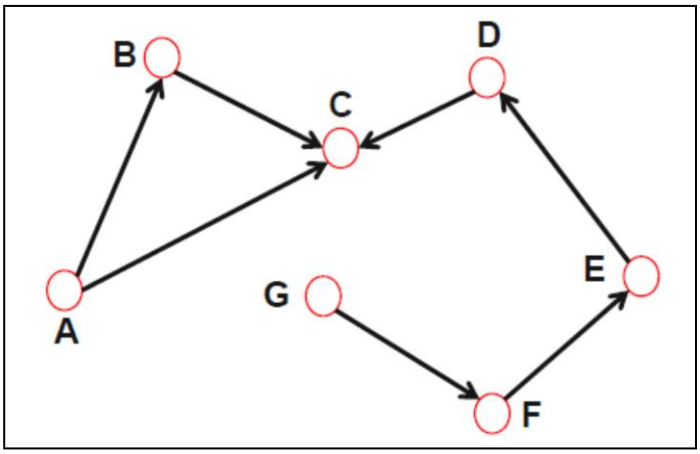
Directed graph

Lines: oriented

➔ *Arcs*

Undirected Graph **Di-Graph= Directed Graph**





Definition of a valued graph

A valued graph is a graph G(X, U, C) such that we associate with the graph G(X, U) a function F defined as follows:

$F: U \longrightarrow R$

F(u) = c is called the weight of the arc u, and we denote

c(x, y) Or c(u)

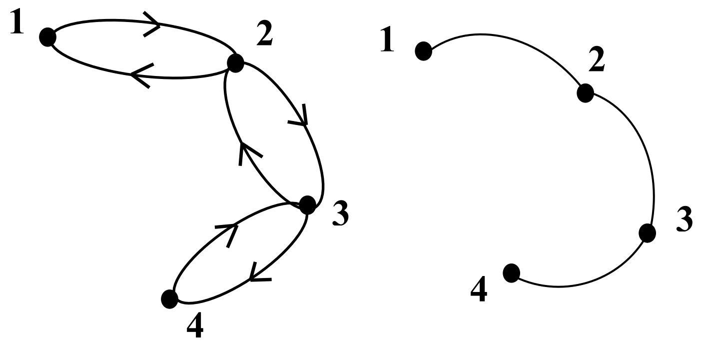
With:

- c(x, y): length of the road section (x, y).
- Capacity of the road section (x, y).
- Flow rate of a drinking water pipe.
- Travel price between cities x, y.

Types of graphs

1. Symmetric graph

A graph G(X, U) is said to be symmetric if ∀x, y∈X, if (x, y)∈U ===> (y, x) ∈U

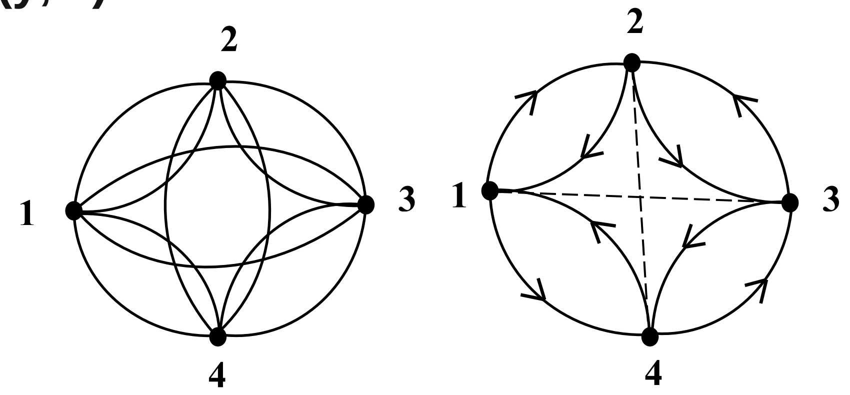


A symmetric graph is usually represented without orientation of the arcs. We talk about edges instead of arcs.

2.Complete graph

A graph G(X, U) is said to be complete if only if:

∀x, y∈X, (x, y)∈U and (y, x) ∈U



Complete graph **Graph not complete**

3. Simple graph

A graph G(X, U) is said to be simple if only if:

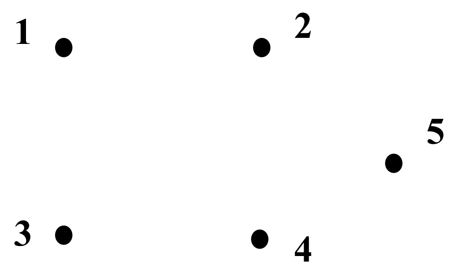
- **Does not contain any loops.**
- **$\forall x, y \in X, \exists$ at most $u=(x, y) \in U$**

4. Empty graph

A graph G(X, U) is said to be empty if there are no vertices, arcs or edges. ($X=\phi$, $U=\phi$).

5. Trivial graph

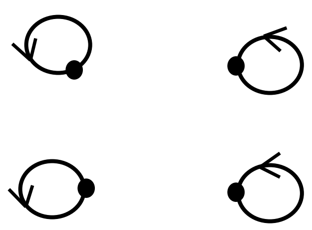
A graph G(X, U) is said to be trivial if there are vertices, but there are no arcs or edges. ($U=\phi$).



6. Reflexive graph

A graph G(X, U) is said to be reflexive, if there exists a loop at each vertex x of G.

- **$\forall x \in X \implies (x, x) \in U$**



7. Anti-symmetric graph

A graph G(X, U) is said to be anti-symmetric if:

- **$\forall x, y \in X$, if $(x, y) \in U$ ===> $(y, x) \notin U$**

8. Transitive graph

A graph G(X, U) is said to be transitive if:

- **$\forall x, y, z \in X$, if $(x, y) \in U$ and $(y, z) \in U$ ===> $(x, z) \in U$**

9. Inverse of a graph

The inverse of a graph G(X, U) is the graph G'(X, U') deduced from G by reversing the direction of its arcs.

OBS: the inverse of G' is G itself.

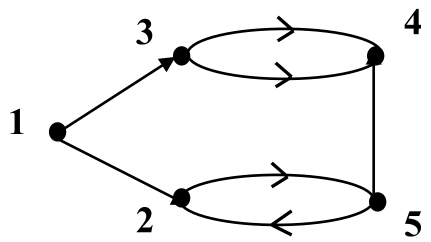
10. Complementary graph

The complementary graph of the graph G(X, U) is the graph G*(X, U*) such that:

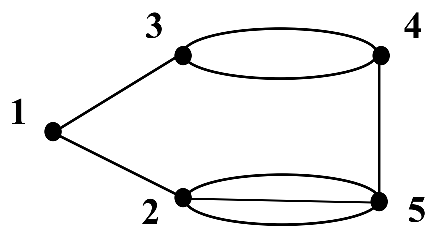
$(x, y) \in U$ ===> $(x, y) \notin U^*$

11. Multigraph

if G is a directed graph, x and y are two vertices of this graph, if x and y are linked by two arcs u1, u2 of the same direction, G is said to be a multigraph and the arc (x, y) is said to be a multiple arc.



a directed multiple graph



an undirected multiple graph

- **(3, 4) a multiple arc**
- **(2, 5) a symmetrical arc**

12. The bipartite graph

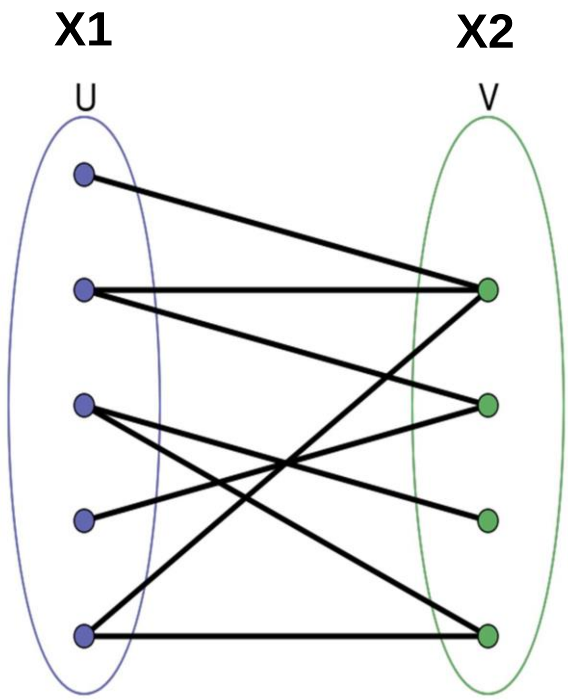
A bipartite graph (or bi-graph) is a graph whose set of vertices can be divided into two disjoint subsets of vertices X1 and X2 such that no two vertices in the same subset are adjacent (i.e., any arc of G has one endpoint in X1 and the other in X2).

And we note:

G(X, U) with: $X = X1 \cup X2$ and $X1 \cap X2 = \phi$

And if u=(x1, x2) and $x1 \in X1$ ===> $x2 \in X2$

OR $x1 \in X2$ ===> $x2 \in X1$

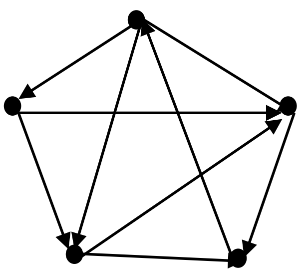


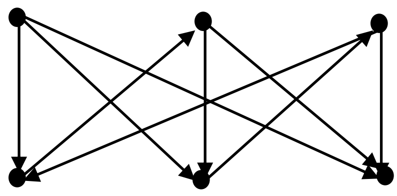
13. Planar graph

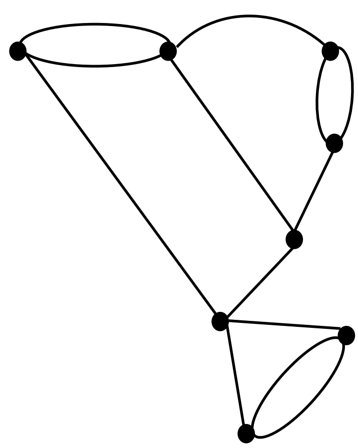
A graph G is said to be planar if it is possible to draw it in the plane in such a way that any two arcs do not intersect except at their ends.

This trace in the plane is called a planar representation..

Example:







G1, G2: two non-planar graphs

G3 a planar graph

14. Isomorphic graphs

Two graphs G(X, U), G'(X', U') are said to be isomorphic if there exists a bijection f:

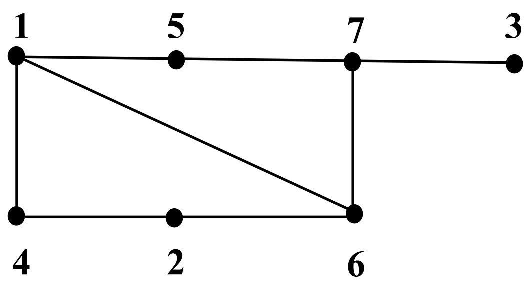
f: X ----→ X'

Such that: (x, y)$\in$U ⇔ (f(x), f(y))$\in$U'

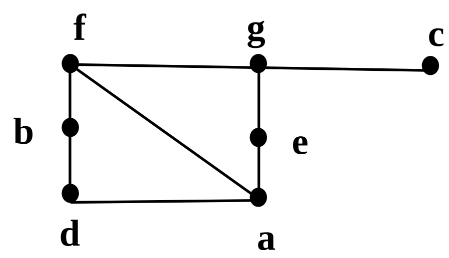
i.e. if the two vertices x, y are connected by an arc in G ===> the image of x and the image of y by f are also connected by an arc in G.'

Example 1:

Let the two graphs G, H be defined as follows:



(G)



(G')

And let a bijection f be defined as follows:

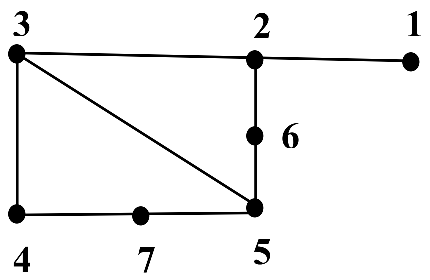
f: {1, 2, ..., 7} -------→ {a, b, ..., g}

With: f(1)=a, f(2)=b, ..., f(7)=g

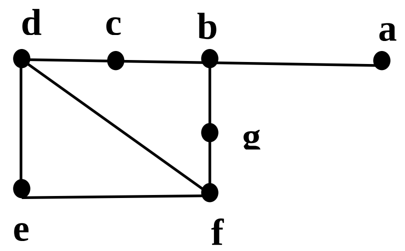
G and G' are isomorphic.

Example 2:

Let the two graphs G, H be defined as follows:



(G)



(H)

(G), (H) two undirected and non-isomorphic graphs.

Because (3, 5)$\in$U===>(f(3), f(5))=(c, e) $\notin$U'

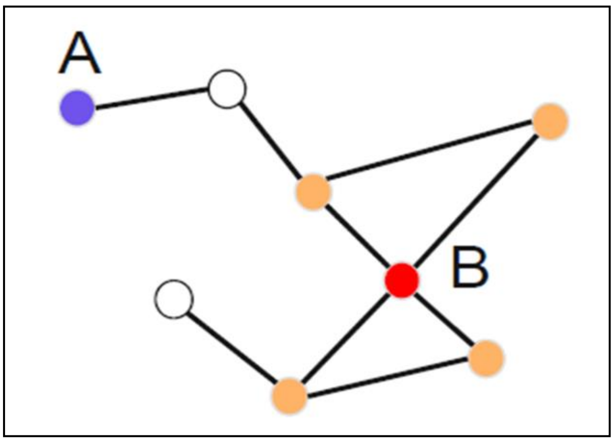
Degree of a vertex

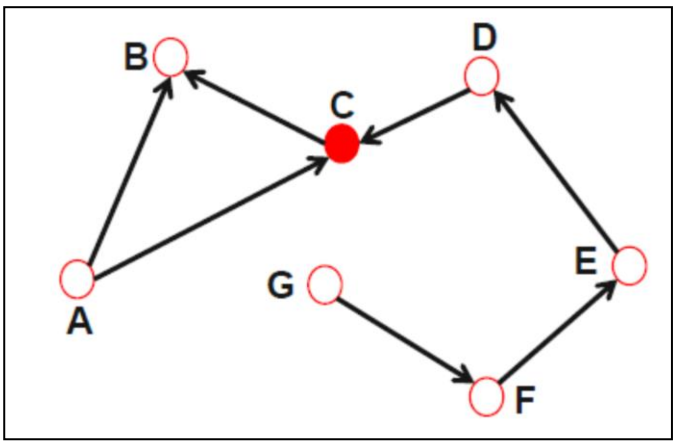
Undirected graph

Degree: the number of edges which connect vertices.

Directed/Oriented Graph

in a directed graph, we defines external degree and internal degree
The total degree of a vertex is the sum of the external degree and the internal degree.





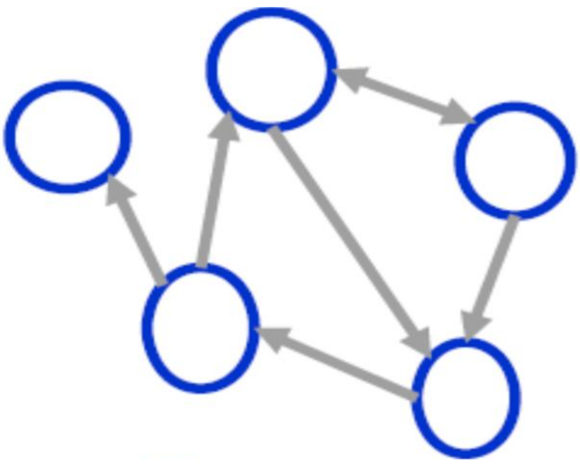
Degree distribution

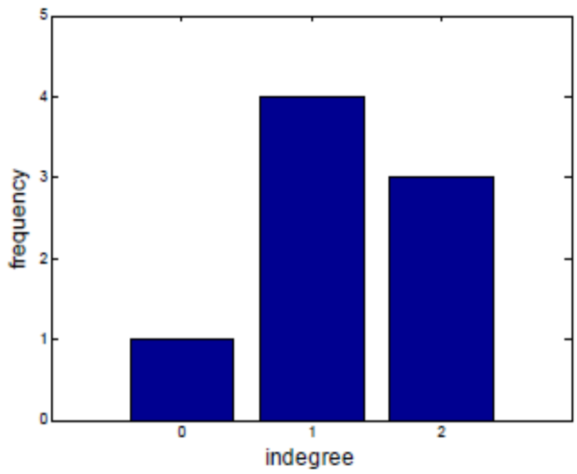
Summarizes the degrees of all vertices in the graph.

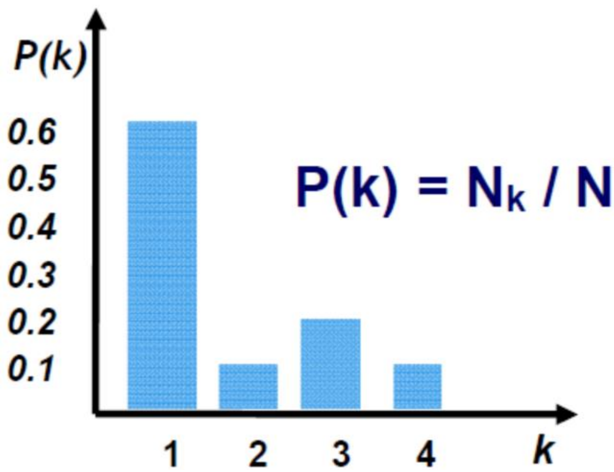
Two distribution methods are proposed:

1. Counting the frequencies of the vertices associated with each degree.

2. Calculating the probability P(k): that a chosen vertex arbitrarily possesses the degree k.

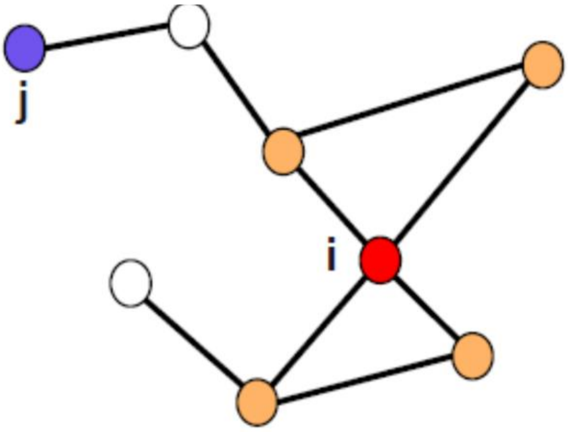




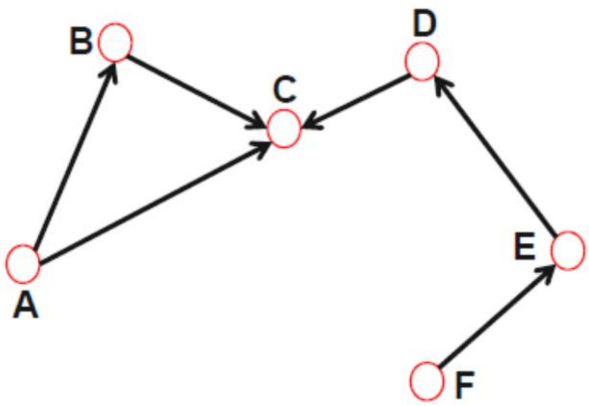


Average degree

Undirected graph



Directed graph



$$\langle k \rangle = \frac{1}{N}\sum_{i=1}^{N} k_i \qquad \langle k \rangle = \frac{2L}{N}$$

$$\langle k^{in} \rangle = \frac{1}{N}\sum_{i=1}^{N} k_i^{in}, \quad \langle k^{out} \rangle = \frac{1}{N}\sum_{i=1}^{N} k_i^{out}, \quad \langle k^{in} \rangle = \langle k^{out} \rangle$$

N: number of vertices in the graph.
L: number of (arcs/edges) in the graph

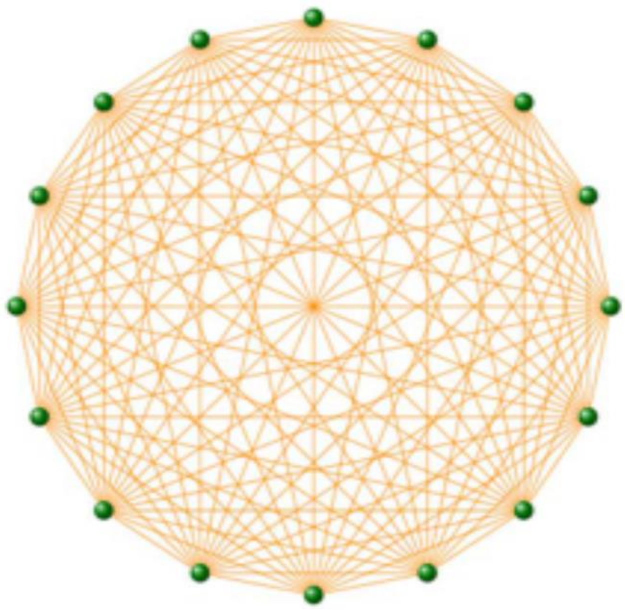
Density of a graph

■ **Number of connections that can exist between vertices:**

• **Directed graph**

Lmax = N*(N-1)

Since each of the N vertices can connect with (N-1) other vertices



• **Undirected graph**

Lmax = N*(N-1)/2

Since the edges are not oriented, then count each edge only once.

■ **Density value?**

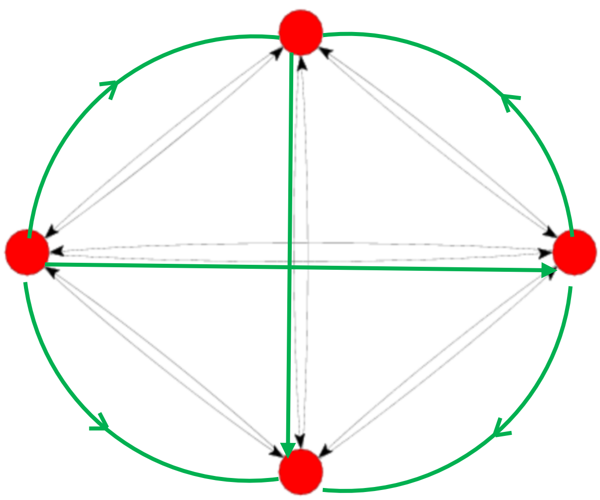
Density = L/ Lmax

• **For example:**

In this graph, we have:

N=4; L = Nb.lines = 6 (in green)
➔Lmax = 4*(4-1)=12

Among the 12 possible connections
Lmax , this graph has 6, giving a density of: 6/12 = 0.50



Definition of a path

We call a path of G(X, U) the sequence of vertices

$a_1, a_2, \ldots, a_p$ (without cut) such that:

- $(a_i, a_{i+1}) \in U$ with $i=1, 2, \ldots, p-1$.
- a_i is the initial endpoint of the path
- a_p is the terminal end of the path.

Definition of a simple path

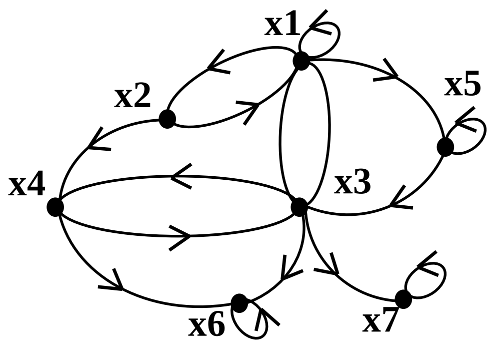
$a_1, a_2, \ldots, a_p$ is a simple path such that:

$\forall i \neq j, (a_i, a_{i+1}) \neq (a_j, a_{j+1})$ (i.e., an arc is taken only once).

Definition of an elementary path

is a path that only takes vertices once.

Examples:



x2x1x5x3x1x5x5x3x6 a path that is not simple, nor elementary.

x2x1x5x3x1x2x4 a simple path not elementary

x1x2x4x3x6 elementary path

Remark

If $a_1a_2a_3 \ldots.a_p$ is a path

- **a_1 is an ascendant of a_p**
- **a_p is a descendant of a_1**

Definition of a circuit

A circuit is a path $a_1a_2a_3 \ldots.a_p$ such that: $a_1=a_p$

i.e., the two initial and terminal ends are merged (a closed path).

Observations

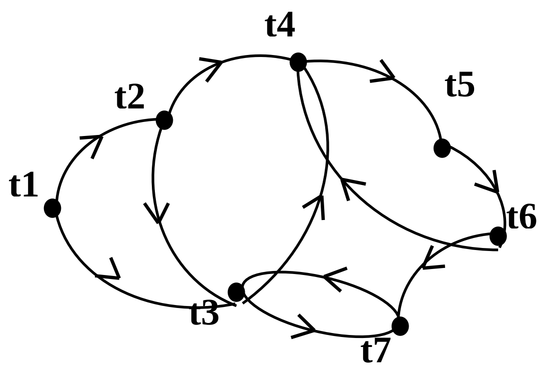
- **In the same way we can define a simple circuit, an elementary circuit.**
- **The notion of the circuit path takes into consideration the direction of the arcs in the graph.**
- **A simple circuit is a closed path that takes arcs only once.**
- **An elementary circuit is a closed path that takes vertices only once.**

Special paths and circuits

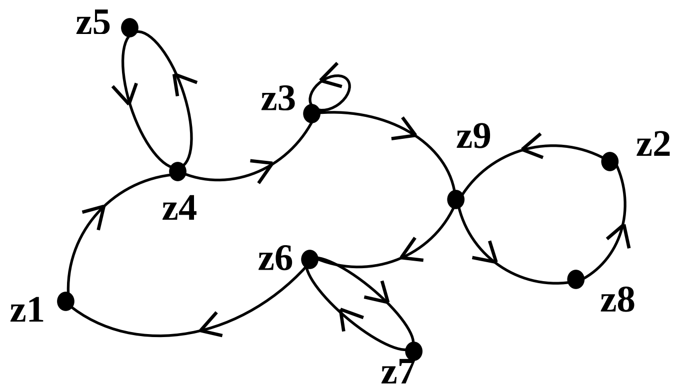
1. **A Hamiltonian path: is an elementary path that passes through all the vertices of the graph.**
2. **An Eulerian path: is a simple path that passes through all the arcs of the graph once and only once.**
3. **A Hamiltonian circuit: is an elementary circuit that passes through all the vertices of the graph.**
4. **An Eulerian circuit: is a simple, closed path that passes through all the arcs of the graph once and only once.**

Examples:

$t_1t_2t_3t_4t_5t_6t_7$ is a Hamiltonian path



$z_1z_4z_5z_4z_3z_3z_9z_8z_2z_9z_6z_7z_6z_1$ is an Eulerian circuit



Definition of a chain

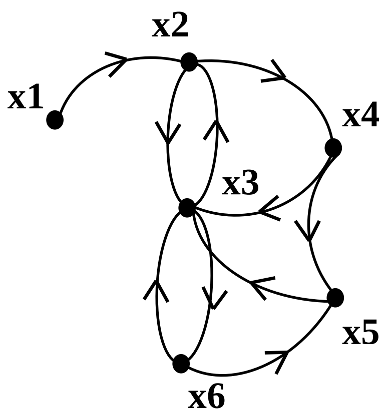
A chain of an undirected graph (or directed but the orientation is not taken into consideration) G(X, U) is a sequence of vertices $a_1, a_2, \ldots, a_p$ (without cut) such that:

$(a_i, a_{i+1}) \in U$ or $(a_{i+1}, a_i) \in U$ with $i=1, 2, \ldots, p-1$.

Example:

$x_1x_2x_4x_3x_6x_5x_3x_2$ is a chain (a path)

$x_1x_2x_5$ is not a chain



Definition of a simple chain

It is a chain where the (arcs/edges) are taken only once.

Example:

x1x2x3x2x4x3x5x6x3 is a simple chain.

Definition of an elementary chain

It is a chain whose all vertices of the graph are taken only once.

Example:

$x_1x_2x_4x_3x_6$ is an elementary chain.

Definition of a cycle

A cycle is a chain $a_1a_2a_3$ ….a_p, such that: a_1=a_p

i.e., the initial end and the terminal end are merged

Example:

$x_2x_4x_3x_6x_5x_3x_2$ is a cycle.

Observations

- **In the same way we can define a simple cycle, an elementary cycle.**
- **In defining a chain or cycle the orientation of the graph is not important.**

Special chains and cycles

1. **A Hamiltonian chain: is an elementary chain that passes through all the vertices of the graph.**
2. **A Eulerian chain: is a chain that passes through all the arcs of the graph once and only once.**
3. **A Hamiltonian cycle: is an elementary cycle that passes through all the vertices of the graph.**
4. **An Eulerian cycle: is a simple, closed chain that passes through all the arcs of the graph once and only once.**

Definition of a subgraph

Let G(X, U) be a graph

G'(X', U') is a subgraph of G if X'$\subset$X

U'={u$\in$U/ both ends of u belong to X'}

(i.e., we delete vertices and arcs from the original graph)

Definition of a partial graph

Let G(X, U) be a graph, we call partial graph of G, the graph G'(X', U') such that: X'=X, U'$\subset$U

(it is a graph having the same set X of vertices, but a different set of arcs U'$\subset$U).

Definition of a partial subgraph

Let G(X, U) be a graph, we call a partial subgraph of G, a partial graph of a subgraph of G.

(i.e. we delete vertices and arcs)

Example:

Let the graph G(X, U) be:

X={a, b, c, d, e, f, g}; U={(a, b), (a, c), (c, d), (b, f), (f, g),

(e, a), (e, f), (e, d), (f, d)}

1. **G'(X', U') is a subgraph (Fig 1)**
   X'={a, e, f, d}; U'={(e, a), (e, f), (d, e), (f, d)}

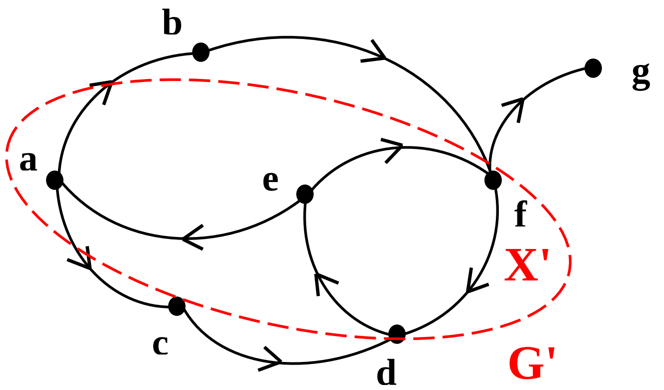


Fig 1

2. **G"(X, U") is a partial graph(Fig 2)**

   U"={(a, b), (a, c), (c, d), (d, e), (e, f), (f, d)}

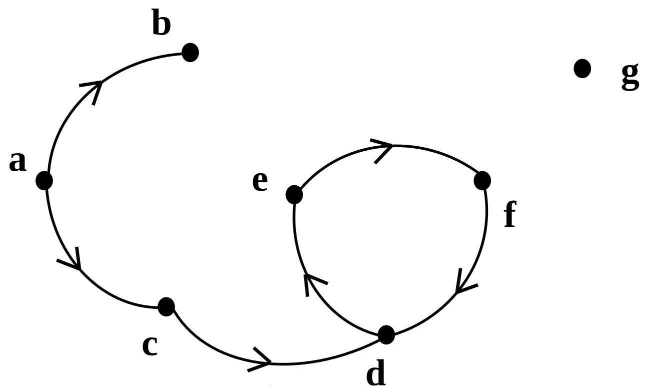


Fig 2

3. G'''(X', U''') is a partial subgraph(Fig 3)

   X'={a, e, f, d}; U'''={(a, e), (e, f), (f, d)}

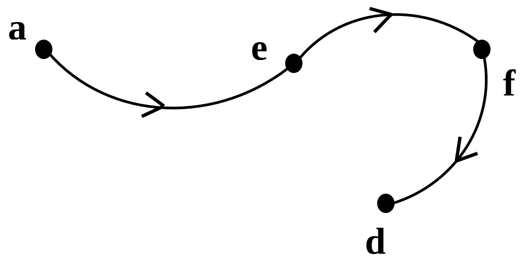


Fig 3

Practical example:

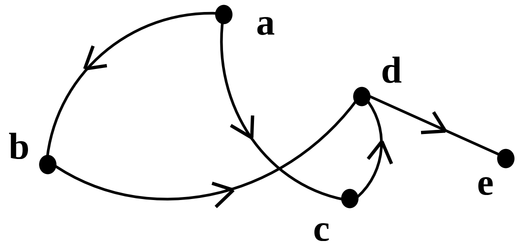
For a country's road network

1. **All the roads in a wilaya are a subgraph of the road network.**
2. **The set of national roads is a partial graph of the road network.**
3. **All the national roads in a wilaya are a partial subgraph of the road network.**

Definition of a connected graph

A graph G(X, U) is said to be connected if:

- **$\forall x, y \in X$, there is a chain between x and y (the direction of the arcs is not important)**



A connected graph; but is not strongly connected

Definition of a maximal connected component

Is a connected subgraph and maximal for this property. That is, if we add a vertex to this component we destroy the connectivity.

OBS: if G is a connected graph, it itself constitutes the only maximal connected component.

Definition of a strongly connected graph

A graph G(X, U) is said to be strongly connected if:

$\forall x, y \in X$, there is a path from x to y and another from y to x

Definition of a strongly connected component

A strongly connected component of a graph G(X, U) is called a strongly connected subgraph G'(X', U').

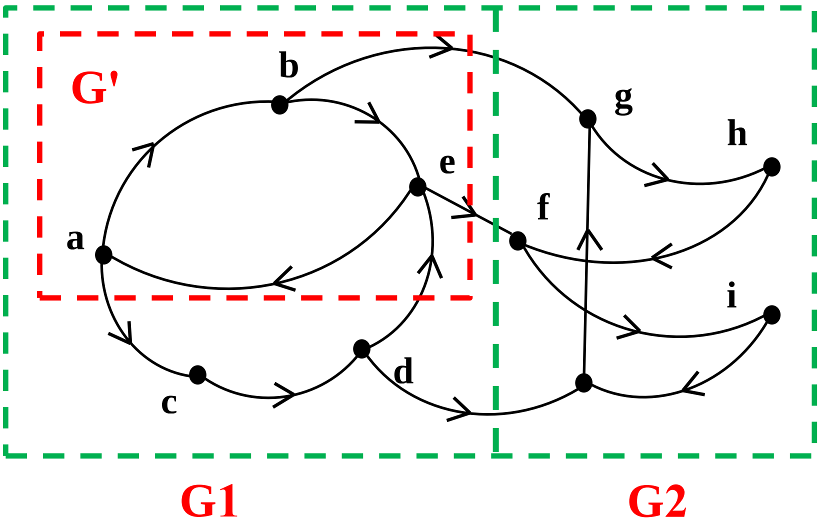
Example:

Let G(X, U) be the following graph such that:

X={a, b, c, d, e, f, g, h, I, j}

U={(a, c), (c, d), (d, e), (e, f), (d, j), (b, g), (g, h), (h, f) , (f, i),

(I, j), (j, g)}



G(X, U) is not strongly connected.

G'(X', U') is a subgraph of G.

X'={a, b, e}; U' = {(a, b), (b, e), (e, a)}

G' represents a strongly connected component (not maximal)

G1(X1, U1), G2(X2, U2) are two maximal strongly connected components.

Observations:

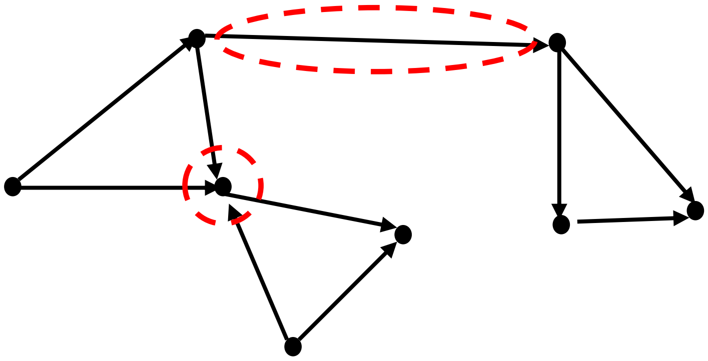
G(X, U) a graph, if ∀x, y∈X, there exists a chain between x and y we say that G(X, U) is simply connected.
G(X, U) is a connected graph, if after eliminating more than k edges, G becomes unconnected we say that G is k-edge connected.

Definition of an articulation point

An articulation point of a graph G is a vertex whose deletion increases the number of connected components.

Definition of an isthmus:

Is an arc whose deletion increases the number of CCs.



Algorithm for finding a Simply Connected Component SCC of a vertex S

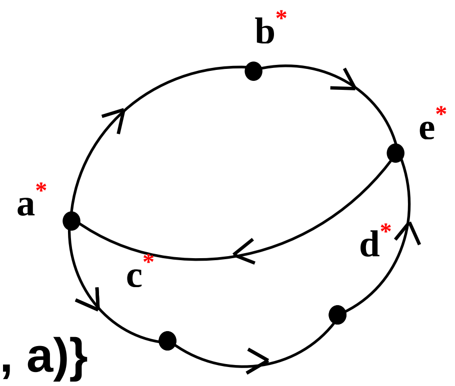
Let G(X, U) be a graph

(1) Mark the vertex S (by *)
(2) Mark any adjacent (successor/predecessor) vertex of an already marked vertex (by *)
(3) Repeat (2) until no more vertices can be marked.
(4) The vertices marked with (*) form the simply connected component of S

Example:

Let G(X, U) be a graph with:

X={a, b, c, d, e}



U = {(a, b), (a, c), (b, e), (c, d), (d, e), (e, a)}

Let's build the CSC of b

(1) Mark b (b*)
(2) Mark the adjacent vertices of b (successors and predecessors of b)
(3) No vertex remained unmarked.

{a, b, c, d, e} constitutes a simply connected component, i.e., G forms a SCC

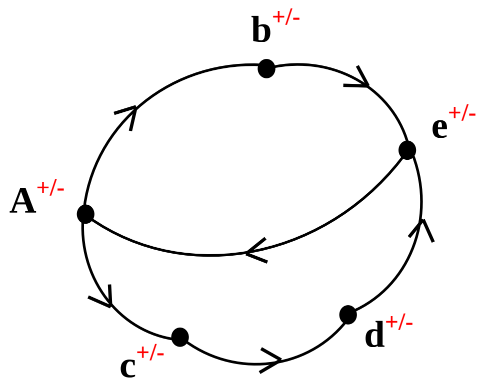
Algorithm for finding a strongly connected component of a vertex S

Let G(X, U) be a graph

(1) Mark the vertex S with (+ and -)
(2) (a) Mark with (+) any successor (not yet marked +) from a vertex already marked (+)
    (b) Mark with (-) any predecessor (not yet marked -) of a vertex already marked (-)
(3) Vertices marked with both + and – form a Str.CC containing S.

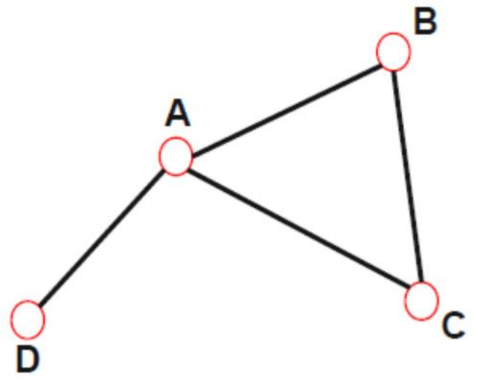
Example:

in the graph opposite
CFC(b)={b, a, e}

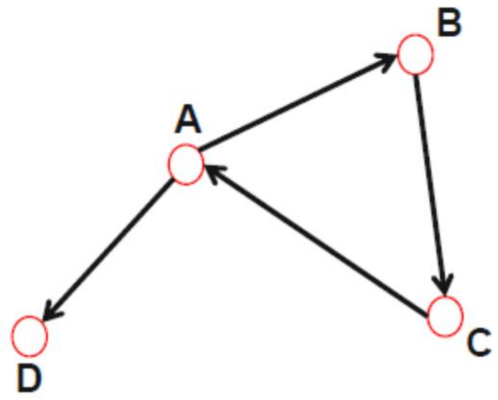


The shortest path / The distance

- **Distance (shortest path)** between two vertices is defined by the number of edges along the shortest path that connects these two vertices. If the two vertices are disconnected, the distance is infinite. If they are the same, the distance is zero.



- In a directed graph every path must follow the direction of the arcs. So the distance from vertex A to vertex B (on path AB) is usually different from the distance from vertex B to vertex A (on path BCA).



Diameter of a network and average distance

Diameter dmax

***The diameter dmax*is the maximum distance between any pair of vertices in the graph.**

Average Distance

<*d*> for a connected graph is given by:

$$\langle d \rangle = \frac{1}{N(N-1)} \sum_{i,j} d_{ij}$$

Where:

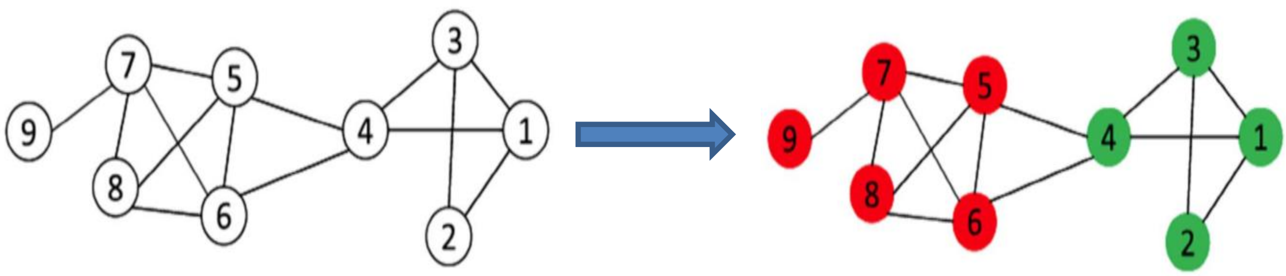
- *d :* is the distance from vertex i to vertex j
- N : is the number of vertices in the graph.
- <d>: the sum of the distances between the different vertices i, j divided by N(N-1)

2. Community Detection

A community is a set of vertices between which interactions are (relatively) frequent.

- **Communities are also called groups, segments, or modules.**

- **Finding a community in a graph consists of identifying a set of vertices that have interactions with each other more than with other vertices that do not belong to the same group.**

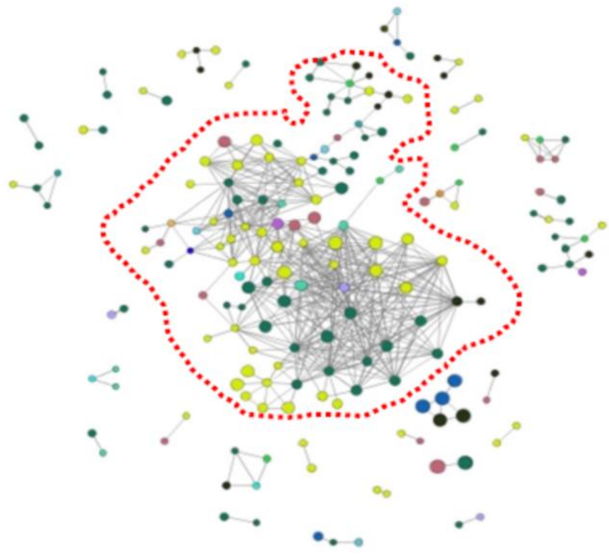


Examples of Communities

Giant Components and Isolated Components

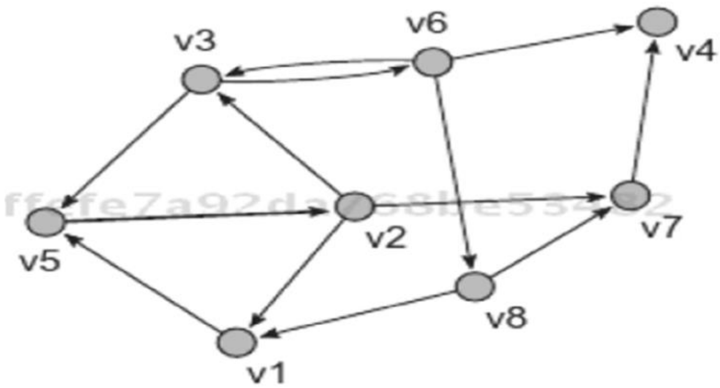
- **If the largest component in a graph occupies a significant part of it, it is therefore called a giant component.**

- **The other small components of the same graph are called isolated components.**



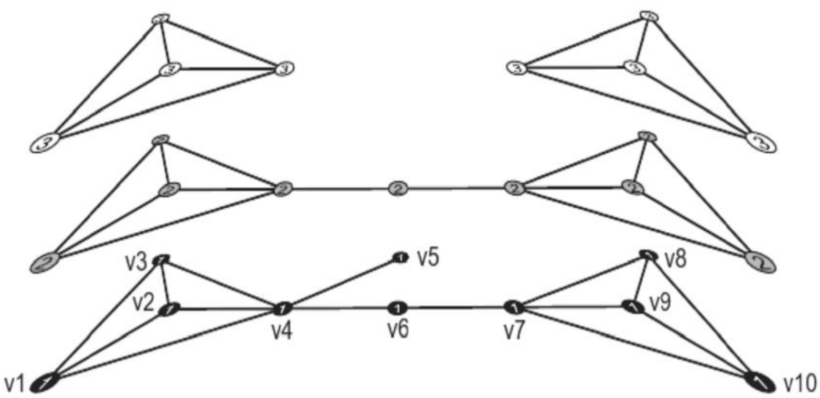
Exercise: Connected components in a graph

- ***How many strongly connected components are there in the following graph?***



K-Cores

A K-Cores is a maximal subgraph in which each vertex has at least a degree k in the graph.

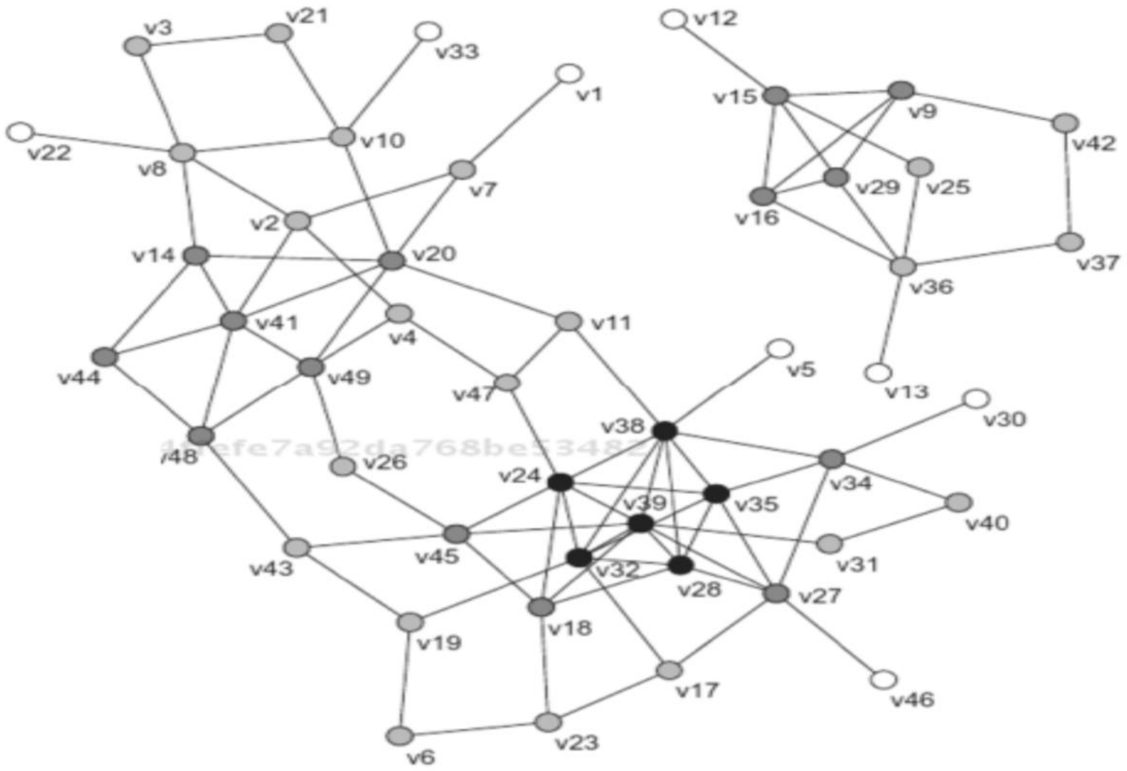


- ❖ **A K-Cores is not necessarily a connected graph. For example one of the 3-cores above consists of two parts (two components).**

Exercise: K-Cores

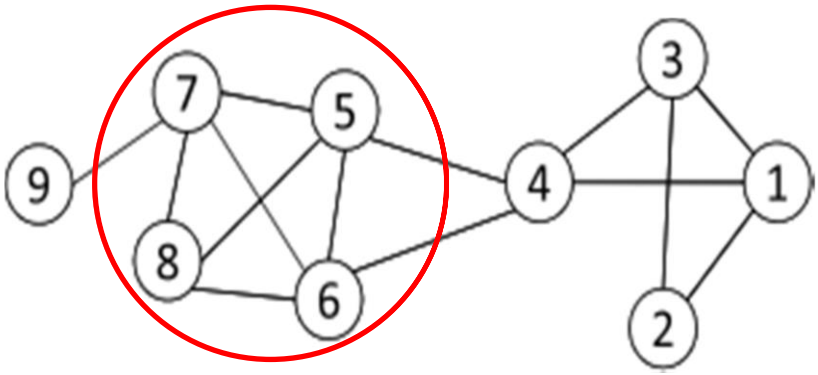
Count the number of 3-cores in the graph (vertex colors indicate the level of k):

White: k=1 (1-core); Light gray: k=2 (2-cores); Dark gray: k=3 (3-cores); Black: k=4 (4-cores)



Complete clique and subgraphs

A clique is a maximal complete subgraph containing at least three (03) vertices.

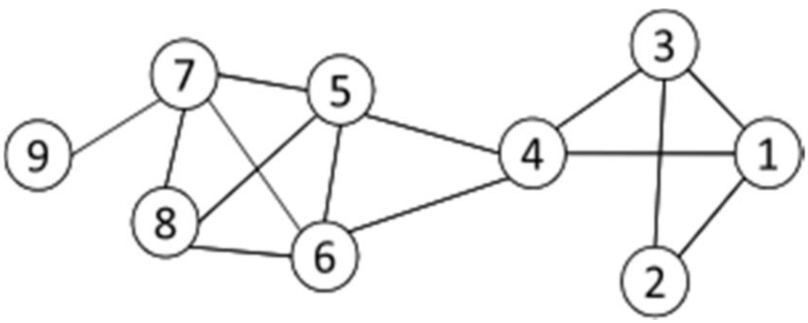


Vertices 5, 6, 7 and 8 form a clique

Structural equivalence between vertices

Two vertices are structurally equivalent if they are connected to the same vertex sets.

Example: the two vertices 5 and 6



Similarity between summits

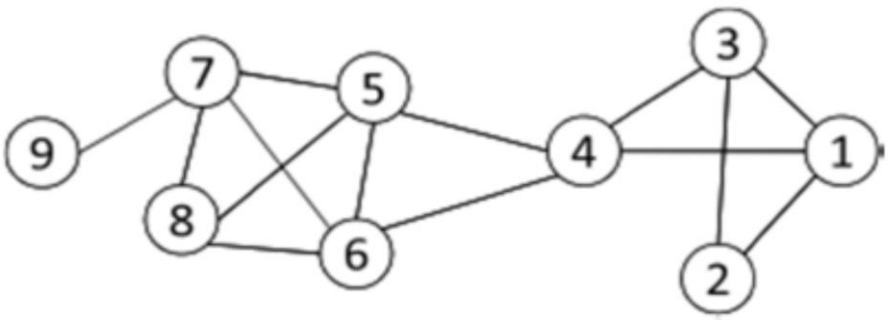
Jaccard similarity

$$Jaccard(v_i, v_j) = \frac{|N_i \cap N_j|}{|N_i \cup N_j|}$$

Cosine Similarity

$$Cosine(v_i, v_j) = \frac{|N_i \cap N_j|}{\sqrt{|N_i| \cdot |N_j|}}$$

Example



$$Jaccard(4,6) = \frac{|\{5\}|}{|\{1,3,4,5,6,7,8\}|} = \frac{1}{7}$$

$$cosine(4,6) = \frac{1}{\sqrt{4 \cdot 4}} = \frac{1}{4}$$

Centrality (central vertex)

Centrality

- Which vertices are more central?

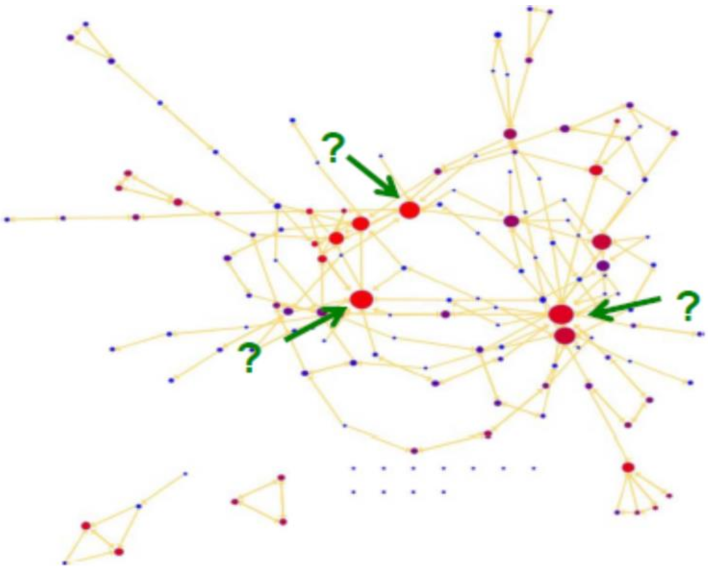
  Calculated for undirected graphs

- A central vertex (central actor) is a vertex involved in several edges.
- The direction of the arcs is not considered.

Concept of centrality

Prestige (prestigious vertex)

- A prestigious vertex (prestigious actor) is a vertex which represents the destination of several arcs (receives arcs).
- The direction of the arcs is considered in this case.

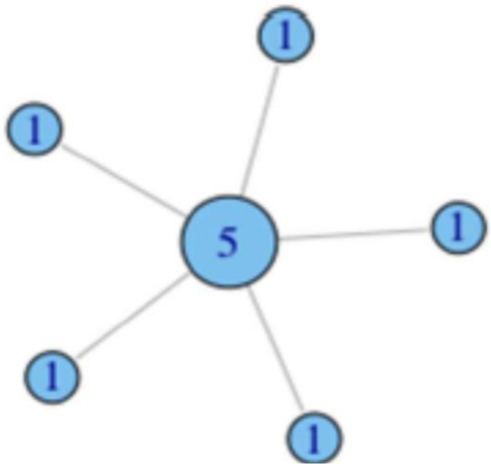


Prestige concept

Centrality Degree

Idea: Measures the centrality as the number of edges towards other vertices in the graph.

Example: ***How many people can be directly influenced by one person?***



Degree of centrality

$$C_D(n_i) = d(n_i)$$

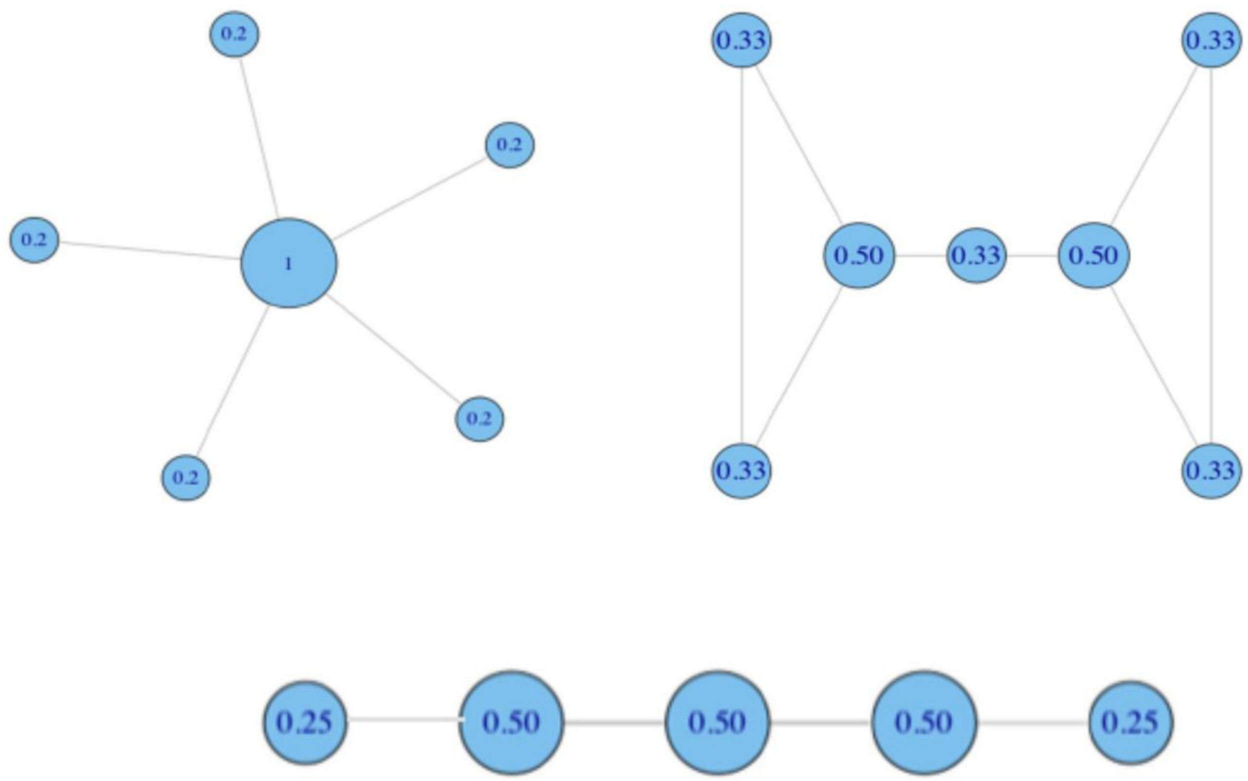
Normalized degree centrality

$$C'_D(n_i) = d(n_i)/N\text{-}1$$

- **The degree divided by the maximum possible degree, i.e., the number of vertices – 1**

- **Proportion of all vertices that are adjacent to n_i**

Example: **Normalized Degree of Centrality C'_D**



Centralization

How many variations is there in the centrality score among vertices?

Freeman's General Formula for Centralization:

Maximum centrality value in the graph

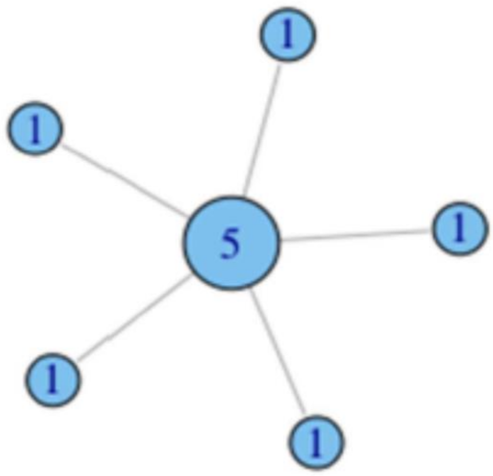
$$C_D = \frac{\sum_{i=1}^{g}\left[C_D(n^*) - C_D(i)\right]}{(N-1)(N-2)}$$

Theoretical maximum of the graph

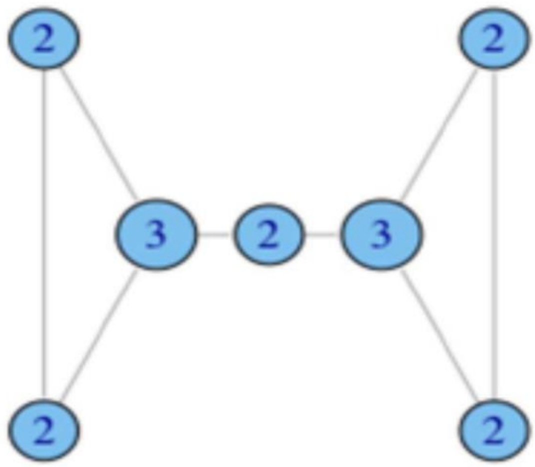
1. **Calculate the sum of the centrality differences between the most central vertices of the graph G and all other vertices.**
2. **Divide the resulting value in (1) by the maximum theoretical sum of the graph.**

NB: Centralization value$\in$[0.1]

Example: Degree of centralization CD



high



CD= [(5-1)+(5-1)+(5-1)+(5-1)+(5-1)] /(6-1)(6-2) = 20/20 = 1.0

CD= [(3-2)+(3-2)+(3-2)+(3-3)+(3-2)+(3-2)] /(7-1)(7-2) = 5/30 = 1/6≈0.167

CD =[(2-1)+ (2-2)+ (2-2)+ (2-2)+ (2-1)]/(5-1)*(5-2)
= 2/12 = 1/6≈0.167

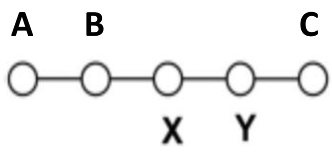
Betweenness Centrality/Binary Centrality

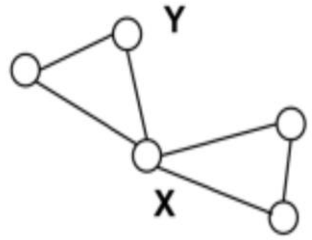
Definition

Interactions between two non-adjacent vertices X and Y may depend on other vertices in the graph, especially those that belong to paths connecting the two vertices X and Y.

■ **Which of the two vertices X or Y has a high binary centrality value?**

Example 1:

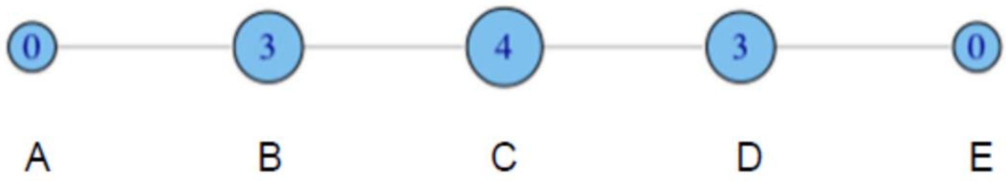




In fig 1: X has a high binary centrality value (4) against Y(3)

In fig 2: X has a high binary centrality value (5) against Y(3)

Example 2:



☐ **A is a vertex that does not connect any other two vertices in the graph.**

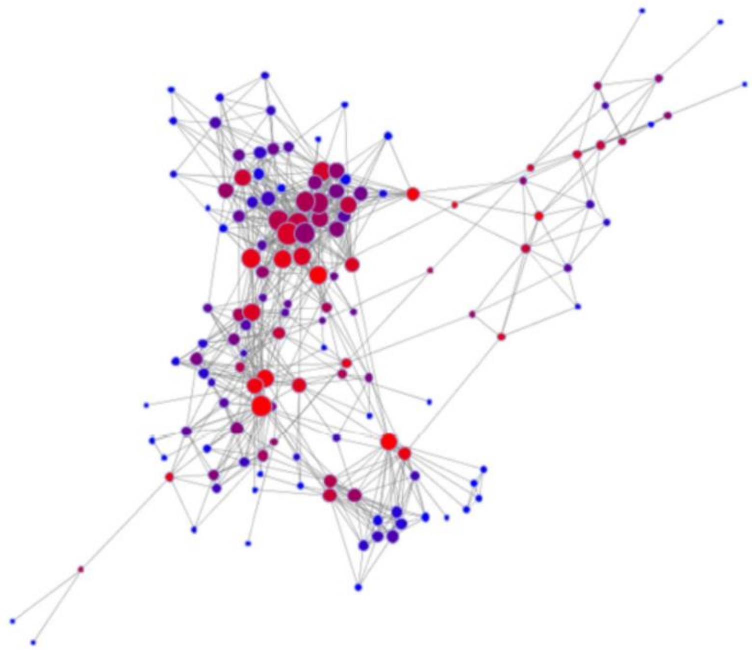
☐**B connects between A and 3 other vertices: C, D, and E**

☐ **C connects between 4 pairs of vertices (A, D),(A, E),(B, D), (B, E).**

We note here that there are no other alternative paths connecting these pairs of vertices except those passing through vertex C.

Example 3: (centrality Degree/binary centrality) on the Facebook network

☐ **The vertices are sized by degree of centrality and colored by binary centrality**



Proximity Centrality

☐ **This measure focuses on how close a vertex is to other vertices in the graph.**

☐ **Proximity centrality is based on the average length of the shortest path between a vertex and all other vertices in the graph.**

Proximity Centrality:

$$C_c(i) = \left[\sum_{j=1}^{N} d(i,j) \right]^{-1}$$

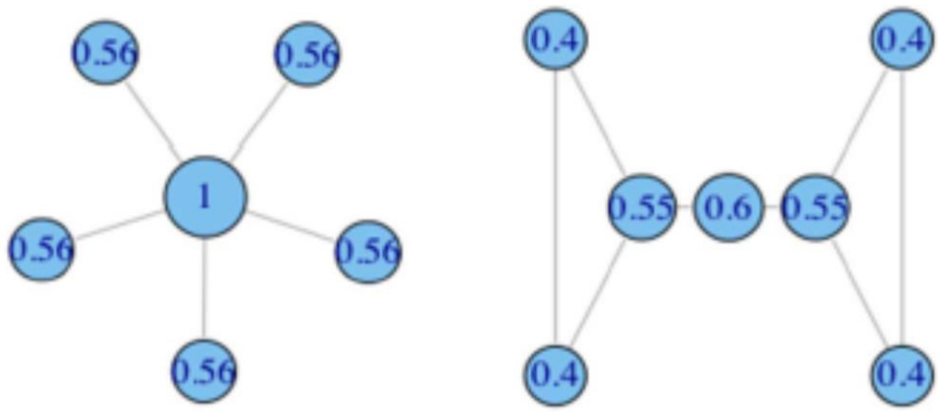
Normalized Proximity Centrality:

$$C'_C(i) = C_C(i)(N-1)$$

Example: Proximity centrality

$$C'_c(A) = \left[\frac{\sum_{j=1}^{N} d(A,j)}{N-1} \right]^{-1} = \left[\frac{1+2+3+4}{4} \right]^{-1} = \left[\frac{10}{4} \right]^{-1} = 0.4$$

More examples:



Prestige

Prestige is a centrality metric that takes into account the direction of arcs.

Prestige Degree/Popularity

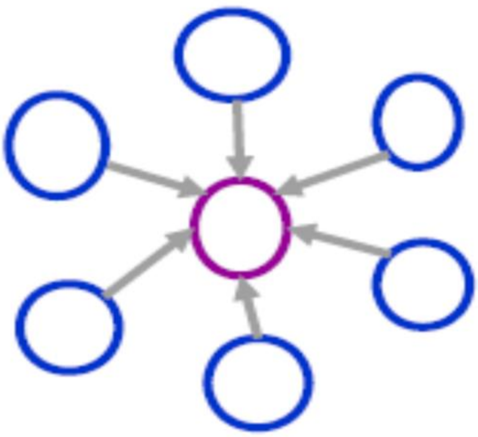
- **The prestige of a vertex is calculated based on the vertex in-degree**

Prestige Degree/popularity

$P_D(n_i) = d^{in}(n_i)$

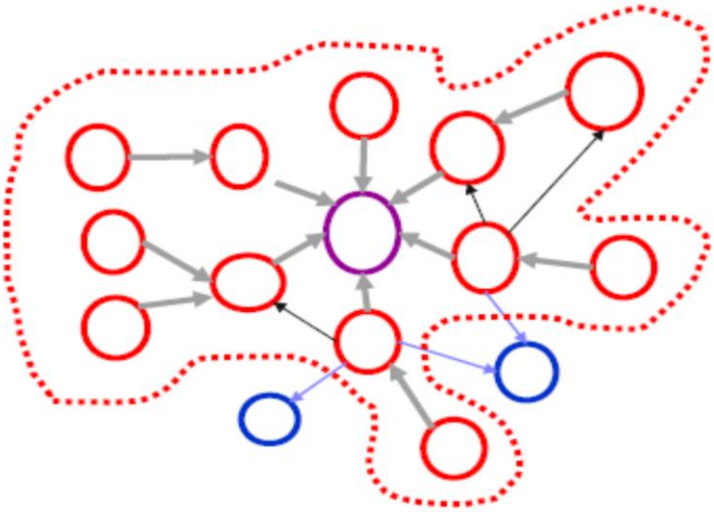
Standardized prestige degree

$P'_D(n_i) = d^{in}(n_i)/(N-1)$



Input (influence) domain

■ The prestige degree only calculates actors that are adjacent or directly linked to actor n_i, but one can also consider vertices that are indirectly linked to n_i.



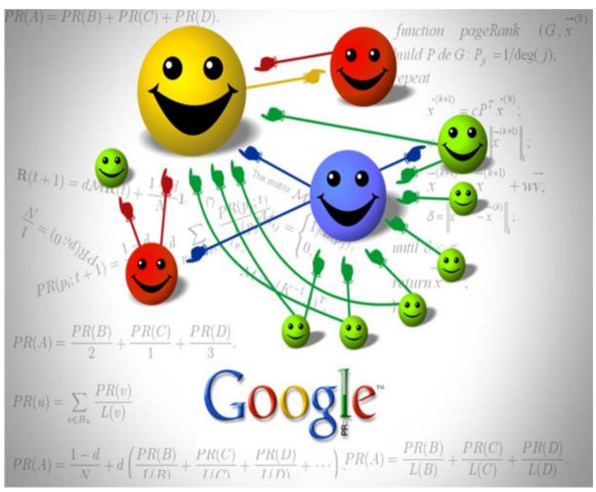
■ The input domain of a vertex in a directed graph is the number or percentage of all other vertices that are linked by Paths to this vertex. It is also called domain of influence.

Prestige Rank

☐ A new metric that takes into account the prestige of actors (vertices) belonging to the same domain of influence.

☐ A vertex will be more prestigious if it has many other prestigious summits belonging to the same area of influence.

$$P_R(i) = \sum_{(j,i)\in E} P_R(j)$$



Where:

I: a vertex in the input domain of i

$P_R(i)$: Prestige Rank of the vertex i

$P_R(j)$: prestige of the vertex j