MOHAMED BOUDIAF UNIVERSITY OF M'SILA

DEPARTMENT OF COMPUTER SCIENCE

Advanced Graph Theory

Chapter 6: Hypergraphs and Matroids

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2017 - 2018

Hypergraphs: Extensions of Graphs

- **Hypergraphs are mathematical objects that generalize the notion of a graph. They were named by Claude Berge in the 1960s.**
- **Hypergraphs generalize the notion of an undirected graph in the sense that the edges don't connect one or two vertices, but any number of vertices (a traditional graph is a special case of a hypergraph).**
- **Some theorems of graph theory can be generalized to hypergraphs.**

Definition of a hypergraph

Let = {$x_1, x_2, \ldots, x_n$} be a finite set of vertices

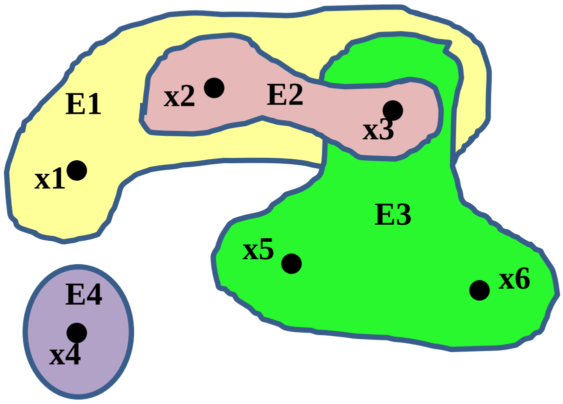
A hypergraph on the set X is a family H = ($E_1, E_2, \ldots, E_m$) of non-empty (and generally finite) subsets of X such that:

- $E_i \neq \phi \quad (i=1, 2, \ldots, m) \qquad (1)$
- $\bigcup_{i=1}^{m} E_i = X \qquad (2)$

Like graphs, we say that:

- **The elements of X are the vertices of H.**
- **The number of vertices *n* is the order of the hypergraph H, denoted n(H)=|X|**
- **The elements of E are the edges (or hyper-edges) of H.**
- **The number *m* of edges is the size of the hypergraph H, denoted m(H) = |E|**

- We also say H = (E_1, E_2, …, E_m) is a hypergraph on X



✓ X = $\{x_1, x_2, x_3, x_4, x_5, x_6\}$

✓ E = $\{E_1, E_2, E_3, E_4\}$

$= \{ \{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_3, x_5, x_6\}, \{x_4\} \}$

✓ $E_1 = \{x_1, x_2, x_3\}$ is a non-binary edge (or hyper-edge) of the hypergraph H

Likewise for E_2, E_3, E_4 (the edges in H are subsets of X)

Hypergraphs correspond precisely to matrices with coefficients 0 or 1 (for which, each column contains at least one 1). Effectively, any hypergraph H corresponds uniquely to the matrix $A_{n,m}$ such that:

$$\forall a_{i,j} \in A, \quad a_{i,j} = \begin{cases} 1 & if\ v_i \in E_j \\ 0 & Otherwise \end{cases}$$

Remarks :

- **In a hypergraph, there can be no isolated vertices that do not belong to any edge (subset *Ei*) of H.**
- **A simple graph is a hypergraph whose subsets *Ei* (edges) has cardinality 2**
- **A multigraph (with multiple loops and edges) is a hypergraph for which each edge has a cardinality≤2**
- **A hypergraph is represented by a set of points representing the vertices x_i, on the other hand the edges *Ei* are represented by:**
  - **A curve or continuous line connecting the two ends of the edge *Ei* if |*Ei*| = 2**
  - **By a loop if |*Ei*| = 1**
  - **By a simple closed curve enclosing the different vertices if |*Ei*|≥*3***

Areas of use of hypergraphs

Hypergraphs are manipulated in all areas where graph theory is used, including:

- **Solving constraint satisfaction problems.**
- **Image processing.**
- **Optimization of network architectures.**
- **Modeling, etc.**

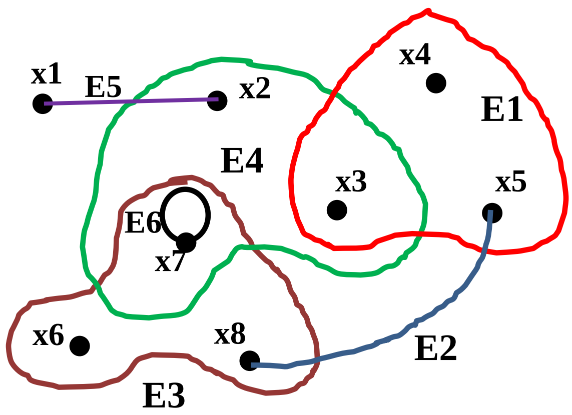
Representation of a hypergraph

A hypergraph can be represented by an incidence matrix $A(n, m)$, such that the columns represent the edges $E_1, E_2, E_3, \ldots, E_m$ and the rows represent the vertices $x_1, x_2, \ldots, x_n$

With

$$a_j^i = \begin{cases} 1 & \text{if } x_i \in E_j \\ 0 & \text{Otherwise} \end{cases}$$

Example: Representation of a hypergraph



Example of a hypergraph H

	E1	*E2*	*E3*	*E4*	*E5*	*E6*
x1	0	0	0	0	1	0
x2	0	0	0	1	1	0
x3	1	0	0	1	0	0
x4	1	0	0	0	0	0
x5	1	1	0	0	0	0
x6	0	0	1	0	0	0
x7	0	0	1	1	0	1
x8	0	1	1	0	0	0

Incidence matrix associated with the hypergraph H

Definition of a simple hypergraph

A simple hypergraph on X (or Sperner hypergraph) is a hypergraph H=($E_1, E_2, \ldots, E_m$) such that:

In addition to conditions (1) and (2) seen previously, we must also have the following condition:

- $E_i \subset E_j \Longrightarrow i = j$ (i.e., no edge contains another) (3)

Definition of the dual of a hypergraph

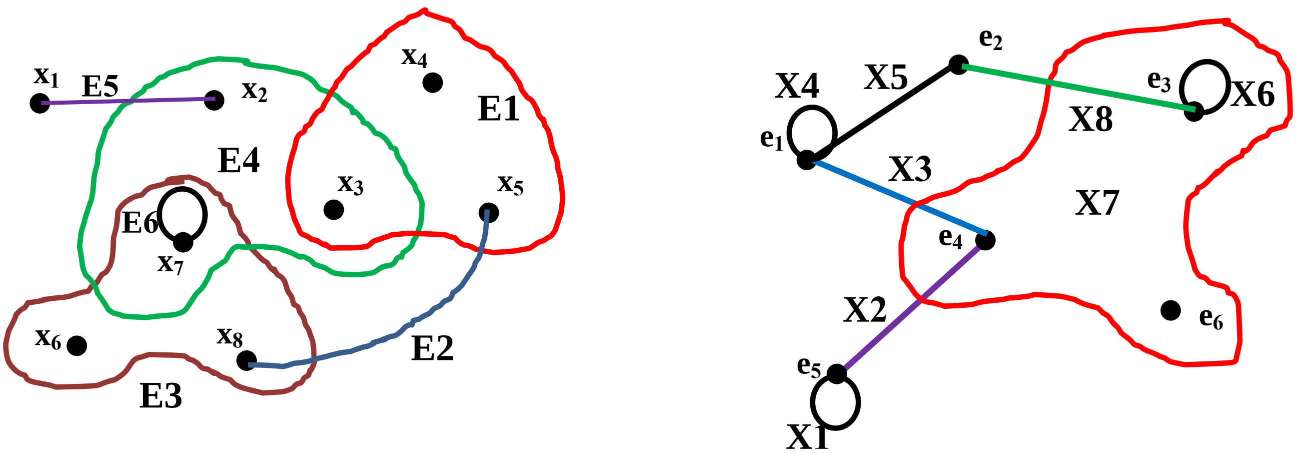
The dual of a hypergraph H = {E_1, E_2, …, E_m} on X = {x_1, x_2, …, x_n} is also a hypergraph H^* = {X_1, X_2, …, X_n} whose vertices e_1, e_2, ..., e_m correspond to the edges E_1, E_2, …, E_m of H and the edges are defined by the vertices of H as follows:

$X_i = \{E_j / x_i \in E_j$ *in H*$\}$ or the edge X_i in H* consists of the set of edges E_j in H to which its corresponding vertex x_i (in H) belongs.

i.e., vertices become edges and edges become vertices.

We can easily see that H* satisfies the conditions of a hypergraph seen previously.

Exemple: **Dual of a hypergraph**



Un hypergraphe et son dual

Ex : $X_7 \in H^*$ <=> $x_7 \in H$, $x_7 \in \{E_6, E_3, E_4\}$ dans H ==> $X_7 = \{e_6, e_3, e_4\}$ dans H^*

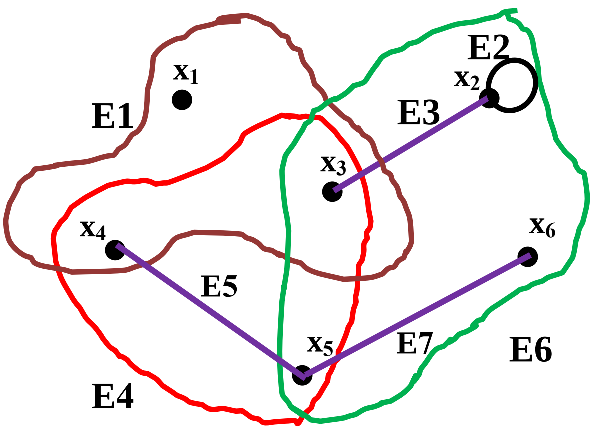
Remark : **The incidence matrix of H* is the transpose matrix of H, and we therefore have:**

$$M(H^*) = M^T(H) \Longrightarrow M^T(H^*) = M(H)$$

Properties of a hypergraph

- **The order of a hypergraph H and we note n(H) = |X| is the number of vertices**
- **The size of a hypergraph H and we note m(H) = |E| is the number of edges**
- **The rank of a hypergraph H and we denote r(H) = Max_j|Ej| (the size of the largest edge in H)**
- **The anti-rank of a hypergraph and we note δ(H) = $\mathrm{Min}_j|E_j|$ (the size of the smallest edge in H)**

Example: Properties of a hypergraph



n(H) = 6 (number of vertices)

m(H) = 7 (number of edges)

r(H) = |E6| = 4 (Max edge size)

δ(H) = |E2| = 1 (Min edge size)

RemarK : if r(H) =δ(H) we say that H is uniform

Definition of a partial hypergraph

Let H be a hypergraph of order *n*=|*X*| and size *m*=|*E*|, and let be a subset J⊂{1, 2, …, m}

H_j = {E_j/j∈J} is called a partial hypergraph generated by the set J and having the same vertex set X. (we take all the vertices and some edges of H)

Example: **A partial hypergraph**

In the example of the previous figure, we define a partial hypergraph H_j as follows:

X = {x_1, x_2, x_3, x_4, x_5, x_6} ; Ej = {E_1, E_4, E_6} = {{x_1, x_3, x_4}, {x_3, x_4, x_5}, {x_2, x_3, x_5, x_6}}

Definition of a Subhypergraph

H(X, E) is a hypergraph, A is a subset of vertices (A⊂X)

The family H_A={E_j/E_j⊆A pour $1 \leq j \leq m$} is called a subhypergraph generated by the subset A.

(We take some vertices x_i of A⊂X and some edges $E_j \subset E$ of H)

Example: **A sub-hypergraph**

In the example of the previous figure, we define a subhypergraph H_A as follows:

A= {x_1, x_2, x_3, x_4} ; Ej = {E_1, E_2, E_3} = {{x_1, x_3, x_4}, {x_2}, {x_2, x_3}}

Definition of a Partial Subhypergraph

It is a partial hypergraph of a subhypergraph.

Example: A partial subhypergraph

From the previous example, we deduce a partial subhypergraph *H'* as follows:

A= $\{x_1, x_2, x_3, x_4\}$; $E_j = \{E_1, E_3\} = \{\{x_1, x_3, x_4\}, \{x_2, x_3\}\}$

Remark:

- **The dual of a subhypergraph of H is a partial hypergraph of the dual hypergraph H*.**
- **If we have a hypergraph of *rank(H)=2*, or when all the edges of H are of cardinality at most 2, we return to the definition of a traditional graph, and in this case, all the concepts of graph theory seen previously are valid.**

Definition of degree in a hypergraph

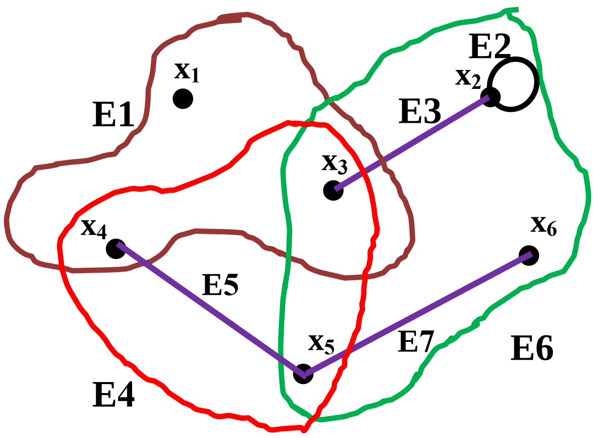
Let H(X, E) be a hypergraph, for $x \in X$, we define the star centered at x and we note H(x) as being the partial hypergraph formed of the edges containing x (centered on x).

The degree of vertex x in H and we denote by $d_H(x)$ is the number of edges of H(x).

We write: $d_H(x)=m(H(x))$ or the number of edges linked to x.

The maximum degree of H is therefore $\Delta(H) = Max_{x \in X}[d_H(x)]$

Example: **The centered star and the degree of a hypergraph**



Let's take the vertex x_3

- **The star centered at x_3 noted $H(x_3)$ is the hypergraph Partial formed of:**
  - **The set of vertices X = $\{x_1, x_2, x_3, x_4, x_5, x_6\}$**
  - **The set of edges $E(x_3)$ containing x_3 ==>**
  - **$E(x_3) = \{E_1, E_3, E_4, E_6\}$**
- **The degree of vertex x_3 denoted $d_H(x_3) = |E(x_3)| = 4$ or the number of edges of $H(x_3)$**

  The maximum degree of H is therefore $\Delta(H) = Max_{x_i \in X}[d_H(x_i)] = d_H(x_3) = d_H(x_5) = 4$ (car $E(x_5) = \{E_5, E_7, E_4, E_6\}$ ==> $|E(x_5)| = 4$)

Remark :

A hypergraph for which all vertices x_i have the same degree is said to be a regular hypergraph

i.e., $d_H(x_1) = d_H(x_2) = d_H(x_i) = \ldots\ldots\ldots = d_H(x_n)$

Definition of Cross Families

Let H be a hypergraph, we define *__a crossing family__* as being a *__set of edges__* having *__non-empty pairwise intersections.__*

For example, for any vertex $x_i \in H$, the star centered in x_i $H(x_i) = \{E/E \in H, x_i \in E\}$ is a cross family of H.

The maximum cardinality of a crossed family, and we note $\Delta_0(H)$ must satisfy the condition:

$$\Delta_0(H) \geq Max_{x_i \in X}|H(x_i)| = \Delta(H)$$

Example: Crossed families

In the previous example, we define crossed families as follows:

$FC = E(x_3) = \{E_1, E_3, E_4, E_6\}$

Property of a colored edge

Let H = ($E_1, E_2, \ldots, E_m$) be a hypergraph, the chromatic index of H is the smallest number of colors needed to color the edges of H, such that: two adjacent edges are colored differently. This same index q(H) has been extended to traditional graphs.

If $\Delta_0(H) = k$, so we need at least k different colors to color the edges of a family of k-crossed edges, and we therefore have:

$$q(H) \geq \Delta_0(H) \geq \Delta(H)$$

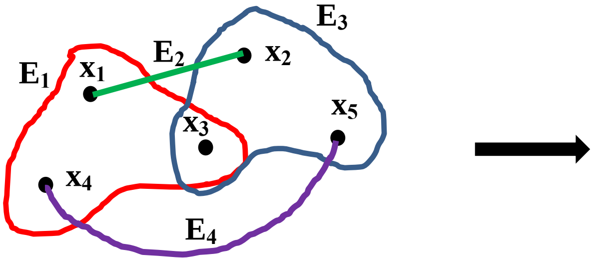
We say that H has the property of a colored edge if q(H) =*Δ(H)*, i.e., it is possible to color the edges of H with *Δ(H)* colors.

Graphs representing a hypergraph

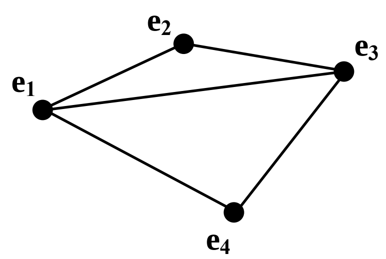
Let H = ($E_1, E_2, \ldots, E_m$) be a hypergraph on X, its representative graph or *Line-graph* L(H) is a graph whose vertices are the points $e_1, e_2, \ldots, e_m$ representing the edges $E_1, E_2, \ldots, E_m$ of H. Under these conditions, two vertices e_i, e_j are adjacent if and only if $E_i \cap E_j \neq \phi$

Example: ***Graph representing a hypergraph***

Let the hypergraph H be characterized by: X = {x_1, x_2, x_3, x_4, x_5} ; E = (E_1, E_2, E_3, E_4)



Hypergraph H



Associated representative graph
(Line graph L(H))

Theorem:

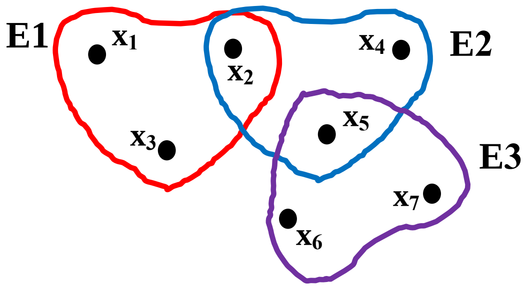
Every graph is a representative graph of a linear hypergraph, i.e., whose edges are of non-empty intersection two by two.

Definition of a uniform hypergraph

H(X, E) is a hypergraph, H is said to be uniform if its rank and anti-rank are equal. i.e., r(H) =*δ(H)*, and we say that H is r-uniform if it is simple (the edges are totally disjoint) and uniform of rank r.

Note that the rank of a hypergraph is bounded by its order.

Example: **A uniform hypergraph**



We note that:

The rank r(H) = the anti-rank δ(H) = $|Ei|_{i=1,\,2,\,3}$ = 3

==> H is a uniform hypergraph

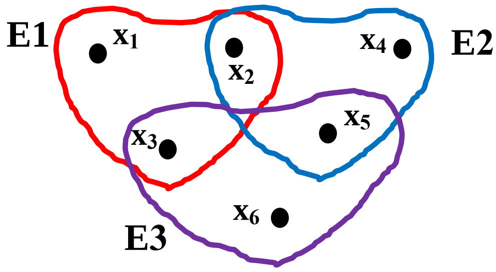
Definition of a complete hypergraph

H(X, E) is said to be r-complete, if H is the set of edges of X and of cardinality *r* with non-empty intersection two by two.

For *n* = |*X*|, H is denoted k_n^r

For r = 2, we will return to the notion of the complete graph.

Example :A complete hypergraph



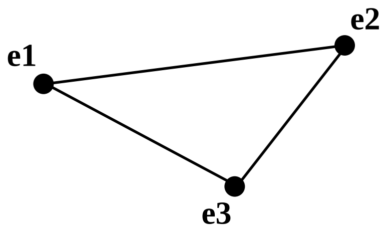
The figure represents a complete hypergraph H

Of order $n = |X|=6$ and cardinality $r = 3$

And we note: k_6^3

Remark

The representative graph of a complete hypergraph is also a complete graph



Definition of a k-section

H(X, E) is a hypergraph on X, the k-section of H is the hypergraph denoted $[H]_k$ whose edges are the sets $F_j \subseteq X$ and verify one of the following two conditions:

- **Soit $|F_j| = k$ et $F \subseteq E$ pour un E∈H**
- **Soit $|F_j| < k$ et $F = E$ pour un E∈H**

For example, the 3-section denoted $[H]_3$ is the hypergraph whose edges are the subsets F⊆X with

- **Let | F_j | = 3 and $F \subseteq E$**
- **Let | F_j |<3 and F=E**

Theorem: the representative graph of a hypergraph H is the 2-section $[H]_2$

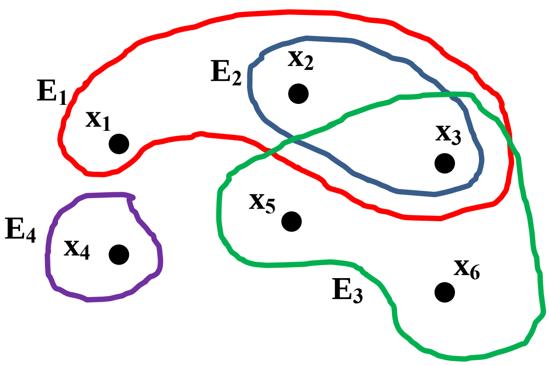
Example: k-section of a hypergraph

Let X = {$x_1, x_2, x_3, x_4, x_5, x_6$}

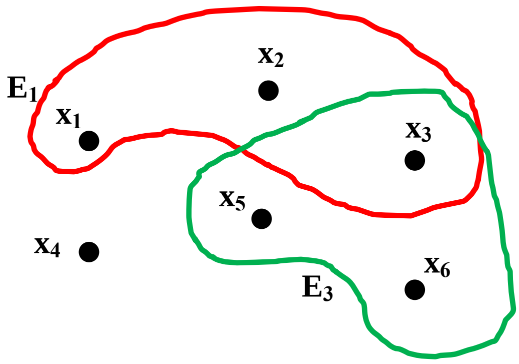
E = {$E_1 = \{x_1, x_2, x_3\}$; $E_2 = \{x_2, x_3\}$; $E_3 = \{x_3, x_5, x_6\}$; $E_4 = \{x_4\}$}

We define a 3-section $[H]_3$ as follows:

- **$|F| = 3$ and $F \subseteq E$**
- **$|F|<3$ and $F=E$**



Hypergraph H



A 3-section $[H]_3$

Matroids

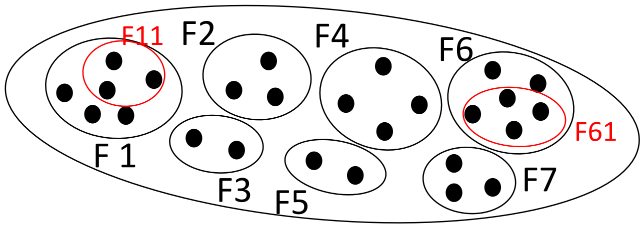
Matroids are a special class of hypergraphs.

Definition of a hereditary family

<u>*A hereditary family F*</u> or also called <u>*an ideal I*</u>, is a collection of sets, such that any subset of a set of F belongs to F (the notion of inheritance).

We note that:

- **$F11 \subset F1 \in F$**
- **$F61 \subset F6 \in F$**



A hereditary family ***F*** or an Ideal ***I***

Definition of a hereditary system

A hereditary system M on the set E is a hereditary family F_m or an ideal I_m of subsets of E.

The elements of I_m are called <u>*the independents*</u> of M, the other subsets are called the <u>*dependents*</u> of M and we denote D_m

We also note:

- **B_m: <u>*Maximal independents*</u> of M called the bases of M**
- **C_m: <u>*Minimal dependents*</u> of M called <u>*circuits*</u> of M**
- **The rank of a set X of E is the cardinality of a maximum independent of X.**
- **The rank function $r_m(X)$ is defined by:**

$$r_m(X) = Max\{|Y| / Y \subseteq X,\ Y \in I_m\}$$

- **The super-bases S_m are the sets containing a base.**

- **The H_m subbases are the maximal sets containing no base.**

So, to define a hereditary system M, it is sufficient to define one of its parameters: B_m, I_m, D_m, S_m, H_m

Remarks :

- **A hereditary family defines a hypergraph.**
- **The family of bases of a hereditary system is also a hypergraph of rank the maximum cardinality of a base.**

Definition of a matroid

Let E be a finite set, let I be a non-empty collection of parts of E (subsets of E)

The pair M(E, I) is a matroid if the following three (03) conditions are satisfied:

1. $\phi \in I$
2. If $I_1 \in I$ Then $\forall J \subseteq I_1$, $J \in I$
3. If $(I_1, J) \in I$ Such that: $|I_1|<|J|$, then $\exists x \in (J-I_1)$ Such that : $I_1 \cup \{x\} \in I$

Some properties of a matroid

- **The elements of I are called *<u>the independent sets</u>*.**
- **A set satisfying condition (2) is called *<u>a hereditary family</u>*.**
- **Property (3) is called the *<u>axiom of augmentation</u>*.**

- **If M(E, I) is a matroid, we call the elements of M *__the members__* of the set E.**
- **The subsets of E which belong to I are said to be *__independent__***
- **The independents of maximum size are called *__bases__* (having the same cardinality)**
- **A subset of E that is not independent (does not belong to I) is said to be *__dependent__*.**
- **We generally denote as follows: E(M) = E and I(M) = I, and we say that M is a matroid on E.**
- **The minimum size dependents ($\notin$I) are called *__circuits__*.**

Remark :

A matroid is a hereditary system M on E satisfying the following property called the augmentation property.

$\forall(I_1, I_2) \in I_M$; si $|I_2|>|I_1|$ then, there exist $e \in (I_2 - I_1)$, tel que : $I_1 \cup \{e\} \in I_M$

Example: Definition of a matroid

Let A(m, n) be a coefficient matrix in a field F, defined as follows:

$$A = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad m=2 \;;\; n=5$$

Let E = {e_1, e_2, e_3, e_4, e_5} be the set of column vectors of this matrix A ==> E = {{1, 0}, {0, 1}, {0, 0}, {1, 0}, {1, 1}}

Let I be the collection (family) of subsets $\{I_1, I_2, \ldots, I_m\} \subseteq \{1, 2, \ldots, n\}$ = E, such that the set of columns {e_{I1}, e_{I2}, …, e_{Im}} are linearly independent on F.

Then M(E, I) is a matroid on the matrix A defined as follows:

- **E is defined on the columns of A as follows:**
  - **E = {{1, 0}, {0, 1}, {0, 0}, {1, 0}, {1, 1}}**
  - $I = \{\phi, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$

Then, the matrix A gives rise to a matroid M(A) whose independents form the predefined set I.

The subsets: $\phi, \{1\}, \{1, 2\}, \ldots \in I$ **are called the independents.**

The subsets: $\phi, \{3\}, \{3, 4\}, \{3, 5\}, \ldots \notin I$ **are called dependents.**

The subsets: {{1, 2}, {1, 5}, {2, 4}, {2, 5}, {4, 5}} $\in I$ **and of equal maximum size ==> are independent and also called bases**

The subset {3} $\notin I$ **and of minimal size ==> is a dependent and also called a circuit**

Transverse matroid

Let E = {e_1, e_2, …, e_n}, and let A = {A_1, A_2, …, A_k}, with $A_i \subseteq E$ and $n \geq k$

A transversal of A is a subset {e_{j1}, e_{j2}, …, e_{jk}} of E such that:

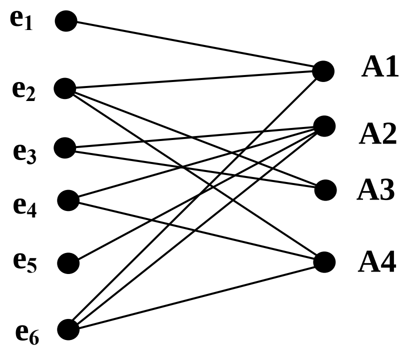
$e_{ji} \in A_i$

If we have the set X⊆E, X is said to be a partial transversal of A, if there exists a subset {i_1, i_2, …, i_l}⊆{$1, 2, …, K$}, such that X is a transversal of {A_{i1}, A_{i2}, …, A_{il}}

Example : Transverse matroid and partial transverse matroid

Let E = {e_1, e_2, e_3, e_4, e_5, e_6} and A = {A_1, A_2, A_3, A_4} with:

$A_1 = \{ e_1, e_2, e_6\}$; $A_2 = \{ e_3, e_4, e_5, e_6\}$; $A_3 = \{e_2, e_3\}$; $A_4 = \{ e_2, e_4, e_6\}$



Transversal matroid and partial transversal

- **The set T₁ = {e_1, e_3, e_2, e_6} is a transversal from A because**

  we have: $e_1 \in A_1$; $e_3 \in A_2$; $e_2 \in A_3$; $e_6 \in A_4$

- **The set T₂ = $\{e_6, e_4, e_2\} \subseteq E$ is a partial transversal from A because T₂ is not transversal to A, but to $\{A_1, A_2, A_3\}$ <==> ($e_6 \in A_1$; $e_4 \in A_2$; $e_2 \in A_3$)**

Uniform matroid

For k a positive integer, E a finite set

Let $I = \{ J \subseteq E : |J| \leq k \}$

The pair M(E, I) is a matroid called a uniform matroid.

Example : **Uniform Matroid**

Let E = $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ **and k=3 I =** $\{\{ e_1, e_2, e_3\}, \{ e_2, e_3\}, \{e_2, e_4, e_5\}, \{ e_5, e_6\}\}$ **==> M(E, I) is a uniform matroid.**