MOHAMED BOUDIAF UNIVERSITY OF M'SILA

DEPARTMENT OF COMPUTER SCIENCE

Advanced Graph Theory

Chapter 4: Couplings and Assignment Problem

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Couplings in graphs

Definition of a coupling

Let G(X, U) be a simple undirected graph.

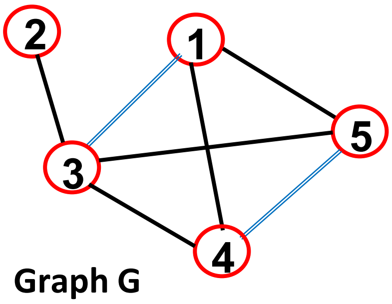
A coupling C of the graph G is a partial subgraph of G, or a subset of non-adjacent edges. (without common endpoints)

Interest in coupling

- **Allows us to solve assignment problems.**
- **Usable for solving a wide class of problems in linear programming of integers.**

Properties of a coupling

- **The cardinality of the coupling C, and we denote by |C| is the number of its edges.**
- **A vertex $s_i \in X$ is said to be saturated by the coupling C if s_i is the end of an edge of C. Otherwise, s_i is said to be unsaturated.**
- **A maximum coupling is a coipling containing the largest possible number of edges (of maximum cardinality |C| = max).**
- **In a graph, we can find several maximum couplings.**

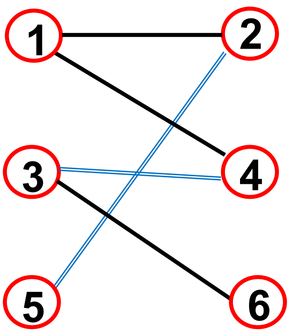


Graph G

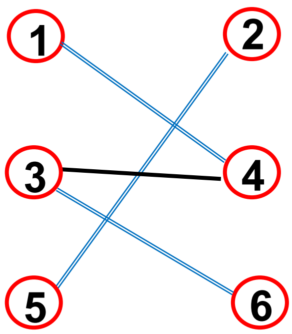
- **In blue, a maximum coupling of G of cardinality |C|=2**
- **Vertices 1, 3, 4, 5 are saturated vertices.**

- **A perfect coupling is a coupling where every vertex of the graph is saturated.**

Example:



A coupling

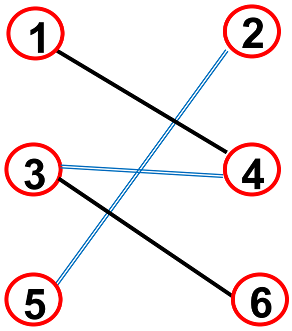


Maximum and perfect coupling

Definition of an alternating chain

- **Let C$\subset$U a coupling of G(X, U), we call an alternating chain relative to C, an elementary chain of G whose edges belong to C and outside of C (to U - C).**
- **An alternating chain is said to be augmenting or improving if it connects two unsaturated vertices.**

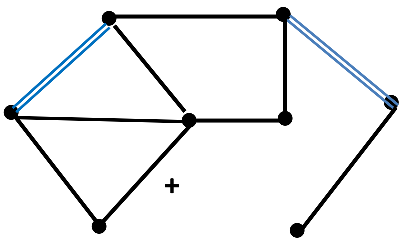
Example:



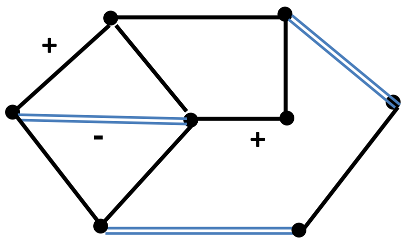
A coupling

For example the chain c=(1, 4, 3, 6) is an augmenting chain. We can then say that a coupling C is maximum if and only if there is no augmenting chain relative to C.

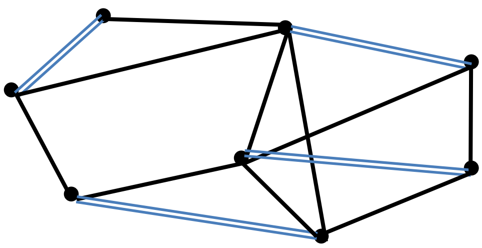
Examples:



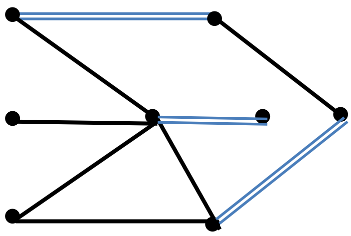
Non-maximal coupling



Maximal coupling
not-maximum



A perfect coupling



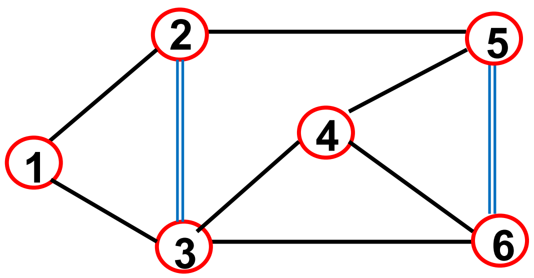
Maximal coupling
not perfect

Coupling transfer along an alternating chain

Given a coupling C⊂U, and let L be an alternating chain for which each endpoint is an unsaturated vertex (and/or) such that the unique edge of C incident to it, is in the chain L. Then the coupling C' obtained by exchanging the role of the edges of C and (U - C) along the chain L is still a coupling of G. This operation which makes the transition from the coupling C to the coupling C' is called transfer along the alternating chain L. This operation increases the cardinality of the coupling by one unit.

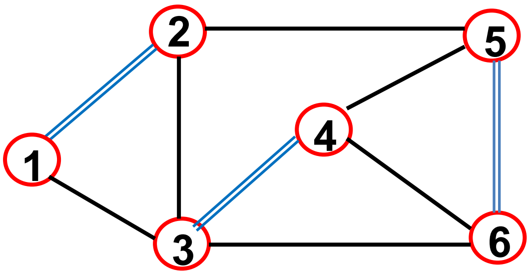
Example:

Let G(X, U) be the following graph:



- **C={(2, 3), (5, 6)} is a coupling of cardinality equal to 2.**
- **The chain L={(1, 2), (2, 3), (3, 4)} or L={1, 2, 3, 4} is an alternating chain whose ends 1 and 4 are unsaturated vertices➔It is an alternating augmenting chain.**

- **A transfer along this chain produces the new coupling C'={(1, 2), (3, 4), (5, 6)} of cardinality equal to 3.**



Remark:

The coupling C' is a maximum (and maximal) coupling, because for a graph G of order n the cardinality of a maximum coupling is the integer part of (n/2)

Definition of maximum coupling

A coupling C is maximum if only if there is no alternating augmenting chain relative to C.

Construction of a maximum coupling

To find a maximum coupling, it is sufficient to find an alternating augmenting chain (if it exists) relative to any coupling $C \subset U$, for this fact we follow the following procedure:

1. **Initialize C=ϕ**
2. **Find an improving chain c_a of C, do the transfer along this chain and replace the coupling C by the new coupling C', thus obtained (with card(C')=card(C)+1).**
3. **Until it is not possible to find a new augmenting chain, and then $k^{(i)}$ is a maximum (and maximal) coupling.**

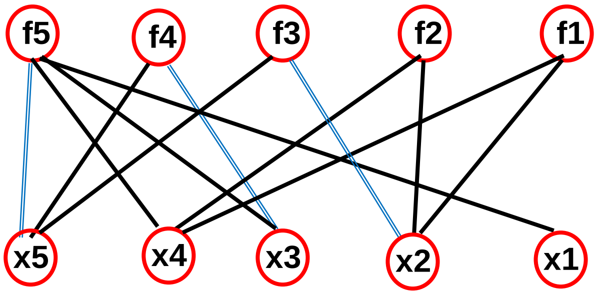
A simple method to find an augmenting alternating chain for a given coupling.

1. **We fix on the graph an origin x_0 for all the alternating chains.**
2. **We construct the tree T(X, U') which is a connected partial subgraph of G and without cycles.**
3. **For all $x_i \in X$, the unique chain $c_t(x_0 x_i)$ of the tree T is an alternating chain.**

Construction of an alternating tree T_{alt}

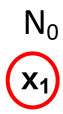
1. We choose as root of the tree T_{alt} (i.e., level N_0) an unsaturated vertex x_0 ($x_0 \notin C$)
2. At the odd level *i* (N_1, N_3, N_5, …), we insert into the tree T_{alt} a vertex adjacent to one of the vertices inserted at level *(i-1)* by an edge not belonging to the coupling C, as well as this edge.
3. At the even level $_i$ (N_2, N_4, N_6, …), we insert into the tree T_{alt} a vertex adjacent to one of the vertices inserted at level *(i-1)* by an edge belonging to the coupling C, as well as this edge.
4. We continue to gradually build this tree until we insert - at an odd level - an unsaturated vertex or until we can not insert any vertex.
5. If there exists an alternating augmenting (improving) chain of the current coupling C, we necessarily finish with inserting an unsaturated vertex *s*. The chain connecting *s* to the root x_0 of the tree T_{alt} is then an alternating augmenting (improving) chain.

Example



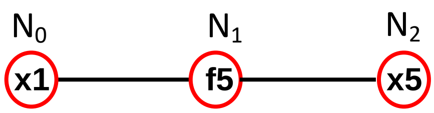
C={(f5, x5), (f4, x3), (f3, x2)}, |C|=3

1. We choose the unsaturated vertex x_1 as the root of the tree T_{alt} ($x_1 \notin C$) (level N_0)

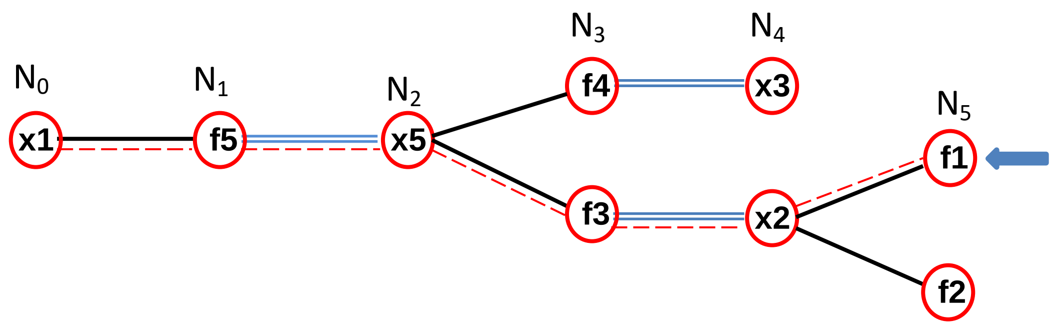


2. In the odd level N_1, we insert into the tree T_{alt} a vertex adjacent to one of the vertices already inserted (here x_1) at level *(i-1)* (i.e., N_0) by means of an edge A1$\notin$C, as well as this edge (here we take the edge $A_1=(x_1, f_5)$)

3. In the even level N_2, we insert into the tree T_{alt} a vertex adjacent to one of the vertices already inserted (x_1, f_5) at level N_1, by means of an edge $A_2 \in C$, as well as this edge (here the edge $A_2=(f_5, x_5)$ which is the only possibility)



4. We continue in the same way to progressively build the tree T_{alt} in an alternating way, i.e., insert an edge $A_i \notin C$ in an odd level, and an edge $Ai \in C$ in an even level, until an unsaturated vertex is inserted in an odd level or until no more vertex can be inserted.



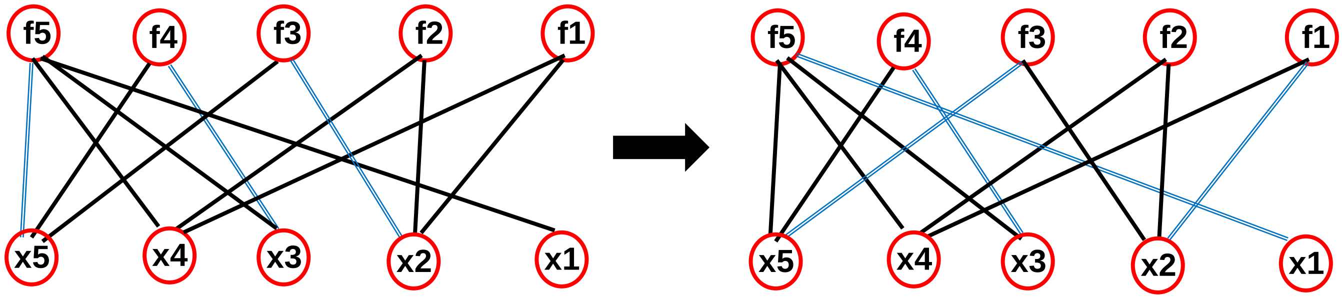
We note that for level N_6, we cannot find an edge $A_i \in C$ to insert, either for f1 or f2.

===> So we stop at level N_5, and we also note that the vertices f1, f2 are two unsaturated vertices.

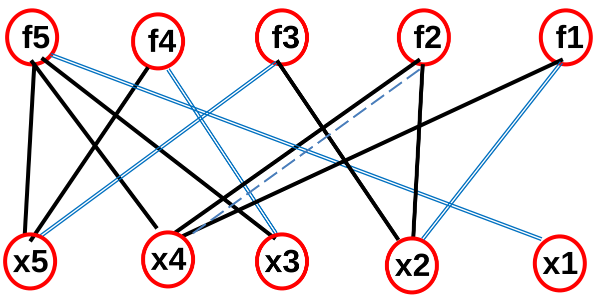
===> According to (5), there are two (02) alternating augmenting chains c_a, c_b (because we ended up with 02 unsaturated vertices f1, f2).

- The augmenting chain c_a connects f1 to the root x_1 of the tree T_{alt}.
- The augmenting chain c_b connects f2 to the root x_1 of the T_{alt} tree.

We therefore improve the following coupling c_a or c_b by bringing into the coupling C the edges which do not already belong to C and removing those which belong to C (except those which do not belong to c_a or c_b we keep them)



- If we follow c_a, we replace in C' the edge (f3, x_2) by the edge (f1, x_2).
- And to maximize this coupling, it is sufficient to add the edge (f2, x_4) to the coupling C' to obtain a maximum coupling C".



- C" is also perfect, because all vertices are saturated.

Assignment problem

Problem statement

Given n tasks t_1, t_2, t_3, …, t_n to be performed on n machines M_1, M_2, …, M_n.

And let C_{ij} be the cost of performing task t_i on the machine m_j. ($C_{ij} = \propto$ task t_i cannot be performed on machine m_j)

The assignment problem is to find a permutation σ of {1, 2, …, n} leading to a total cost : $c_{tot} = \sum_{i=1}^{n} c_{i,\sigma(i)}$ which must be minimal over the set of all $n!$ possible permutations.

The assignment problem can be considered as a special case of :

- **A transportation problem (without capacity).**
- **A minimum weight perfect coupling problem.**

Guidelines for completing the problem.

- **The optimal solution(s) should not be changed by increasing or decreasing all the elements of the same row (or column) of the C_{ij} matrix by the same value v.**
- **If we manage to make enough 0s appear in the C_{ij} matrix (but no negative costs) through these transformations, and if there are n independent zeros**

(a single 0 in each row and each column), we will have found an optimal assignment.

Hangrois algorithm.

Let the assignment problem be considered by the C_{ij} matrix below:

	1	2	3	4	5
A	7	3	5	7	10
B	6	∝	∝	8	7
C	6	5	1	5	∝
D	11	4	∝	11	15
E	∝	4	5	2	10

The Hangrois algorithm is an iterative algorithm with three phases:

Phase 1: Getting null values.

From all the elements of each of the columns, we remove the smallest element of this column, then in the matrix thus obtained, we perform the same operation on all the rows. We obtain a new matrix C_{ij} having at least one zero per row and per column.

	1	2	3	4	5
A	7	3	5	7	10
B	6	∝	∝	8	7
C	6	5	1	5	∝
D	11	4	∝	11	15
E	∝	4	5	2	10
	- 6	-3	-1	-2	-7

	1	2	3	4	5	
A	1	0	4	5	3	-0
B	0	∝	∝	6	0	-0
C	0	2	0	3	∝	-0
D	5	1	∝	9	8	-1
E	∝	1	4	0	3	-0

	1	2	3	4	5
A	1	0	4	5	3
B	0	∝	∝	6	0
C	0	2	0	3	∝
D	4	0	∝	8	7
E	∝	1	4	0	3

Phase 2: Search for maximum possible assignments

From the C_{ij} matrix, we look for a zero-cost solution (a single zero in each row and in each column). If this is the case, we have the optimal solution sought, otherwise, we carry out the following processing:

a. We take the line with a maximum number of 0.

b. We put a box one of the zeros of this line.

c. We cross zeros that are on the same row or column as the boxed zero.

d. We repeat the same previous tasks for all the lines.

	1	2	3	4	5
A	1	$\boxed{0}$	4	5	3
B	$\boxed{0}$	∝	∝	6	~~0~~
C	~~0~~	2	$\boxed{0}$	3	∝
D	4	~~0~~	∝	8	7
E	∝	1	4	$\boxed{0}$	3

Phase 3: Determining interesting assignments

To determine these assignments, we proceed as follows:

a. We mark by (*) All lines that do not contain any framed zeros.

b. We mark with (*) the columns with one or more crossed-out 0s in a row marked with (*).

c. We mark with (*) the lines with a framed zero in a marked column.

d. We repeat the previous tasks until we can not make new markings.

	1	2	3	4	5	
A	1	$\boxed{0}$	4	5	3	* See (c)
B	$\boxed{0}$	∝	∝	6	~~0~~	
C	~~0~~	2	$\boxed{0}$	3	∝	
D	4	~~0~~	∝	8	7	* See (a)
E	∝	1	4	$\boxed{0}$	3	
		* See (b)				

- The interesting assignments to consider (in the previous example) are those issued from the vertices corresponding to the lines marked (*) ===> we therefore cross out the unmarked lines.
- The vertices corresponding to the marked columns are not interesting, because they can be reassigned;

We therefore also cross out the marked columns.

	1	2	3	4	5	
A	1	$\boxed{0}$	4	5	3	*
B	$\boxed{0}$	∝	∝	6	~~0~~	
C	~~0~~	2	$\boxed{0}$	3	∝	
D	4	~~0~~	∝	8	7	*
E	∝	1	4	$\boxed{0}$	3	
		*				

- In the new table obtained after reduction (uncrossed boxes), we look for the smallest element (which must necessarily be ≠ 0)➔ in our example it is 1
- We subtract its value from the uncrossed columns, and add it to the crossed rows, we obtain:

	1	2	3	4	5
A	0	0	3	4	2
B	-1	∝	∝	5	-1
C	-1	2	-1	2	∝
D	3	0	∝	7	6
E	∝	1	3	-1	2
	- 1		-1	-1	-1

	1	2	3	4	5	
A	0	0	3	4	2	
B	0	∝	∝	6	0	+1
C	0	3	0	3	∝	+1
D	3	0	∝	7	6	
E	∝	2	4	0	3	+1

Return to phase 2 (Search for the maximum number of possible assignments) and perform the same tasks from this phase seen previously. When an optimal solution is obtained (one zero per row and per column) we stop, otherwise we start the previous phases of the algorithm again.

	1	2	3	4	5
A	$\boxed{0}$	$\cancel{0}$	3	4	2
B	$\cancel{0}$	∝	∝	6	$\boxed{0}$
C	$\cancel{0}$	3	$\boxed{0}$	3	∝
D	3	$\boxed{0}$	∝	7	6
E	∝	2	4	$\boxed{0}$	3

Optimal solution

The optimal solution is obtained using the elements:

$$C_{A1} = C_{B5} = C_{C3} = C_{D2} = C_{E4} = 0$$

And the possible permutation is:

$$\sigma = \begin{pmatrix} A & B & C & D & E \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix}$$

Positions of boxed zeros

Optimal solution

$$S = C_{A1} + C_{B5} + C_{C3} + C_{D2} + C_{E4}$$

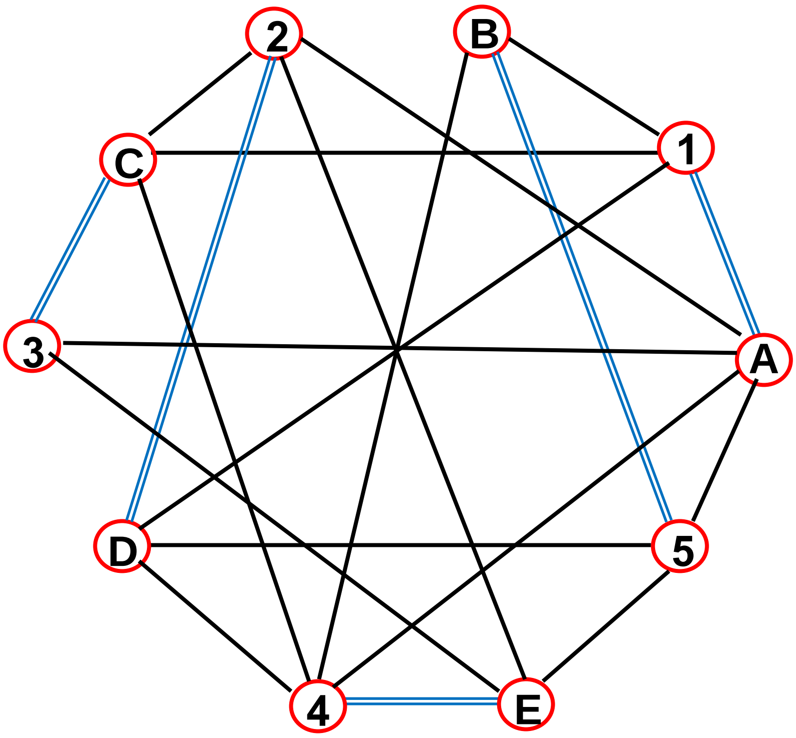
And if we return to the initial data, we obtain:

	1	2	3	4	5
A	$\boxed{7}$	3	5	7	10
B	6	∝	∝	8	$\boxed{7}$
C	6	5	$\boxed{1}$	5	∝
D	11	$\boxed{4}$	∝	11	15
E	∝	4	5	$\boxed{2}$	10

$$C_{A1} = 7;\ C_{B5} = 7;\ C_{C3} = 1;\ C_{D2} = 4;\ C_{E4} = 2$$

$\Rightarrow$ $S = 21$

The following graph considers the optimal coupling obtained



Optimal allocation