University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of **Numerical and Mathematical Analysis S 2**
– ***Distributions Analyse de Fourier II***–

<u>**Final Exam**</u> May 2024

** We use the notations of the lesson.
** We take into account of the presentation of the copy.

Exercise 1. (4 pts.) (1) (Lesson.) Give the definition of :
– $\mathcal{S}(\mathbb{R})$. – $\mathcal{S}'(\mathbb{R})$. – The Fourier transform of a distribution. – The classification of partial differential equations of 2^{ed} order.

Exercise 2. (6 pts.) (1) Let $T = 1 + 2x^2$.
(a) Prove that $T \in \mathcal{S}'(\mathbb{R})$ and calculate $\widehat{T}$.
(b) Is that $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \longrightarrow L^1_{loc}(\mathbb{R})$?

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) := \begin{cases} 1 - x \text{ if } 0 \leq x < 1\,, \\ 0 \text{ if } x \geq 1\,, \text{ even.} \end{cases}$

(a) Calculate $\widehat{f}$ (To be checked). (b) Calculate $\displaystyle\int_0^\infty \frac{\cos(y)\sin^2(y)}{y^2}\,\mathrm{d}y$.

Exercise 2. (10 pts.) (1) In $\mathbb{R}^n$. we put $r^2 = |x|^2 = x_1^2 + ... + x_n^2$ and $f(x) = g(r)$.
(a) If $f \in C^2(\mathbb{R}^n)$, calculate $\Delta f(x)$ on function of g and r.
(b) Solve the equation $\Delta f = 0$ in $\mathbb{R}^n \setminus \{0\}$.

(2) In $\mathbb{R}^2$. Integer the following differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + \frac{2y^2}{x^2 + y^4}\,, \quad 2y\frac{\mathrm{d}y}{\mathrm{d}t} = -y^2 - \frac{2x}{x^2 + y^4}\,.$$

(2) In $\mathbb{R}^2$. Let the two equations

$$y^2\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial y^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{xy}\left(y^3\frac{\partial u}{\partial x} + x^3\frac{\partial u}{\partial y}\right) \quad \cdots (E).$$

$$\frac{\partial v}{\partial y} - y\frac{\partial^2 v}{\partial y^2} = 0 \quad \cdots (F).$$

(a) What is the nature of (E) ? (b) Find the solution of (F).

Scale :
Ex. 1 : 4
Ex. 2 : (2+1)+(1.5+1.5)
Ex. 3 : (2+2)+3+(1+2)

Exercise 1. (4 pts.) (1) (Cours.) Donner la définition de :
– $\mathcal{S}(\mathbb{R})$. – $\mathcal{S}'(\mathbb{R})$. – La transformée de Fourier d'une distribution. – La classification des équations aux dérivées partielles de 2e ordre.

Exercice 2. (6 pts.) (1) Soit $T = 1 + 2x^2$.
(a) Démontrer que $T \in \mathcal{S}'(\mathbb{R})$ et calculer $\widehat{T}$.
(b) Est-ce que $\mathcal{F} :\ \mathcal{S}'(\mathbb{R}) \longrightarrow L^1_{loc}(\mathbb{R})$?

(2) Soit $f : \mathbb{R} \to \mathbb{R}$ telle que $f(x) := \begin{cases} 1 - x \text{ si } 0 \leq x < 1\,, \\ 0 \text{ si } x \geq 1\,, \text{ paire.} \end{cases}$
(a) Calculer $\widehat{f}$ (justifier la réponse). (b) Calculer $\displaystyle\int_0^{\infty} \frac{\cos(y)\sin^2(y)}{y^2}\,\mathrm{d}y$.

Exercise 2. (10 pts.) (1) Dans $\mathbb{R}^n$. On pose $r^2 = |x|^2 = x_1^2 + ... + x_n^2$ et $f(x) = g(r)$.
(a) Si $f \in C^2(\mathbb{R}^n)$, calculer $\Delta f(x)$ en fonction de g et r.
(b) Résoudre l'équation $\Delta f = 0$ dans $\mathbb{R}^n \setminus \{0\}$.

(2) Dans $\mathbb{R}^2$. Intégrer le système différentiel

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + \frac{2y^2}{x^2 + y^4}\ , \quad 2y\frac{\mathrm{d}y}{\mathrm{d}t} = -y^2 - \frac{2x}{x^2 + y^4}\ .$$

(3) Dans $\mathbb{R}^2$. Soient les équations

$$y^2\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial y^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{xy}\Big(y^3\frac{\partial u}{\partial x} + x^3\frac{\partial u}{\partial y}\Big) \quad \cdots (E).$$

$$\frac{\partial v}{\partial y} - y\frac{\partial^2 v}{\partial y^2} = 0 \quad \cdots (F).$$

(a) Quelle est la nature de (E) ? (b) Trouver la solution de (F).